ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 258 (2010) 814–851

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Coercive inequalities on metric measure spaces ☆

W. Hebisch [a], B. Zegarliński [b,*]

[a] *Institute of Mathematics, University of Wrocław, Wrocław, Poland*
[b] *Department of Mathematics, Imperial College, London, UK*

Received 11 May 2009; accepted 15 May 2009

Available online 30 May 2009

Communicated by L. Gross

Abstract

In this paper we study coercive inequalities on finite dimensional metric spaces with probability measures which do not have the volume doubling property.

Keywords: Coercive inequalities; Logarithmic Sobolev; U-bounds; Heisenberg group

1. Introduction

In this paper we study coercive inequalities on finite dimensional metric spaces with probability measures which do not have the volume doubling property. This class of inequalities includes the well-known Poincaré inequality

$$M\mu|f - \mu f|^q \leqslant \mu|\nabla f|^q$$

with some constants $M \in (0, \infty)$, $q \in (1, \infty)$ independent of the function f. The (metric) length of the gradient $|\nabla f|$ is assumed to be well defined here. This class also includes a variety of stronger coercive inequalities with the variance on the left-hand side replaced by a functional

☆ Supported by EPSRC EP/D05379X/1 & Royal Society. The first author was partially supported by KBN grant 1 P03A 03029.
* Corresponding author.
E-mail address: b.zegarlinski@imperial.ac.uk (B. Zegarliński).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.05.016

with a stronger growth, as for example in case of the celebrated Log-Sobolev inequality, which is of the following form

$$\mu f^2 \log \frac{f^2}{\mu f^2} \leqslant c\mu |\nabla f|^2$$

with some constant $c \in (0, \infty)$ independent of the function f.

We are interested in probability measures on non-compact spaces, like for example finite products of real lines $\mathbb{R}^n$, as well as certain non-compact groups, including the Heisenberg group.

For probability measures on the real line the necessary and sufficient condition for Poincaré inequality characterising the density (of the absolutely continuous part with respect to the Lebesgue measure) were established a long time ago by Muckenhoupt [31] ([29]). More recently such criteria were established for other coercive inequalities (Log-Sobolev type: (LS_2) [8], (LS_q) [11], for distributions with weaker tails [6], etc., and others). In the multidimensional case the situation is rather different and more intricate. First of all, since the inequalities of interest to us have a natural tensorisation property, there is a number of perturbative techniques which allow to obtain classes of interesting examples in higher and even in infinite dimensions (see e.g. [21,11,33,28], etc., [12,36,37,41,42], etc., and references given there). We would like to mention a work [34] in which the coercive inequalities for probability measures on $\mathbb{R}^n$, $n \geqslant 3$, with a variety of decay of the tails (slower as well as faster than the Gaussian) were systematically studied with the help of classical Sobolev inequalities providing in particular an effective sufficient criteria (in terms of certain non-linear differential inequalities for the log of the density function), for related coercive inequalities (see also reviews [35,18] and references therein). In the mid 80'ties Bakry and Emery [4] introduced a very effective criterion based on convexity (curvature) which allowed to enlarge the class of examples where a Log-Sobolev inequality holds, including the situation with measures on certain finite dimensional Riemannian manifolds (as well as some infinite dimensional cases, however with a compact configuration space [14]). Following a similar line of reasoning, in [3] the authors provided an effective criteria for (generalisation) of Brascamp–Lieb inequality [13] as well as Log-Sobolev inequality (with possibly more general entropy functional and weighted Dirichlet form dependent on the measure).

More recently, in [9], certain convexity ideas (including Brunn–Minkowski inequality), were exploited to recover in the special case of the space $\mathbb{R}^n$ similar results as in [3] and obtained additionally inequalities (LS_q) which are naturally related to metrics different from the Euclidean metric (and in particular involving a different length of the gradient in the preceding inequalities). These results concerned principally the probability measures with tails decaying faster than the Gaussian. We point out that while such distributions were also discussed in [34], in [9] they involved in a natural way Lipschitz functions with respect to a non-Euclidean metric (while in Rosen's work the emphasis of improvement was on different functionals on the left-hand side). The corresponding results for measures on $\mathbb{R}^n$ with slower distribution tails were obtained in [6] (see also references therein), which included in particular those of Rosen [34] for a similar class of measures.

Part of the motivation for the current paper was provided by [28] in which the coercive inequalities involving Hörmander fields instead of the (non-degenerate full gradient) were studied. Such a situation is naturally related to a more general Carnot–Caratheodory metric associated to the family of fields and the interest here is to obtain coercive inequalities involving length of the corresponding metric gradient. While in [28] a rich family of examples on compact spaces was provided, non-compact situation is more difficult. In this paper we develop an efficient technology which not only recovers interesting results in $\mathbb{R}^n$ briefly reviewed in the above, but also allows us to extend to interesting metric spaces, such as certain non-compact Lie groups, including in particular the Heisenberg group. Part of our approach is directed at proving inequalities, which we call U-bounds, of the following form

$$\int |f|^q U\,d\mu \leqslant C\int |\nabla f|^q\,d\mu + D\int |f|^q\,d\mu$$

with a suitable increasing unbounded function U of the metric and the length of the metric gradient $|\nabla f|$; see Section 2. We show later in Sections 3 and 4 that such an inequality implies corresponding Poincaré as well as other coercive inequalities; in fact as we illustrate in some of the cases the U-bounds are equivalent to the coercive inequalities. (This requires an extension of a result on a Gaussian exponential bound of [1] for other measures and functions with possibly unbounded gradient.)

