

CRITICAL CONCEPTS IN DOMINATION

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Introduction

Graphs which are minimal or critical with respect to a given property frequently play an important role in the investigation of that property. Not only are such graphs of considerable interest in their own right, but also a knowledge of their structure often aids in the development of the general theory.

In particular, when investigating any finite structure, a great number of results are proven by induction. Consequently it is desirable to learn as much as possible about those graphs that are critical with respect to a given property so as to aid and abet such investigations.

In this paper we will survey two such concepts. The first, *domination critical* graphs, deals with those graphs that are critical in the sense that their domination number drops when any missing edge is added. The other, *domination perfect* graphs, is analogous to the idea of perfect graphs in the chromatic sense, and deals with those graphs that have all their induced subgraphs satisfying $\gamma(G) = i(G)$ where $i(G)$ is the independent domination number of G .

All our graphs will be finite, undirected and without loops or multiple edges. We will write $x \perp y$ to indicate that x is adjacent to y and $x \not\perp y$ when x is not adjacent to y . We will denote the neighborhood of a vertex v by $N(v)$. The minimum degree and maximum degree of G will be indicated by $\delta(G)$ and $\Delta(G)$ respectively. We will use n to represent the number of vertices in a graph and q to denote the number of edges.

Domination critical graphs

A graph is said to be *domination critical* if for every edge $e \notin E(G)$, $\gamma(G + e) < \gamma(G)$. If G is a domination critical graph with $\gamma(G) = k$ we will say G is k -domination critical or, in the context of this paper, just k -critical. The 1-critical graphs are (vacuously) K_n for $n \geq 1$.

It is also a simple matter to characterize the 2-critical graphs as was shown in Sumner–Blitch [12].

Theorem 1. *A graph G is a 2-domination critical iff $\bar{G} = \bigcup H_i$ where each H_i is a star $K_{1,n}$, $n \geq 1$. Here \bar{G} denotes the complement of G .*

A similar concept was studied by Bauer, Harary, Niemenin and Suffel [3]. They defined a graph to be critical if the deletion of any edge increased the domination number; i.e. $\gamma(G) < \gamma(G - e)$ for every $e \in E(G)$. These graphs turn out to be precisely the complements of the 2-domination-critical graphs defined here. Hence Theorem 1 above also provides a characterization of these graphs. Walikar and Acharya [1] also characterized this class of graphs.

For $k > 2$, the structure of the k -critical graphs is more complex. We will concentrate here primarily on the concept of 3-critical graphs. Many of the difficulties in understanding critical graphs are already present at this level.

The disconnected 3-critical graphs are easily characterized and so we will generally assume that our 3-critical graphs are connected.

Theorem 2. *G is a disconnected 3-critical graph iff $G = A \cup B$ where either A is trivial and B is any 2-critical graph or A is complete and B is a complete graph minus a 1-factor.*

The characterization in Theorem 2 involves the concept of a complete graph with a 1-factor removed. This type of graph occurs frequently in the study of 3-critical graphs. This phenomenon may be better understood when it is pointed out that these graphs are precisely those with the property that $\gamma(G) = 2$, and $\gamma(G - v) = 1$ for every vertex $v \in V(G)$ (i.e. the 2-vertex-critical graphs).

It is possible to generate by computer a large number of 3-critical graphs at random. We have done this and many conjectures appearing in this paper are supported by heuristic evidence compiled from these examples. Since the procedure used is not truly random in the sense of producing each 3-critical graph on n vertices with equal likelihood, we will explain how the examples are generated.

Algorithm to generate a connected random 3-critical graph. For a given number of vertices n : Generate a random permutation π of $1, 2, 3, \dots, n(n-1)/2$. Associate with each integer $i = 1, 2, 3, \dots, n(n-1)/2$ an edge $e(i)$. Pick a tree T on n vertices at ‘random’ subject only to $\gamma(T) > 2$. For each value $i = 1$ to $n(n-1)/2$, add the edge $\pi(e(i))$ to T if the domination number does not drop to 2 as a consequence. At the completion of this loop, the resulting graph must be 3-critical.

