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Periodic Solutions of a Class of Non-autonomous Second-Order Systems*

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Some existence theorems are obtained by the least action principle for periodic solutions of nonautonomous second-order systems with a potential which is the sum of a subconvex function and a subquadratic function. © 1999 Academic Press

Key Words: periodic solution; Sobolev's inequality, subconvex potential; sublinear nonlinearity; second order system; coercivity; minimizing sequence.

1. INTRODUCTION AND MAIN RESULTS

Consider the second-order systems

$$\begin{aligned} \ddot{u}(t) &= \nabla F(t, u(t)) \qquad \text{a.e. } t \in [0, T] \\ u(0) - u(t) &= \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{1}$$

where $T > 0$ and $F\colon [0, T] \times R^n \to R$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for each $x \in R^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(R^+, R^+)$, $b \in L^1(0, T; R^+)$ such that

$$|F(t, x)| + |\nabla F(t, x)| \le a(|x|) b(t)$$

for all $x \in R^N$ and a.e. $t \in [0, T]$.

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The corresponding functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2\, dt + \int_0^T F(t, u(t))\, dt$$

is a continuously differentiable and weakly lower semicontinuous on H_T^1 (see [1]), where

$$H_T^1 = \{u: [0,T] \to R^N | u \text{ is absolutely continuous}, \\ u(0) = u(T) \text{ and } \dot{u} \in L^2(0,T; R^N)\}$$

is a Hilbert space with a norm defined by

$$\|u\| = \left(\int_0^T |u(t)|^2\, dt + \int_0^T |\dot{u}(t)|^2\, dt\right)^{1/2}$$

for $u \in H_T^1$.

It has been proved by the least action principle that problem (1) has at least one solution which minimizes φ on H_T^1 (see [1–8]). Specifically, [1, 2] consider problem (1) with convex potential and γ-subadditive potential, respectively. In this paper we consider problem (1) with a potential which is the sum of a subconvex function and a subquadratic function by the least action principle. The main results are the following.

A function G: $R^N \to R$ is called to be (λ, μ)-subconvex if

$$G(\lambda(x+y)) \le \mu(G(x) + G(y))$$

for some $\lambda, \mu > 0$ and all $x, y \in R^N$. A function G: $R^N \to R$ is called γ-subadditive if it is $(1, \gamma)$-subconvex. A function G: $R^N \to R$ is called subadditive if it is 1-subadditive. The convex function and the γ-subadditive function are special subconvex functions. There are subconvex functions which are neither convex nor γ-subadditive. For example, let

$$G(x) = e^{|x|^2} + 450 \ln(1 + |x|^2).$$

Then G is $(1/2, 1)$-subconvex and neither convex nor γ-subadditive.

THEOREM 1. *Assume that* $F = F_1 + F_2$, *where* F_1 *and* F_2 *satisfy assumption* (A) *and the following conditions*:

(i) $F_1(t, \cdot)$ *is* (λ, μ)-*subconvex with* $\lambda > 1/2$ *and* $\mu < 2\lambda^2$ *for* a.e. $t \in [0,T]$, *and there exist* $0 \le \alpha < 1$, $f, g \in L^1(0,T; R^+)$ *such that*

$$|\nabla F_2(t,x)| \le f(t)|x|^\alpha + g(t)$$

for a.e. $t \in [0, T]$ *and all* $x \in R^N$;

(ii)

$$\frac{1}{|x|^{2\alpha}}\left[\frac{1}{\mu}\int_0^T F_1(t, \lambda x)\, dt + \int_0^T F_2(t, x)\, dt\right] \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem (1) *has at least one solution which minimizes* φ *on* H_T^1.