In Section 5 we explore also a family of weighted Poincaré and Log-Sobolev inequalities on Riemannian manifolds including measures with ultra slow tails. In such a context we can effectively employ a Laplacian comparison theorem (see e.g. [16]), which in particular allows us to extend recent results of [10] where convexity ideas in Euclidean spaces were used.

As an application of our technique we also prove (see Sections 6–7) the Log-Sobolev inequality for the heat kernel measure on the Heisenberg group (a topic which attracted recently some extra attention [26,27,19]).

2. U-bounds

By ∇ we denote a subgradient in $\mathbb{R}^N$, that is a finite collection of possibly non-commuting fields. It is assumed that the divergence of each of these fields with respect to the Lebesgue measure Λ on $\mathbb{R}^N$ is zero. (While this provides some simplification in our expositions, it is possible to extend our arguments to a more general setting.)

We begin with proving the following result.

Theorem 2.1. *Let* $d\mu_p = \frac{e^{-\beta d^p}}{Z}\,d\lambda$ *be a probability measure defined with* $\beta \in (0,\infty)$ *and* $p \in (1,\infty)$ *(*Z *being the normalisation constant). Suppose* $0 < \frac{1}{\sigma} \leqslant |\nabla d| \leqslant 1$*, for some* $\sigma \in [1,\infty)$*, and* $\Delta d \leqslant K + \beta p\varepsilon d^{p-1}$ *outside the unit ball* $B \equiv \{d(x) < 1\}$ *for some* $K \in [0,\infty)$ *and* $\varepsilon \in [0, \frac{1}{\sigma^2})$*. Then there exist constants* $C, D \in (0,\infty)$ *such that the following bound is true:*

$$\int |f| d^{p-1}\,d\mu_p \leqslant C\int |\nabla f|\,d\mu_p + D\int |f|\,d\mu_p. \tag{1}$$

Remark. In particular the assumptions of the theorem are satisfied for d being the Carnot–Caratheodory distance and ∇ the (horizontal) gradient of the Heisenberg group.

Proof of Theorem 2.1. For a smooth function $f \geqslant 0$ such that $f = 0$ on the unit ball, by the Leibniz rule we have

$$(\nabla f)e^{-\beta d^p} = \nabla\big(fe^{-\beta d^p}\big) + \beta pf\big(d^{p-1}\nabla d\big)e^{-\beta d^p}. \tag{2}$$

Put

$$\boldsymbol{\alpha}(\cdot) \equiv \int (\nabla d)(\cdot)\, d\lambda.$$

Acting with this functional on the expression (2) we get

$$\boldsymbol{\alpha}\big((\nabla f)e^{-\beta d^p}\big) = \boldsymbol{\alpha}\big(\nabla\big(f e^{-\beta d^p}\big)\big) + \beta p \int f d^{p-1} |\nabla d|^2 e^{-\beta d^p}\, d\lambda. \tag{3}$$

Using Hölder inequality, the left-hand side of (3) can be estimated from above as follows

$$\boldsymbol{\alpha}\big((\nabla f)e^{-\beta d^p}\big) = \int (\nabla d)\cdot(\nabla f)e^{-\beta d^p}\, d\lambda \leqslant \int |\nabla d||\nabla f| e^{-\beta d^p}\, d\lambda \leqslant \int |\nabla f| e^{-\beta d^p}\, d\lambda \tag{4}$$

where we have used the fact that $|\nabla d| \leqslant 1$. The first term on the right-hand side of (3) can be treated with the help of integration by parts as follows

$$\begin{aligned}\boldsymbol{\alpha}\big(\nabla\big(f e^{-\beta d^p}\big)\big) &= \int (\nabla d)\cdot \nabla\big(f e^{-\beta d^p}\big)\, d\lambda = -\int (\Delta d) f e^{-\beta d^p}\, d\lambda \\ &\geqslant -K \int f e^{-\beta d^p}\, d\lambda - \beta p \varepsilon \int f d^{p-1} e^{-\beta d^p}\, d\lambda \end{aligned}\tag{5}$$

where we have used the assumption that $\Delta d \leqslant K + \beta p \varepsilon d^{p-1}$. Combining (3), (4) and (5), we get

$$\beta p \int f d^{p-1}\big(|\nabla d|^2 - \varepsilon\big) e^{-\beta d^p}\, d\lambda \leqslant \int |\nabla f| e^{-\beta d^p}\, d\lambda + K \int f e^{-\beta d^p}\, d\lambda$$

from which the inequality (1) follows with $C = \frac{1}{(1/\sigma^2 - \varepsilon)\beta p}$ and $D = \frac{K}{(1/\sigma^2-\varepsilon)\beta p}$, provided $\varepsilon \in [0, \frac{1}{\sigma^2})$.

Now, the estimate (1) is proven for smooth non-negative f which vanish on the unit ball. We can handle non-smooth functions approximating them by smooth ones (on compact sets via convolution and splitting f into compactly supported pieces using a smooth partition of unity — details are tedious but do not pose any essential difficulty).

We can handle f of arbitrary sign replacing f by $|f|$ and using equality $\nabla|f| = \operatorname{sgn}(f)\nabla f$.