While not every connected 3-critical graph on n vertices is equally likely to be produced by this procedure, every connected 3-critical graph *can* result, and after a long run, it is reasonable to expect a good collection of examples for study. This seems to hold up in practice. Of course essentially the same procedure can also be used to generate random examples of k -critical graphs for $k > 3$.

Fig. 1 shows a selection of examples of 3-critical graphs of small order (at most 9 vertices).

The fundamentals. If x, y are non-adjacent vertices of the 3-critical graph G , then adding the edge xy to G must result in a graph having domination number 2. Consequently, there must be a vertex a in G such that either $\{x, a\}$ dominates $G - y$ or else $\{y, a\}$ dominates $G - x$. If $\{x, a\}$ dominates $G - y$, we write $[x, a] \mapsto y$. This relation imposes a natural ordering on the complement \bar{G} of G . We simply orient the edge xy of \bar{G} from x to y in case $[x, a] \mapsto y$. Note that it is possible to have both the arcs (x, y) and (y, x) for a given edge in \bar{G} . We refer to this orientation of \bar{G} as the *domination ordering* on G .

The following lemmas, established in Sumner–Blicht [12] have proven to be of considerable utility in dealing with 3-critical graphs.

Lemma 1. *If G is a 3-critical graphs and S is an independent set of $r \geq 4$ vertices in G , then the elements of S can be ordered as $a_1, a_2, a_3, \dots, a_r$ in such a way that there exists a path $x_1, x_2, x_3, \dots, x_{r-1}$ with each $x_i \notin S$, and such that for each $i = 1, 2, \dots, r - 1$, $[a_i, x_i] \mapsto a_{i+1}$.*

The proof of this lemma appears in [12] and is a direct consequence of considering the domination ordering on \bar{G} restricted to S , and noting that since this ordering on the edges in \bar{S} contains a tournament, there is a spanning directed path for \bar{S} .

Lemma 2. *If v is a cutpoint of the 3-critical graph G , then v is adjacent to an endpoint of G .*

Lemma 3. *If G is a 3-critical graph, then no two endpoints of G have a common neighbor.*

Lemma 4. *If S is an independent set of r vertices in the connected, 3-critical graph G , then S contains a vertex v with degree $\delta(v) \geq r - 2$.*

Degree sequences

It is reasonable to expect that a 3-critical graph on n vertices, where n is large, cannot have many vertices of small degree. In fact letting d_k denote the number of vertices in G having degree at most k , we can show that when n is sufficiently

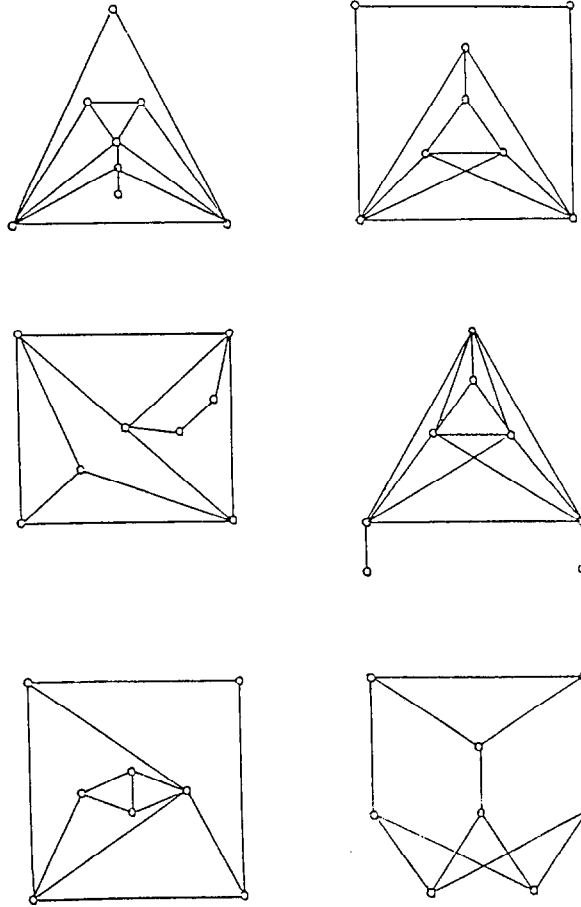


Fig. 1.