COROLLARY 1. *Assume that* $F = F_1 + F_2$, *where* F_1 *and* F_2 *satisfy assumption* (A) *and the following conditions*:

(iii) $F_1(t, \cdot)$ *is subadditive for* a.e. $t \in [0, T]$, *and there exist* $0 \le \alpha < 1$, $f, g \in L^1(0, T; R^+)$ *such that*

$$\left|\nabla F_2(t, x)\right| \le f(t)|x|^{\alpha} + g(t)$$

for a.e. $t \in [0, T]$ *and all* $x \in R^N$;

(iv)

$$\frac{1}{|x|^{2\alpha}}\int_0^T F(t, x)\, dt \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem (1) *has at least one solution which minimizes* φ *on* H_T^1.

COROLLARY 2. *Assume that F satisfies assumption* (A) *and the following conditions*:

(v) $F(t, \cdot)$ *is* (λ, μ)-*subconvex with* $\lambda > 1/2$ *and* $\mu < 2\lambda^2$ *for* a.e. $t \in [0, T]$;

(vi)

$$\int_0^T F(t, x)\, dt \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem (1) *has at least one solution which minimizes* φ *on* H_T^1.

Remark 1. Theorem 1 in [2] is a special case of Corollary 1 corresponding to $\alpha = 0$ and Theorem 1 in [3] corresponds to the case $F_1 = 0$. There are functions $F(t, x)$ satisfying our Corollary 1 and not satisfying the results in [1–8]. For example, let $\alpha = 1/2$ and

$$F_1(t, x) = 3 + \sin|x|^4, \qquad F_2(t, x) = \left(\tfrac{2}{3}T - t\right)|x|^{3/2} + (h(t), x),$$

where $h \in L^1(0, T; R^N)$. Then F_1 is subadditive, ∇F_2 is sublinear, $F = F_1 + F_2$ is not convex, not γ-subadditive, not periodic, and not a.e. uniformly coercive, and ∇F is not sublinear.

THEOREM 2. *Assume that* $F = F_1 + F_2$, *where* F_1 *and* F_2 *satisfy assumption* (A) *and the following conditions*:

(vii) $F_1(t, \cdot)$ *is* (λ, μ)-*subconvex for* a.e. $t \in [0, T]$ *satisfying*

$$F_1(t, x) \geq (h(t), x) + \gamma(t)$$

for a.e. $t \in [0, T]$, *all* $x \in R^n$, *and some* $\gamma \in L^1(0, T; R)$, $h \in L^1(0, T; R^N)$ *with* $\int_0^T h(t)\, dt = 0$, *and there exist* $g \in L^1(0, T; R^+)$ *and* $C_0 \in R$ *such that*

$$|\nabla F_2(t, x)| \leq g(t)$$

for a.e. $t \in [0, T]$ *and all* $x \in R^N$, *and*

$$\int_0^T F_2(t, x)\, dt \geq C_0$$

for all $x \in R^N$.

(viii)

$$\frac{1}{\mu} \int_0^T F_1(t, \lambda x)\, dt + \int_0^T F_2(t, x)\, dt \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem (1) *has at least one solution which minimizes* φ *on* H_T^1.

COROLLARY 3. *Assume that F satisfies assumption* (A) *and the following conditions*:

(ix) $F(t, \cdot)$ *is* (λ, μ)-*subconvex for* a.e. $t \in [0, T]$ *satisfying*

$$F(t, x) \geq (h(t), x) + \gamma(t)$$

for a.e. $t \in [0, T]$, *all* $x \in R^N$, *and some* $\gamma \in L^1(0, T; R)$, $h \in L^1(0, T; R^N)$ *with* $\int_0^T h(t)\, dt = 0$;

(x)

$$\int_0^T F(t, x)\, dt \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem (1) *has at least one solution which minimizes* φ *on* H_T^1.