To handle f which are non-zero on the unit ball we write $f = f_0 + f_1$ where $f_0 = \phi f$, $f_1 = (1-\phi) f$ and $\phi(x) = \min(1, \max(2 - d(x), 0))$. Then

$$\begin{aligned}\int |f| d^{p-1}\, d\mu_p &= \int_{d(x)\leqslant 2} |f| d^{p-1}\, d\mu_p + \int_{d(x)>2} |f| d^{p-1}\, d\mu_p \\ &\leqslant 2^{p-1} \int_{d(x)\leqslant 2} |f|\, d\mu_p + \int_{d(x)>2} |f|_1 d^{p-1}\, d\mu_p \\ &\leqslant 2^{p-1} \int |f|\, d\mu_p + \int |f|_1 d^{p-1}\, d\mu_p.\end{aligned}$$

Next

$$|\nabla f_1| \leqslant |\nabla f| + |f|,$$
$$\int |f_1| d^{p-1}\, d\mu_p \leqslant C \int |\nabla f_1|\, d\mu_p + D \int |f_1|\, d\mu_p$$
$$\leqslant C \int |\nabla f|\, d\mu_p + (D + C) \int |f|\, d\mu_p.$$

Combining inequalities above we see that (1) is valid without restriction on the support of f if we replace D by $D + 2^{p-1} + C$. □

Using our result and a perturbation technique we obtain the following generalisation.

Theorem 2.2. *Let $d\mu = \frac{e^{-W-V}}{Z'}\, d\mu_\theta$ be a probability measure defined with a differentiable potential W satisfying*

$$|\nabla W| \leqslant \delta d^{p-1} + \gamma_\delta \tag{6}$$

with some constants $\delta < 1/C$ and $\gamma_\delta \in (0, \infty)$, and suppose that V is a measurable function such that $osc(V) \equiv \max V - \min V < \infty$. Then there exist constants $C', D' \in (0, \infty)$ such that the following bound is true:

$$\int |f| d^{p-1}\, d\mu \leqslant C' \int |\nabla f|\, d\mu + D' \int |f|\, d\mu. \tag{7}$$

Remark. In particular the assumption (6) of the theorem is satisfied if W is a polynomial of lower order in d. Another example, in the spirit of [20] and [11], with deep wells is as follows

$$W = \vartheta d^{p-1} \cos(d)$$

with a small constant $\vartheta > 0$ (but $\vartheta d^{p-1} \cos(d^{1+\varepsilon})$ would not work for any $\varepsilon > 0$ no matter how small $\vartheta > 0$ would be).

Proof of Theorem 2.2. We consider first the case $V = 0$ and start from substituting fe^{-W} in the inequality (1) for the measure μ_p. Using Leibniz rule

$$\int |f| d^{p-1} e^{-W}\, d\mu_p \leqslant C \int |\nabla f| e^{-W}\, d\mu_p + D \int |f| e^{-W}\, d\mu_p + C \int |f| |\nabla W| e^{-W}\, d\mu_p.$$

Now our assumption (6) about W implies

$$\int |f| |\nabla W| e^{-W}\, d\mu_p \leqslant \delta \int |f| d^{p-1} e^{-W}\, d\mu_p + \gamma_\delta \int |f| e^{-W}\, d\mu_p.$$

Thus combining these bounds we arrive at

$$\int |f| d^{p-1} e^{-W}\, d\mu_p \leqslant \bar{C} \int |\nabla f| e^{-W}\, d\mu_p + \bar{D} \int |f| e^{-W}\, d\mu_p$$

with

$$\bar{C} \equiv C/(1 - C\delta) \quad \text{and} \quad \bar{D} = (D + \gamma_\delta)/(1 - C\delta).$$

Next we note that if $V \neq 0$ we have

$$\begin{aligned}
&\int |f| d^{p-1} \frac{e^{-W-V}}{Z'} \, d\mu_p \\
&\leqslant e^{osc(V)} \int |f| d^{p-1} \frac{e^{-W}}{\int e^{-W} \, d\mu_p} \, d\mu_p \\
&\leqslant e^{osc(V)} \bar{C} \int |\nabla f| \frac{e^{-W}}{\int e^{-W} \, d\mu_p} \, d\mu_p + e^{osc(V)} \bar{D} \int |f| \frac{e^{-W}}{\int e^{-W} \, d\mu_p} \, d\mu_p \\
&\leqslant e^{2osc(V)} \bar{C} \int |\nabla f| \frac{e^{-W-V}}{Z'} \, d\mu_p + e^{2osc(V)} \bar{D} \int |f| \frac{e^{-W-V}}{Z'} \, d\mu_p. \qquad \square
\end{aligned}$$

Theorem 2.3. *Let μ be a probability measure for which conclusion of Theorem* 2.1 *holds. Let $p \in (1, \infty)$. Then for each $q \in [1, \infty)$ there exist constants $C_q, D_q \in (0, \infty)$ such that the following bound is true*:

$$\int |f|^q d^{q(p-1)} \, d\mu \leqslant C_q \int |\nabla f|^q \, d\mu + D_q \int |f|^q \, d\mu. \tag{8}$$

Proof. Let $d_1(x) = \max(1, d(x))$. Enlarging constants D if necessary we may assume that

$$\int |f| d_1^{p-1} \, d\mu \leqslant C \int |\nabla f| \, d\mu + D \int |f| \, d\mu.$$

Put $h = |f|^q d_1^{(p-1)(q-1)}$. We have

$$\begin{aligned}
\int |f|^q d^{q(p-1)} \, d\mu \leqslant \int |f|^q d_1^{q(p-1)} \, d\mu &= \int h d_1^{p-1} \, d\mu \\
&\leqslant C \int |\nabla h| \, d\mu + D \int h \, d\mu.
\end{aligned}$$