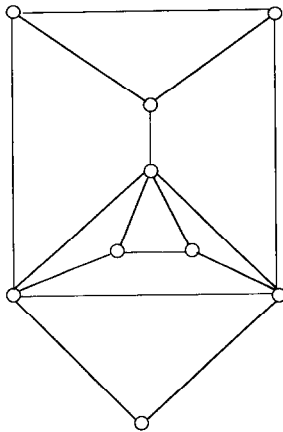


Fig. 1. (contd.)

larger than k , d_k must be bounded by a simple linear function of k . In Sumner–Blitch [12], the following bound was determined for d_k .

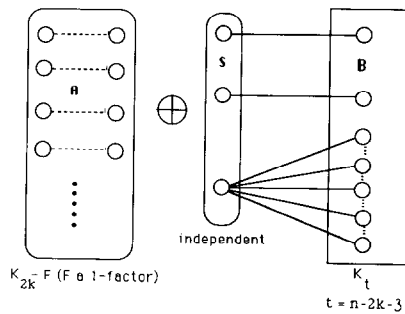
Theorem 3. *If G is 3-critical on $n \gg k$ vertices then $d_k \leq k + 1$.*

We first believed that this bound was best possible. In fact it is easy to see that d_1 can be 2 for arbitrarily large graphs, and the example in Fig. 2 shows that for every odd $k \geq 3$, d_k can be $k + 1$ for arbitrarily large graphs. The most general graphs of this type are formed by taking a 1-factor F out of a complete graph K_{2k} of even order and adjoining a set S of three independent vertices adjacent to all of the vertices in $K_{2k} - F$. Finally, add a complete graph K_t with each vertex of S adjacent to at least one vertex of K_t , and so that for each $a, b \in S$, $N(a) \cap N(b) \cap K_t = \emptyset$. The graphs in Fig. 2 are special in that two of the vertices in S are required to be adjacent to exactly one vertex of K_t . This is to force those two vertices to have degree $2k + 1$.

However, the result of Theorem 3 is not best for $k = 2$, as the next theorem shows.

Theorem 4. $d_2 \leq 2$ for $n \gg 2$.

Proof. Suppose that G is a 3-critical graph with $S = \{a, b, c\}$ the set of vertices having degree at most 2. Let $M = \{v : v \text{ is adjacent to some element of } S\}$, and let $W = V(G) - M \cup S$. We first note that no element of S can be adjacent to the other two. For suppose that $b \perp a$ and $b \perp c$, then since G is connected, there exists $x \in M$ such that $x \perp a$. But now $x \not\perp c$ as otherwise x would have to be a cutpoint of G that is not adjacent to an endpoint. Thus there would have to exist



A 3-critical graph with conjectured minimum number of edges.

The vertices in A and the top two in S all have degree $2k + 1$

Fig. 2.

y such that either (i) $[x, y] \mapsto c$ or (ii) $[y, c] \mapsto x$. In either case, a simple consideration of the possibilities—realizing that each of the vertices a and b have to be dominated—shows that y must be a . So either $[x, a] \mapsto c$ which is impossible since then we would have $\{x, b\}$ dominating G , or $[a, c] \mapsto x$ which is impossible since $\{a, c\}$ can dominate at most one vertex other than those discussed so far. Hence we will suppose that no vertex in S is adjacent to the other two. An extension of this argument shows that in fact we may assume that S is independent.

Now for any $x \in W$, x is not adjacent to a , and so there exists a $y \in M$ such that either $[x, y] \mapsto a$ or $[a, y] \mapsto x$. In the latter case it must be that y is not adjacent to x but is adjacent to all of $W - \{x\}$. Hence there can be at most one such x for each element of M . But since M can have only at most six elements and $n \gg 2$, we can assume that there exists $x \in W$ and $y \in M$ with $[x, y] \mapsto a$. Similarly we can assume that there is some $r \in W$ and $s \in M$ such that $[r, s] \mapsto b$, and a $u \in W$ and $v \in M$ such that $[v, u] \mapsto c$. But then we have $N(a) = \{s, v\}$, $N(b) = \{v, y\}$, and $N(c) = \{y, s\}$. In particular s , v , and y are all distinct.