Remark 2. Theorem 2 slightly generalizes Theorem 2 of [2]. Corollary 3 is an extension of Theorem 1.7 in [1], for the assumption of the later implies (ix) with $\lambda = \mu = 1/2$ and $h(t) = \nabla F(t, \bar{x})$, $\gamma(t) = 0$, where $\bar{x}$ is a minimum of the real function defined by

$$x \to \int_0^T F(t, x)\, dt.$$

There are functions $F(t, x)$ satisfying our Theorem 2 and not satisfying the results in [1–8]. For example, let

$$F_1(t, x) = t\left(e^{|x|^2} + 450 \ln\left(1 + |x|^2\right)\right) \qquad F_2(t, x) = (h(t), x),$$

where $h \in L^1(0, T; R^N)$ with $\int_0^T h(t)\, dt = 0$. Then F_1 is $(1/2, 1)$-subconvex, F is not convex, not γ-subadditive, not periodic, and not a.e. uniformly coercive, and ∇F is not sublinear.

THEOREM 3. *Assume that* $F = F_1 + F_2$, *where* F_1 *and* F_2 *satisfy assumption* (A) *and the following conditions*:

(xi) $F_1(t, \cdot)$ *is* (λ, μ)*-subconvex for* a.e. $t \in [0, T]$ *satisfying*

$$F_1(t, x) \geq (h(t), x) + \gamma(t)$$

for a.e. $t \in [0, T]$, *all* $x \in R^n$, *and some* $\gamma \in L^1(0, T; R)$, $h \in L^1(0, T; R^N)$ *with* $\int_0^T h(t)\, dt = 0$, *and there exist* $0 \leq \alpha < 1$, $f, g \in L^1(0, T; R^+)$ *such that*

$$\left|\nabla F_2(t, x)\right| \leq f(t)|x|^\alpha + g(t)$$

for a.e. $t \in [0, T]$ *and all* $x \in R^N$;

(xii)

$$\frac{1}{|x|^{2\alpha}} \int_0^T F_2(t, x)\, dt \to +\infty \quad \text{as } |x| \to \infty.$$

Then problem 1 *has at least one solution which minimizes* φ *on* H_T^1.

Remark 3. Theorem 3 is an extension of Theorem 1 in [3] from $F_1 = 0$ to general F_1; it is an extension of Theorem 1.5 of [1], too. There are functions $F(t, x)$ satisfying our Theorem 3 and not satisfying the results in [1–8]. For example, let

$$F_1(t, x) = t\left(e^{|x|^2} + 450 \ln\left(1 + |x|^2\right)\right)$$
$$F_2(t, x) = \left(\frac{2}{3}T - t\right)|x|^{3/2} + (h(t), x),$$

where $h \in L^1(0, T; R^N)$. Then F_1 is $(1/2, 1)$-subconvex, ∇F_2 is sublinear, $F = F_1 + F_2$ is not convex, not γ-subadditive, not periodic, and not a.e. uniformly coercive, and ∇F is not sublinear.

2. PROOFS OF THEOREMS

For $u \in H_T^1$, let $\bar{u} = (1/T)\int_0^T u(t)\,dt$ and $\tilde{u} = u(t) - \bar{u}$. Then one has Sobolev's inequality $\|\tilde{u}\|_\infty \le C\|\dot{u}\|_{L^2}$ for all $u \in H_T^1$ and some $C > 0$ (see Proposition 1.3 in [1]).

Proof of Theorem 1. Let $\beta = \log_{2\lambda}(2\mu)$. Then $\beta < 2$. For $|x| > 1$ there exists a positive integer n such that

$$n - 1 < \log_{2\lambda}|x| \le n.$$

Then one has $|x|^\beta > (2\lambda)^{(n-1)\beta} = (2\mu)^{n-1}$ and $|x| \le (2\lambda)^n$. Hence we have

$$F_1(t,x) \le 2\mu F_1\left(t, \frac{x}{2\lambda}\right) \le \cdots \le (2\mu)^n F_1\left(t, \frac{x}{(2\lambda)^n}\right) \le 2\mu|x|^\beta a_0 b(t)$$

for a.e. $t \in [0,T]$ and all $|x| > 1$ by (i) and assumption (A), where $a_0 = \max_{0\le s\le 1} a(s)$. Moreover one obtains

$$F_1(t,x) \le \left(2\mu|x|^\beta + 1\right)a_0 b(t) \tag{2}$$

for a.e. $t \in [0,T]$ and all $x \in R^n$, where $\beta < 2$.