By Leibniz formula

$$|\nabla h| = q |\nabla f| |f|^{(q-1)} d_1^{(q-1)(p-1)} + (q-1)(p-1) |\nabla d_1| |f|^q d_1^{(q-1)(p-1)-1}$$

and

$$\begin{aligned}
&\int q |\nabla f| |f|^{(q-1)} d_1^{(q-1)(p-1)} \, d\mu \\
&\leqslant q \left(\int |\nabla f|^q \, d\mu \right)^{1/q} \left(\int \left(|f|^{q-1} d_1^{(q-1)(p-1)} \right)^{q/(q-1)} \, d\mu \right)^{(q-1)/q} \\
&\leqslant \alpha^q \int |\nabla f|^q \, d\mu + \frac{q-1}{\alpha^{q/(q-1)}} \int |f|^q d_1^{q(p-1)} \, d\mu.
\end{aligned}$$

Next

$$\int h\,d\mu = \int |f|^q d_1^{(q-1)(p-1)}\,d\mu \leqslant \left(\int |f|^q\,d\mu\right)^{1/q}\left(\int |f|^q d_1^{q(p-1)}\,d\mu\right)^{(q-1)/q}$$
$$\leqslant \frac{\beta^q}{q}\int |f|^q\,d\mu + \frac{q-1}{\beta^{q/(q-1)}q}\int |f|^q d_1^{q(p-1)}\,d\mu.$$

If $(q-1)(p-1) \leqslant 1$, then

$$\int (q-1)(p-1)|\nabla d_1||f|^q d_1^{(q-1)(p-1)-1}\,d\mu \leqslant \int |f|^q\,d\mu.$$

If $(q-1)(p-1) > 1$, then

$$\int (q-1)(p-1)|\nabla d_1||f|^q d_1^{(q-1)(p-1)-1}\,d\mu$$
$$\leqslant (q-1)(p-1)\int |f|^q d_1^{(q-1)(p-1)-1}\,d\mu$$
$$\leqslant (q-1)(p-1)\left(\int |f|^q\,d\mu\right)^{p/(q(p-1))}\left(\int |f|^q d_1^{q(p-1)}\,d\mu\right)^{((p-1)(q-1)-1)/(q(p-1))}$$
$$\leqslant \frac{(q-1)p}{q}\gamma^{q(p-1)/p}\int |f|^q\,d\mu + \frac{(q-1)^2(p-1)-(q-1)}{q\gamma^{q(p-1)/((q-1)(p-1)-1)}}\int |f|^q d_1^{q(p-1)}\,d\mu.$$

Combining inequalities above, if $(q-1)(p-1) \leqslant 1$ we get

$$\left(1 - C\frac{q-1}{\alpha^{q/(q-1)}} - D\frac{q-1}{\beta^{q/(q-1)}q}\right)\int |f|^q d_1^{q(p-1)}\,d\mu$$
$$\leqslant C\alpha^q\int |\nabla f|^q\,d\mu + \left(C + D\frac{\beta^q}{q}\right)\int |f|^q\,d\mu$$

which gives the claim with

$$C_q = \frac{C\alpha^q}{1 - C\frac{q-1}{\alpha^{q/(q-1)}} - D\frac{q-1}{\beta^{q/(q-1)}q}}$$

and

$$D_q = \frac{C + D\frac{\beta^q}{q}}{1 - C\frac{q-1}{\alpha^{q/(q-1)}} - D\frac{q-1}{\beta^{q/(q-1)}q}}$$

if α and β are big enough. Similarly, for $(q-1)(p-1) > 1$ we get the claim with

$$C_q = \frac{C\alpha^q}{1 - C\frac{q-1}{\alpha^{q/(q-1)}} - C\frac{(q-1)^2(p-1)-(q-1)}{q\gamma^{q(p-1)/((q-1)(p-1)-1)}} - D\frac{q-1}{\beta^{q/(q-1)}q}}$$

and

$$D_q = \frac{Cp\gamma^{q(p-1)/p} + D\frac{\beta^q}{q}}{1 - C\frac{q-1}{\alpha^{q/(q-1)}} - C\frac{(q-1)^2(p-1)-(q-1)}{q\gamma^{q(p-1)/((q-1)(p-1)-1)}} - D\frac{q-1}{\beta^{q/(q-1)}q}}$$

if α, β and γ are big enough. □

Theorem 2.4. *Let $d\mu_p = \frac{e^{-\beta d^p}}{Z} d\lambda$ be a probability measure defined with $\beta \in (0,\infty)$ and $p \in [2,\infty)$ (Z being the normalisation constant). Suppose $0 < \frac{1}{\sigma} \leqslant |\nabla d| \leqslant 1$, for some $\sigma \in [1,\infty)$, and $\Delta d \leqslant K + \beta p\varepsilon d^{p-1}$ outside the unit ball $B \equiv \{d(x) < 1\}$ for some $K \in [0,\infty)$ and $\varepsilon \in [0, \frac{1}{\sigma^2})$.*

Suppose $\frac{1}{q} + \frac{1}{p} = 1$, then we have

$$\int |f|^q d^p \, d\mu \leqslant C_q \int |\nabla f|^q \, d\mu + D_q \int |f|^q \, d\mu. \tag{9}$$

Remark. In particular the assumptions of the theorem are satisfied for d being the Carnot–Caratheodory distance and ∇ the (horizontal) gradient of the Heisenberg group.

Proof of Theorem 2.4. This is a special case of Theorem 2.3. □

Extension to more general measures is as follows.

Theorem 2.5. *Let $d\mu = \frac{e^{-W-V}}{Z'} d\mu_p$ be a probability measure defined with a differentiable potential W satisfying*

$$|\nabla W|^q \leqslant \delta d^p + \gamma_\delta \tag{10}$$

with some constants $\delta 2^{q-1} q^{-q} C < 1$ and $\gamma_\delta \in (0,\infty)$, and suppose that V is a measurable function such that $osc(V) \equiv \max V - \min V < \infty$. Then there exist constants $C', D' \in (0,\infty)$ such that the following bound is true:

$$\int |f|^q d^p \, d\mu \leqslant C' \int |\nabla f|^q \, d\mu + D' \int |f|^q \, d\mu \tag{11}$$

with q such that $\frac{1}{q} + \frac{1}{p} = 1$.