But now since $a \not\sim b$, we may assume that there exists a w in G such that $[a, w] \mapsto b$. But a consideration of the possibilities shows that w must be s , and so $[a, s] \mapsto b$. But then it follows that $\{s, v\}$ dominates G which is impossible. \square

So the question remains as to whether $d_k \leq k + 1$ for $n \geq k$ is best possible for even values of $k \geq 4$. If not, what is the best bound on d_k for even values of k ?

Fig. 3 shows the degree sequences which are known to be possible for 3-critical graphs of order at most 9. It would be desirable to completely characterize k -critical degree sequences.

A reasonable conjecture relating to degree sequences of 3-critical graphs is:

Conjecture. If $d_1 \leq d_2 \leq \dots \leq d_n$ is the degree sequence of a 3-critical graph G , then for each $i = 0, 2, \dots, [n/2]$, $d_{i+1} + d_{n-i} \geq n - 3$.

Vizing [15] gave an upper bound on the number of edges in a graph on n vertices and having domination number k . In Blich [4] his result was improved for connected 3-critical graphs by the next theorem.

Theorem 5. If G is connected and 3-critical on n vertices then $q \leq \binom{n-2}{2}$.

This result is best possible since the graphs of the form in Fig. 4 all have $\binom{n-2}{2}$ edges.

A much harder problem is to determine the minimum number of edges in a connected domination critical graph. For 3-critical graphs we can conjecture that the graphs in Fig. 2 have as few edges as possible.

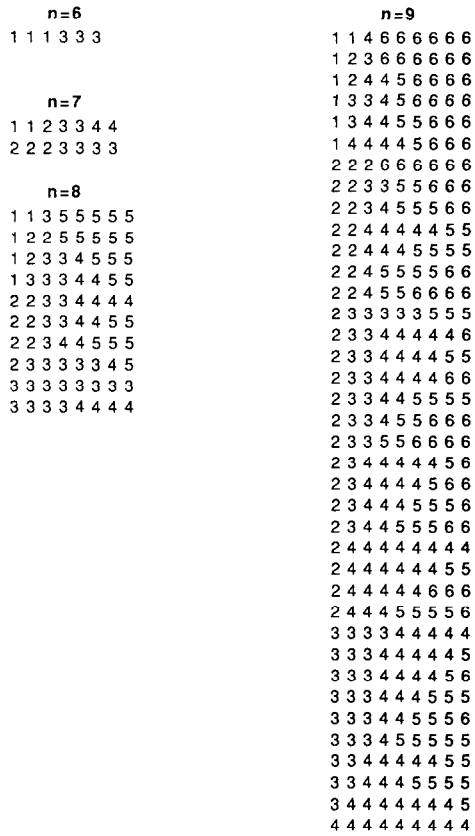


Fig. 3.

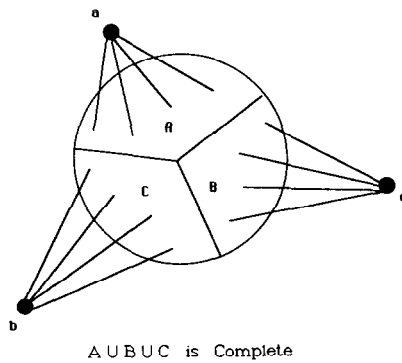


Fig. 4.

Conjecture. If G is a connected, 3-critical graph on n vertices, then the number of edges in G is at least

$$\min\left\{\binom{n-k}{2} + \binom{k}{2} + (k-4)/2\right\}$$

where the minimum is taken over all even k , $2 \leq k \leq n$. For $n \geq 10$, this minimum is achieved at essentially $k = n/2$.

Table 1 shows the conjectured minimum number of edges for 3-critical graphs on at most 14 vertices. This table has been verified by computer search for $n \leq 9$.

1-Factors. A 1-factor or a perfect matching of a graph is a partition of its vertices into adjacent pairs. Clearly such a graph must have even order. For 3-critical graphs this obvious condition was shown to be also sufficient in Sumner–Blitch [12].