It follows from (i) and Sobolev's inequality that

$$\begin{aligned}
&\left|\int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]\,dt\right| \\
&\quad= \left|\int_0^T\int_0^1 \left(\nabla F_2(t,\bar{u} + s\tilde{u}(t)), \tilde{u}(t)\right) ds\,dt\right| \\
&\quad\le \int_0^T\int_0^1 f(t)|\bar{u} + s\tilde{u}(t)|^\alpha |\tilde{u}(t)|\,ds\,dt + \int_0^T\int_0^1 g(t)|\tilde{u}(t)|\,ds\,dt \\
&\quad\le 2\left(|\bar{u}|^\alpha + \|\tilde{u}\|_\infty^\alpha\right)\|\tilde{u}\|_\infty \int_0^T f(t)\,dt + \|\tilde{u}\|_\infty \int_0^T g(t)\,dt \\
&\quad\le \frac{1}{4C^2}\|\tilde{u}\|_\infty^2 + 4C^2|\bar{u}|^{2\alpha}\left(\int_0^T f(t)\,dt\right)^2 \\
&\qquad+ \|\tilde{u}\|_\infty^{\alpha+1}\int_0^T f(t)\,dt + \|\tilde{u}\|_\infty \int_0^T g(t)\,dt
\end{aligned}$$

for all $u \in H_T^1$. Hence we have

$$\left|\int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]\,dt\right| \le \tfrac{1}{4}\|\dot{u}\|_{L^2}^2 + C_1\left(|\bar{u}|^{2\alpha} + \|\dot{u}\|_{L^2}^{\alpha+1} + \|\dot{u}\|_{L^2}\right) \tag{3}$$

for all $u \in H_T^1$ and some positive constant C_1. Moreover, by (2) and (3) we have

$$\begin{aligned}
\varphi(u) \ge{}& \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,dt - \int_0^T F_1(t,-\tilde{u}(t))\,dt \\
&+ \int_0^T F_2(t,\bar{u})\,dt + \int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]\,dt \\
\ge{}& \frac{1}{4}\|\dot{u}\|_{L^2}^2 - \left(2\mu\|\tilde{u}\|_\infty^\beta + 1\right)\int_0^T a_0 b(t)\,dt - C_1\|\dot{u}\|_{L^2}^{\alpha+1} - C_1\|\dot{u}\|_{L^2} \\
&+ \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,dt + \int_0^T F_2(t,\bar{u})\,dt - C_1|\bar{u}|^{2\alpha} \\
\ge{}& \frac{1}{4}\|\dot{u}\|_{L^2}^2 - C_2\|\dot{u}\|_{L^2}^\beta - C_1\|\dot{u}\|_{L^2}^{\alpha+1} - C_1\|\dot{u}\|_{L^2} - C_3 \\
&+ |\bar{u}|^{2\alpha}\left\{\frac{1}{|\bar{u}|^{2\alpha}}\left[\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,dt + \int_0^T F_2(t,\bar{u})\,dt\right] - C_1\right\}
\end{aligned}$$

for all $u \in H_T^1$, which implies that

$$\varphi(u) \to +\infty$$

as $\|u\| \to \infty$ by (ii) because $\alpha < 1$, $\beta < 2$, and the norm $\|\cdot\|$ given by

$$\|u\| = \left(|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2\right)^{1/2}$$

is an equivalent norm on H_T^1. By Theorem 1.1 and Corollary 1.1 in [1] we complete our proof.