The proof is similar to that of Theorem 2.2.

2.1. U-bounds: Sub-quadratic case

Theorem 2.6. *Let $d\mu_\theta = \frac{e^{-\beta d^\theta}}{Z} d\lambda$ be a probability measure defined with $\beta \in (0,\infty)$ and $\theta \in [1,2]$ (Z being a normalisation constant). Suppose $0 < \frac{1}{\sigma} \leqslant |\nabla d| \leqslant 1$, for some $\sigma \in [1,\infty)$, and $\Delta d \leqslant K + \beta p\varepsilon d^{p-1}$ outside the unit ball $B \equiv \{d(x) < 1\}$ for some $K \in [0,\infty)$ and $\varepsilon \in [0, \frac{1}{\sigma^2})$.*

Then there exist constants $C_\theta, D_\theta \in (0,\infty)$ *such that the following bound is true*

$$\int |f|^2 d^{2(\theta-1)}\, d\mu_\theta \leqslant C_\theta \int |\nabla f|^2\, d\mu_\theta + D_\theta \int |f|^2\, d\mu_\theta. \tag{12}$$

Remark. In particular the assumptions of the theorem are satisfied for d being the Carnot–Caratheodory distance and ∇ the (horizontal) gradient of the Heisenberg group.

Proof of Theorem 2.6. Again, this is a special case of Theorem 2.3. □

Extension to more general measures is as follows.

Theorem 2.7. *Let* $d\mu = \frac{e^{-W-V}}{Z'}\, d\mu_\theta$ *be a probability measure defined with a differentiable potential* W *satisfying*

$$|\nabla W|^2 \leqslant \delta d^{2(\theta-1)} + \gamma_\delta \tag{13}$$

with some constants $\delta C/2 < 1$ *and* $\gamma_\delta \in (0,\infty)$, *and suppose that* V *is a measurable function such that* $osc(V) \equiv \max V - \min V < \infty$. *Then there exist constants* $C', D' \in (0,\infty)$ *such that the following bound is true.*

$$\int |f|^2 d^\theta\, d\mu \leqslant C' \int |\nabla f|^2\, d\mu + D' \int |f|^2\, d\mu. \tag{14}$$

Again, the proof is similar to that of Theorem 2.2.

3. Poincaré inequality

Theorem 3.1. *Suppose* $1 \leqslant q < \infty$ *and a measure* λ *satisfies the* q*-Poincaré inequality for every ball* B_R, *that is there exists a constant* $c_R \in (0,\infty)$ *such that*

$$\frac{1}{|B_R|}\int_{B_R} \left| f - \frac{1}{|B_R|}\int_{B_R} f \right|^q d\lambda \leqslant c_R \frac{1}{|B_R|}\int_{B_R} |\nabla f|^q\, d\lambda. \tag{15}$$

Let μ *be a probability measure on* $\mathbb{R}^n$ *which is absolutely continuous with respect to the measure* λ *and such that*

$$\int f^q \eta\, d\mu \leqslant C \int |\nabla f|^q\, d\mu + D \int f^q\, d\mu \tag{16}$$

with some non-negative function η *and some constants* $C, D \in (0,\infty)$ *independent of a function* f. *If for any* $L \in (0,\infty)$ *there is a constant* A_L *such that*

$$\frac{1}{A_L} \leqslant \frac{d\mu}{d\lambda} \leqslant A_L \tag{17}$$

on the set $\{\eta < L\}$ *and, for some* $R \in (0, \infty)$ *(depending on* L*), we have* $\{\eta < L\} \subset B_R$*, then* μ *satisfies the* q*-Poincaré inequality*

$$\mu|f - \mu f|^q \leqslant c\mu|\nabla f|^q. \tag{18}$$

Proof. For any a we have

$$\mu|f - \mu f|^q \leqslant 2^q \mu|f - a|^q. \tag{19}$$

Next

$$\mu|f - a|^q \leqslant \mu|f - a|^q \chi(\eta < L) + \mu|f - a|^q \chi(\eta \geqslant L). \tag{20}$$

Using our assumptions and putting $a = \frac{1}{|B_R|}\int_{B_R} f$, for the first term on the right-hand side of (20) we have

$$\begin{aligned}\mu|f - a|^q \chi(\eta < L) &\leqslant A_L \int_{B_R} \left| f - \frac{1}{|B_R|}\int_{B_R} f\right|^q d\lambda \\ &\leqslant A_L c_R \int_{B_R} |\nabla f|^q \, d\lambda \leqslant A_L^2 c_R \mu|\nabla f|^q.\end{aligned} \tag{21}$$

On the other hand for the second term on the right-hand side of (20) we get

$$\mu|f - a|^q \chi(\eta \geqslant L) \leqslant \frac{1}{L}\mu|f - a|^q \eta. \tag{22}$$

Hence, by (16), we obtain

$$\mu|f - a|^q \chi(\eta \geqslant L) \leqslant \frac{C}{L}\mu|\nabla f|^q + \frac{D}{L}\mu|f - a|^q. \tag{23}$$

Combining (21) and (23), we get

$$\mu|f - a|^q \leqslant \left[A_L^2 c_R + \frac{C}{L}\right]\mu|\nabla f|^2 + \frac{D}{L}\mu|f - a|^q.$$