The next theorem is the foundation for this result.

Theorem 6. *If G is a connected, 3-critical graph, and S is a separating set of vertices for G , then $G - S$ has at most $|S| + 1$ components.*

The following version of Tutte's characterization of graphs with 1-factors appears in Sumner [13].

Theorem 7 (Tutte). *A connected graph of even order has a 1-factor iff it does not contain a set S such that $G - S$ has at least $|S| + 2$ components of odd order.*

The next theorem is an immediate consequence of this version of Tutte's theorem and the previous theorem.

Table 1. The minimum number of edges in a connected 3-critical graph

n	Minimum edges
6	6
7	10
8	12
9	16
10	21
11	26
12	31
13	37
14	44

Theorem 8. *If G is a connected 3-critical graph of even order, then G has a 1-factor.*

Question. Heuristic evidence from computer-generated examples and degree considerations lead us to speculate that perhaps more is true. And in fact we ask if it could be that every connected, 3-critical graph on $n \geq 7$ vertices contains a Hamiltonian path. Another piece of heuristic evidence for this is that, as a consequence of Theorem 6, every 3-critical graph is at least $\frac{1}{2}$ -tough (see Chvátal [6]).

Diameters. For general graphs if $\gamma(G) = k$, then the diameter of G can be at most $3k - 1$. The situation is even more restrictive for k -critical graphs as is shown by the next theorem from Sumner–Blitch [12].

Theorem 9. *The diameter of a k -critical graph is at most $3k - 4$.*

This result is certainly not best possible.

Problem. What is the maximum diameter of a k -critical graph?

For small values of k we can be more precise. The next two results are simple to establish.

Theorem 10. *The diameter of a 3-critical graph is at most 3.*

Theorem 11. *The diameter of a 4-critical graph is at most 7.*

Independent sets and cliques. We will denote the maximum size of an independent set of vertices in a graph G by $\beta(G)$, and we will denote the size of a largest clique by $\omega(G)$.

Theorem 12. *If G is 3-critical, then $\beta(G) \leq \Delta(G)$.*

It might seem that there should be a limit on the size of independent sets in critical graphs. However this is not the case. A construction to demonstrate this was first provided by Trotter [14]. The following improvements on his construction are due to Blitch [4].

Theorem 13. *For every $n \geq 3$, there exists a 3-critical graph G with $3n$ vertices and $\beta(G) = n$.*

This theorem provides half of the proof of the next one.

Theorem 14. *If G is a 3-critical graph with $\beta(G) = t$ and as few vertices as possible, then $2t \leq |G| \leq 3t$.*

Another simple class of 3-critical graphs having arbitrarily large independent sets, and different from those described above, is the family G_n defined by $V(G_n) = \{a, b\} \cup S \cup T$, where $T = \{x_{i,j} : i, j = 1, 2, \dots, n, i \neq j\}$ is complete, and $S = \{1, 2, 3, \dots, n\}$ is independent. Let $N(a) = N(b) = S$, and for each $i, j = 1, 2, \dots, n, i \neq j$, $x_{i,j}$ is adjacent to all of $S - \{i, j\}$. It is simple to check that G_n is 3-critical. See Fig. 5 for illustration.

In [2], Allan and Laskar studied graphs for which the domination number and independent domination number are the same.

The independent domination number for G — denoted by $i(G)$ — is the size of a smallest independent dominating set for G . Since every maximal independent set in G is also a dominating set for G , $i(G)$ exists for every graph. Clearly $\gamma(G) \leq i(G)$. There is a great deal of heuristic evidence for the next conjecture.

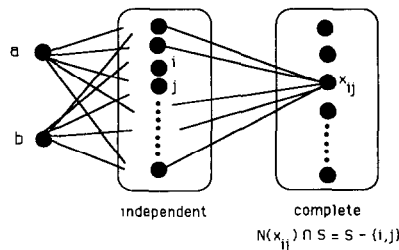
Conjecture. *If G is a connected, 3-critical graph then $\gamma(G) = i(G)$.*

This result is known to be true if G has diameter 3 or $\delta(G) < 3$. For awhile we hoped that an even stronger result might be true; namely that every vertex in a 3-critical graph was contained in an independent dominating set of order 3. This however is not true as we have determined by computer search. The smallest known counter-examples have 10 vertices.