Proof of Theorem 2. Let (u_k) be a minimizing sequence of φ. It follows from (vii) and Sobolev's inequality that

$$\begin{aligned}
\varphi(u_k) \ge{}& \tfrac{1}{2}\|\dot{u}_k\|_{L^2}^2 + \int_0^T (h(t),u_k(t))dt + \int_0^T \gamma(t)\,dt \\
&+ \int_0^T F_2(t,\bar{u}_k)\,dt + \int_0^T\int_0^1 \left(\nabla F_2\left(t,\bar{u}_k + s\tilde{u}_k(t)\right),\tilde{u}_k(t)\right)ds\,dt \\
\ge{}& \tfrac{1}{2}\|\dot{u}_k\|_{L^2}^2 - \|\tilde{u}_k\|_\infty\int_0^T |h(t)|\,dt
\end{aligned}$$

$$+\int_0^T \gamma(t)\,dt - \|\tilde{u}_k\|_\infty \int_0^T g(t)\,dt + C_0$$

$$\geq \tfrac{1}{2}\|\dot{u}_k\|_{L^2}^2 - C_4\|\dot{u}_k\|_{L^2} - C_5$$

for all k and some constants C_4, C_5, which implies that $(\tilde{u}_k)$ is bounded. Hence we have $\|\tilde{u}_k\|_\infty$ bounded by Sobolev's inequality. On the other hand, in a way similar to the proof of Theorem 1, one has

$$\left|\int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]\,dt\right| \leq \tfrac{1}{4}\|\dot{u}\|_{L^2}^2 + C_1(\|\dot{u}\|_{L^2} + 1)$$

for all $u \in H_T^1$ and some positive constant C_1, which implies that

$$\varphi(u) \geq \frac{1}{2}\|\dot{u}_k\|_{L^2}^2 + \frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u}_k)\,dt - \int_0^T F_1(t, -\tilde{u}_k(t))\,dt$$

$$+ \int_0^T F_2(t,\bar{u}_k)\,dt + \int_0^T [F_2(t,u(t)) - F_2(t,\bar{u}_k)]\,dt$$

$$\geq \frac{1}{4}\|\dot{u}_k\|_{L^2}^2 - a(\|\tilde{u}_k\|_\infty)\int_0^T b(t)\,dt - C_1\|\dot{u}_k\|_{L^2} - C_1$$

$$+ \frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u}_k)\,dt + \int_0^T F_2(t,\bar{u}_k)\,dt$$

for all positive integers k and some positive constant C_1. It follows from (viii) and the boundedness of $(\tilde{u}_k)$ that $(\bar{u}_k)$ is bounded. Hence φ has a bounded minimizing sequence (u_k). Now Theorem 2 follows from Theorem 1.1 and Corollary 1.1 in [1].

Proof of Theorem 3. From (xi), (3), and Sobolev's inequality it follows that

$$\varphi(u) \geq \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T (h(t), u(t))\,dt + \int_0^T \gamma(t)\,dt$$

$$+ \int_0^T F_2(t,\bar{u})\,dt + \int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]\,dt$$

$$\geq \frac{1}{4}\|\dot{u}\|_{L^2}^2 - \|\tilde{u}\|_\infty \int_0^T |h(t)|\,dt + \int_0^T \gamma(t)\,dt - C_1\|\dot{u}\|_{L^2}^{\alpha+1}$$

$$- C_1\|\dot{u}\|_{L^2} + \int_0^T F_2(t,\bar{u})\,dt - C_1|\bar{u}|^{2\alpha}$$

$$\geq \frac{1}{4}\|\dot{u}\|_{L^2}^2 - C_1\|\dot{u}\|_{L^2}^{\alpha+1} - C_6(\|\dot{u}\|_{L^2} + 1)$$

$$+ |\bar{u}|^{2\alpha}\left[\frac{1}{|\bar{u}|^{2\alpha}}\int_0^T F_2(t,\bar{u})\,dt - C_1\right]$$

for all $u \in H_T^1$ and some positive constants C_1, C_6. As per the proof of Theorem 1, φ is coercive by (xii), which completes the proof.

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