Choosing $L > D$, simple rearrangement yields

$$\mu|f - a|^q \leqslant \frac{A_L^2 c_R + \frac{C}{L}}{1 - \frac{D}{L}}\mu|\nabla f|^q.$$

This together with (19)–(21) yields

$$\mu|f - \mu f|^q \leqslant c\mu|\nabla f|^q$$

with some constant $c \in (0, \infty)$. $\square$

Corollary 3.1. *If we are on nilpotent Lie group the probability measure μ_q and μ_θ of Theorems 2.5 and 2.7, respectively, satisfies the Poincaré inequality.*

4. From Sobolev inequalities to coercive inequalities with probability measure: The non-compact setting

4.1. Case $p \geqslant 2$

Theorem 4.1. *Let $d\mu = \frac{e^{-U}}{Z}\, d\lambda$. Suppose the following Sobolev inequality is satisfied*

$$\left(\int |f|^{q+\varepsilon}\, d\lambda\right)^{\frac{q}{q+\varepsilon}} \leqslant a\int |\nabla f|^q\, d\lambda + b\int |f|^q\, d\lambda \tag{24}$$

and the following bound is true

$$\mu\big(|f|^q\big[|\nabla U|^q + U\big]\big) \leqslant \bar{C}\mu|\nabla f|^q + \bar{D}\mu|f|^q. \tag{25}$$

Then the following inequality is true

$$\mu\left(f^q \log\frac{f^q}{\mu f^q}\right) \leqslant C\mu|\nabla f|^q + D\mu|f|^q. \tag{26}$$

Moreover, if $q \in (1,2]$ and the following q-Poincaré inequality holds

$$\mu|f - \mu f|^q \leqslant \frac{1}{M}\mu|\nabla f|^q, \tag{27}$$

then one has

$$\mu\left(f^q \log\frac{f^q}{\mu f^q}\right) \leqslant c\mu|\nabla f|^q \tag{28}$$

with some constant $c \in (0,\infty)$ independent of f.

Proof. First we note that for $f \not\equiv 0$, we have

$$\mu\left(f^q \log\frac{f^q}{\mu f^q}\right) = \mu\big(f^q\big)\int g^q \log g^q\, d\lambda + \mu\big(f^q[U + \log Z]\big)$$

with $g \equiv f \cdot \frac{e^{-\frac{1}{q}U}}{Z^{1/q}}$ satisfying $\int g^q\, d\lambda = 1$. Next, by arguments based on Jensen inequality, one gets

$$\int g^q \log g^q\, d\lambda = \frac{q}{\varepsilon}\int g^q \log g^\varepsilon\, d\lambda \leqslant \frac{q(q+\varepsilon)}{\varepsilon}\log\left(\int g^{q+\varepsilon}\, d\lambda\right)^{\frac{1}{q+\varepsilon}}$$

whence, by the Sobolev inequality (24), one obtains

$$\int g^q \log g^q \, d\lambda \leqslant \frac{q+\varepsilon}{\varepsilon} \log\left(\int g^{q+\varepsilon} \, d\lambda\right)^{\frac{q}{q+\varepsilon}} \leqslant a' \int |\nabla g|^q \, d\lambda + b' \int g^q \, d\lambda$$

with $a' \equiv \frac{q+\varepsilon}{\varepsilon} a$ and $b' \equiv \frac{q+\varepsilon}{\varepsilon} b$. Combining all the above we arrive at

$$\mu\left(f^q \log \frac{f^q}{\mu f^q}\right) \leqslant a' \mu\left(f^q\right) \int \left|\nabla\left(f \frac{e^{-\frac{1}{q}U}}{Z^{1/q}}\right)\right|^q d\lambda + \left(b' + \log Z\right) \int f^q \, d\mu + \mu\left(f^q U\right)$$

and, by simple arguments, we obtain

$$\mu\left(f^q \log \frac{f^q}{\mu f^q}\right) \leqslant 2^{q-1} a' \int |\nabla f|^q \, d\mu + \mu\left(f^q\left[2^{q-1} q^{-q} a' |\nabla U|^q + U + b' + \log Z\right]\right). \tag{29}$$

Now using our assumption (25) yields

$$\mu\left(f^q \log \frac{f^q}{\mu f^q}\right) \leqslant \left(2^{q-1} a' + 2^{q-1} q^{-q} a' \bar{C}\right) \mu |\nabla f|^q + \left(b' + \bar{D} + \log Z\right) \mu |f|^q. \tag{30}$$

Since for $q \in (1, 2]$ one has [11]

$$\mu\left(f^q \log \frac{f^q}{\mu f^q}\right) \leqslant \mu\left(|f - \mu f|^q \log \frac{|f - \mu f|^q}{\mu |f - \mu f|^q}\right) + 2^{q+1} \mu |f - \mu f|^q \tag{31}$$

using (30) we arrive at

$$\mu\left(f^q \log \frac{f^q}{\mu f^q}\right) \leqslant \left\{\left(2^{q-1} a' + 2^{q-1} q^{-q} a' \bar{C}\right) + \frac{2^{q+1}\left(b' + \bar{D} + \log Z\right)}{M}\right\} \mu |\nabla f|^q$$

which ends the proof of the theorem. □

Using Theorem 4.1 together with results of Section 3 (q-Poincaré inequality), we arrive at the following result.

Corollary 4.1. *The probability measures $d\mu = e^{-W-V} \, d\mu_p / Z'$, with $p \geqslant 2$, described in Theorem 2.5 satisfies the following coercive inequality*

$$\mu\left(|f|^q \log \frac{|f|^q}{\mu |f|^q}\right) \leqslant c \mu |\nabla f|^q \tag{$\mathbf{LS}_q$}$$

with $\frac{1}{q} + \frac{1}{p} = 1$ and a constant $c \in (0, \infty)$ independent of a function f.