The extreme values for $\omega(G)$ are easy. On the one hand, the next theorem from Sumner–Blitch [12] shows that every 3-critical graph contains a triangle.

Theorem 15. *If G is connected and 3-critical, then $\omega(G) \geq 3$.*

On the other hand, the largest possible size for a clique in a 3-critical graph is clearly $n - 3$. The next result from Blitch [4] shows that this size is readily achievable.



A 3-Critical Graph With A Large Independent Set

Fig. 5.

Theorem 16. *If G is connected and 3-critical with $\omega(G) = n - 3$, then $G = \{a, b, c\} \cup A \cup B \cup C$, where $A \cup B \cup C$ is complete and $N(a) = A$, $N(b) = B$, and $N(c) = C$. (See Fig. 4).*

Vertex deletions. In general removing a vertex from a graph can increase the domination number of the graph dramatically. For instance removing the central vertex of $K_{1,n}$ raises the domination number from 1 to n .

The k -critical graphs are much better behaved than graphs in general.

Theorem 17. *If G is a k -critical graph $k \geq 1$, then for every vertex v in G , $\gamma(G - v) \leq k$.*

The 3-critical graph in Fig. 6 shows that it may not always be possible to find a vertex v such that $\gamma(G - v) = k$. For this example $\gamma(G - v) = 2$ for every v . Of course every k -critical graph contains a vertex v with $\gamma(G - v) = k - 1$.

We call a graph k -vertex-domination critical if $\gamma(G) = k$ and $\gamma(G - v) = k - 1$ for every $v \in V(G)$. As we pointed out earlier, the 2-vertex-domination critical graphs are obtained by removing a 1-factor from a complete graph of even order. Note then that every 2-vertex-critical graph is also 2-critical.

Thus the example in Fig. 6 shows that it is possible for a graph to be both 3-critical and 3-vertex critical. However most 3-critical graphs are not vertex-critical. Also, it is not necessary that a 3-vertex-critical graph be 3-critical; The cycle C_7 is 3-vertex-critical, but not 3-critical.

Other connections with the independence number

There are many relationships between the independence number, $\beta(G)$, and $\gamma(G)$. This should not be too surprising since every maximal independent set in G is also a dominating set for G . In [2], Allan and Laskar showed that every

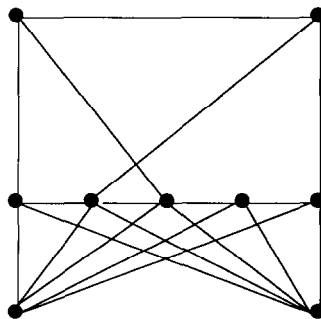


Fig. 6.

claw-free graph G (i.e. G has no induced subgraph isomorphic to $K_{1,3}$) satisfies $\gamma(G) = i(G)$. We give a somewhat different proof of their result. The technique of our proof can be used to obtain slightly more general results.

Theorem 18. *If G has no induced $K_{1,3}$, then $\gamma(G) = i(G)$.*

Proof. Suppose that G is a claw-free graph and let S be a minimum dominating set for G such that $q(S)$, the number of edges in S , is as small as possible. If S is independent, i.e. $q(S) = 0$, then we are finished. So suppose that $q(S) > 0$. Let a, b be adjacent vertices of S . Then since S is minimum, $S - \{a\}$ is not a dominating set for G . Thus there must exist a vertex x in $G - S$ such that x is adjacent to a , but x is not adjacent to any other vertex of S . But now, let $T = S - \{a\} \cup \{x\}$. Then $q(T) < q(S)$, and hence T can not be a dominating set for G . Hence there must exist a vertex y which is adjacent to a , but not to any vertex in $S \cup \{x\}$. Thus, the set $\{a, b, x, y\}$ induces a $K_{1,3}$ in G with a as the center. This completes the proof. \square

The Allan–Laskar theorem was generalized by Bollabás and Cockayne [5].