4.2. Sub-quadratic case

Theorem 4.2. *Suppose $\theta \in [1,2]$ and let $\varsigma = \frac{2(\theta-1)}{\theta}$. Then there exist constants $C, D \in (0,\infty)$ such that*

$$\int f^2 \left|\log \frac{f^2}{\int f^2 d\mu_\theta}\right|^\varsigma d\mu_\theta \leqslant C \int |\nabla f|^2 \, d\mu_\theta + D \int f^2 \, d\mu_\theta. \tag{32}$$

Proof. We note first that if $\theta \in [1,2]$, then $\varsigma \in [0,1]$. Put $g \equiv f \frac{e^{-\frac{\beta}{2}d^\theta}}{Z^{1/2}}$. We have the following inequality

$$\begin{aligned}
&\int f^2 \left|\log \frac{f^2}{\int f^2 \, d\mu_\theta}\right|^\varsigma d\mu_\theta \\
&= \int g^2 \left(\left|\log \frac{g}{\int g^2 \, d\lambda} + \beta d^\theta - \log Z\right|\right)^\varsigma d\lambda \\
&\leqslant \int g^2 \left|\log \frac{g^2}{\int g^2 d\lambda}\right|^\varsigma d\lambda + \int g^2 \big(\beta d^\theta\big)^\varsigma \, d\lambda + |\log Z|^\varsigma \int g^2 \, d\lambda \\
&= \int g^2 \left|\log \frac{g^2}{\int g^2 \, d\lambda}\right|^\varsigma d\lambda + \beta^\varsigma \int f^2 d^{\theta\varsigma} \, d\mu_\theta + |\log Z|^\varsigma \int f^2 \, d\mu_\theta.
\end{aligned} \tag{33}$$

Assume first that $\mu_\theta f^2 = \int g^2 \, d\lambda = 1$. Then we have

$$\begin{aligned}
\int g^2 \left(\left|\log \frac{g^2}{\int g^2 \, d\lambda}\right|\right)^\varsigma d\lambda &\leqslant \int g^2 \big(\log_+ g^2\big)^\varsigma \, d\lambda + D_\varsigma \\
&\leqslant \left(\frac{2+\varepsilon}{\varepsilon}\right)^\varsigma \left(\log_+ \left(\int g^{2+\varepsilon} \, d\lambda\right)^{\frac{2}{2+\varepsilon}}\right)^\varsigma + D_\varsigma
\end{aligned}$$

with $D_\varsigma \equiv \sup_{x\in(0,1)} x (\log \frac{1}{x})^\varsigma$. Choosing suitable $\varepsilon \in (0,1)$, we can apply Sobolev inequality (with constants $\bar{C}, \bar{D} \in (0,\infty)$) to get

$$\begin{aligned}
\int g^2 \left|\log \frac{g^2}{\int g^2 \, d\lambda}\right|^\varsigma &\leqslant \left(\frac{2+\varepsilon}{\varepsilon}\right)^\varsigma \left(\log_+ \left(\bar{C} \int |\nabla g|^2 \, d\lambda + \bar{D} \int g^2 \, d\lambda\right)\right)^\varsigma + D_\varsigma \\
&\leqslant C_1 \int |\nabla g|^2 \, d\lambda + D_1
\end{aligned}$$

with

$$C_1 \equiv s \left(\frac{2+\varepsilon}{\varepsilon}\right)^\varsigma \bar{C}$$

and

$$D_1 \equiv \left\{ s \left(\frac{2+\varepsilon}{\varepsilon}\right)^\varsigma \bar{D} + \gamma_{\varsigma,s} + D_\varsigma \right\}$$

where $s \in (0,\infty)$ and $\gamma_{\varsigma,s} \in (0,\infty)$ is a suitable constant. Using the definition of g, we have

$$\int |\nabla g|^2\, d\lambda \leqslant 2\int |\nabla f|^2\, d\mu_\theta + \frac{1}{2}\beta^2\theta^2 \int f^2 d^{2(\theta-1)}\, d\mu_\theta.$$

Now applying the U-bound of Theorem 2.6, we get

$$\int |\nabla g|^2\, d\lambda \leqslant \left(2 + \frac{1}{2}\beta^2\theta^2 C_\theta\right)\int |\nabla f|^2\, d\mu_\theta + \frac{1}{2}\beta^2\theta^2 D_\theta \int f^2\, d\mu_\theta.$$

Thus we get (for the normalised function g)

$$\int g^2 \left|\log \frac{g^2}{\int g^2\, d\lambda}\right|^\varsigma \leqslant C_2 \int |\nabla f|^2\, d\mu_\theta + D_2 \tag{34}$$

with some constants $C_2, D_2 \in (0,\infty)$. Now coming back to (33), we note that since $\theta\varsigma = 2(\theta-1)$, we can use again the U-bound of Theorem 2.6 to bound the second term from the right-hand side of this relation. Combining this with (34), we arrive at the following bound

$$\int f^2 \left|\log \frac{f^2}{\int f^2\, d\mu_\theta}\right|^\varsigma d\mu_\theta \leqslant C \int |\nabla f|^2\, d\mu_\theta + D \tag{35}$$

with the constants $C = C_2 + \beta^\varsigma C_\theta$ and $D = D_2 + \beta^\varsigma D_\theta + |\log Z|^\varsigma$. At this stage we can remove the normalisation condition to arrive at the desired bound (32). □

Using Theorem 4.2, we prove the following tight inequality.