Theorem 19. *If G does not contain $K_{1,n+1}$ ($n > 1$) as an induced subgraph, then $i(G) \leq \gamma(G)(n - 1) - (n - 2)$.*

As a consequence of this theorem we can bound the domination number in terms of the independence number. First we observe (see Sumner–Moore [11]) that the cardinality of any maximal independent set must be at least an appropriate fraction of that of a maximum independent set.

Theorem 20. *If G does not contain $K_{1,n+1}$ as an induced subgraph, then every maximal independent set S in G satisfies*

$$|S| \geq \beta(G)/n.$$

As a consequence of this we get a bound on the size of $\gamma(G)$.

Theorem 21. *If G does not contain $K_{1,n+1}$ as an induced subgraph, then*

$$\gamma(G) \geq \frac{\beta(G) + n(n - 2)}{n(n - 1)}.$$

Nebesky [9] defined the concept of a partial square of a graph. Two extreme cases of partial squares of the graph G are G^2 and the line graph $L(G)$. It is possible to bound $\gamma(G)$ between the independence numbers of these two types of partial squares.

Theorem 22. $\beta(G^2) \leq \gamma(G) \leq \beta(L(G))$.

The lower bound was established in Meir–Moon [8] and the upper bound in Hedetniemi [7]. It is an interesting problem to determine whether for any graph G , there exists a partial square H of G such that $\gamma(G) = \beta(H)$.

There are other ramifications of the Allan–Laskar theorem. The following result of Sumner (unpublished) is easy to prove using the proof technique of Theorem 18.

Theorem 23. *If G is a graph such that at least one of G and G^2 does not contain an induced $K_{1,3}$, then there is a minimum dominating set S for G^2 such that S is independent in G .*

Question. A natural question that arises at this point is: When does G^n ($n \geq 3$) contain a minimum dominating set which is independent in G ?

Domination perfect graphs. Motivated by the Allan–Laskar theorem and the concept of a perfect graph, Sumner–Moore [11] defined a graph to be *domination perfect* if $\gamma(H) = i(H)$ for every induced subgraph H of G . The proofs of the results mentioned in this section will appear in [11].

As a consequence of the Allan–Laskar theorem, we have

Theorem 24 (Allan–Laskar). *Every claw-free graph is domination perfect.*

It turns out that it is not necessary to check every induced subgraph of a graph in order to determine if it is domination perfect.

Theorem 25. *A graph G is domination perfect if $\gamma(H) = i(H)$ for every induced subgraph H of G with $\gamma(H) = 2$.*

There are many classes of domination perfect graphs.

Theorem 26. *If G is chordal, then G is domination perfect iff G does not contain an induced subgraph isomorphic to the graph in Fig. 7.*

Let $\mathcal{S} = \{H : |H| \leq 8, \gamma(H) = 2, i(H) > 2\}$.

Theorem 27. *If G does not contain any member of \mathcal{S} as an induced subgraph and also does not contain an induced copy of either of the graphs in Fig. 8, then G is domination perfect.*

It is possible to completely characterize the planar domination perfect graphs by a forbidden subgraph condition.

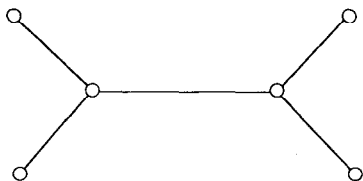


Fig. 7.

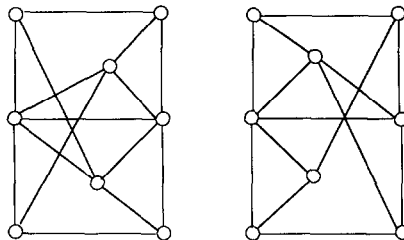


Fig. 8.

Theorem 28. *A planar graph is domination perfect iff it does not contain any graph from \mathcal{S} as an induced subgraph.*

This theorem is a direct consequence of Theorem 26 and the observation that the graphs in Fig. 8 are both nonplanar.

Unfortunately it is impossible to provide a finite forbidden characterization of the entire class of domination perfect graphs.

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