Theorem 4.3. *For $\theta \in [1,2]$ and $\varsigma = \frac{2(\theta-1)}{\theta}$, let*

$$\Phi(x) \equiv x\big(\log(1+x)\big)^\varsigma.$$

Under the assumption of Theorem 4.2, if additionally μ_θ satisfies Poincaré inequality, there exists a constant $c_\theta \in (0,\infty)$ such that

$$\mu_\theta\Phi\big(f^2\big) - \Phi\big(\mu_\theta f^2\big) \leqslant c_\theta \int |\nabla f|^2\, d\mu_\theta. \tag{36}$$

Proof. First we note that

$$\mu_\theta\Phi\big(f^2\big) - \Phi\big(\mu_\theta f^2\big) \leqslant \mu_\theta f^2 \left|\log \frac{1+f^2}{1+\mu_\theta f^2}\right|^\varsigma \tag{37}$$

and

$$\mu_\theta f^2 \left|\log \frac{1+f^2}{1+\mu_\theta f^2}\right|^\varsigma = \mu_\theta \chi\big(f^2 \geqslant \mu_\theta f^2\big) f^2 \left|\log \frac{1+f^2}{1+\mu_\theta f^2}\right|^\varsigma + \mu_\theta \chi\big(f^2 \leqslant \mu_\theta f^2\big) f^2 \left|\log \frac{1+f^2}{1+\mu_\theta f^2}\right|^\varsigma. \tag{38}$$

On the set $\{f \geqslant \mu_\theta f^2\}$ we have $\frac{1+f^2}{1+\mu_\theta f^2} \leqslant \frac{f^2}{\mu_\theta f^2}$ and so

$$\mu_\theta \chi\left(f^2 \geqslant \mu_\theta f^2\right) f^2 \left|\log \frac{f^2}{\mu_\theta f^2}\right|^\varsigma \leqslant \mu_\theta f^2 \left|\log \frac{f^2}{\mu_\theta f^2}\right|^\varsigma.$$

On the other set $\{f \leqslant \mu_\theta f^2\}$, we have $\frac{1+\mu_\theta f^2}{1+f^2} \leqslant 1 + \frac{\mu_\theta f^2}{f^2}$, and therefore

$$\mu_\theta \chi\left(f^2 \leqslant \mu_\theta f^2\right) f^2 \left|\log \frac{1+f^2}{1+\mu_\theta f^2}\right|^\varsigma \leqslant 2\mu_\theta f^2.$$

Using these relations together with (38) we have

$$\mu_\theta \Phi\left(f^2\right) - \Phi\left(\mu_\theta f^2\right) \leqslant \mu_\theta f^2 \left|\log \frac{f^2}{\mu_\theta f^2}\right|^\varsigma + 2\mu_\theta f^2 \tag{39}$$

and thus, by Theorem 4.2, we obtain

$$\mu_\theta \Phi\left(f^2\right) - \Phi\left(\mu_\theta f^2\right) \leqslant C\mu_\theta |\nabla f|^2 + (D+2)\mu_\theta f^2. \tag{40}$$

Now according to Lemma A.1 of [28], one has the following analog of Rothaus lemma for a probability measure with Orlicz function Φ given in the theorem: $\exists a, b \in (0, \infty)$

$$\nu\Phi\left(f^2\right) - \Phi\left(\nu f^2\right) \leqslant a\left[\nu\Phi\left((f-\nu f)^2\right) - \Phi\left(\nu(f-\nu f)^2\right)\right] + b\nu(f-\nu f)^2. \tag{41}$$

Combining (40) and (41) with the Poincaré inequality for the measure μ_θ

$$\mu_\theta (f - \mu_\theta f)^2 \leqslant \frac{1}{M}\mu_\theta |\nabla f|^2$$

we arrive at the following result

$$\mu_\theta \Phi\left(f^2\right) - \Phi\left(\mu_\theta f^2\right) \leqslant \left[aC + \frac{D+b}{M}\right]\mu_\theta |\nabla f|^2. \qquad \square$$

Summarising, in the current section in essence our methods were based on the fact that the primary part of the interaction where a nice function of certain unbounded function d which length of the gradient $|\nabla d|$ (with respect to a given set of fields) was bounded from above and stayed strictly away from zero. We also used number of times the Leibniz rule for the fields.

4.3. From coercive inequalities to U-bounds

For a probability measure $d\mu \equiv e^{-U}\, d\lambda/Z$, we have shown that if for $q \in (1, 2]$ the following bound is satisfied

$$\int f^q\left(|\nabla U|^q + U\right) d\mu \leqslant C\int |\nabla f|^q\, d\mu + D\int |f|^q\, d\mu$$

together with q-Poincaré inequality

$$M\mu|f - \mu f|^q \leqslant \mu|\nabla f|^q,$$

then the following $\mathbf{LS}_q$ inequality holds

$$\mu|f|^q \log \frac{|f|^q}{\mu|f|^q} \leqslant c\mu|\nabla f|^q.$$

We show that the following result in the converse direction is true as well.

Theorem 4.4. *Suppose $q \in (1, 2]$ and for some constants $a, b \in (0, \infty)$, we have*

$$|\nabla U|^q \leqslant aU + b$$

and assume that the measure $d\mu \equiv e^{-U} d\lambda/Z$ satisfies $\mathbf{LS}_q$. Then the following U-bound is true

$$\int |f|^q U \, d\mu \leqslant C \int |\nabla f|^q \, d\mu + D \int |f|^q \, d\mu$$

with some constants $C, D \in (0, \infty)$ independent of f.

Proof. We note that by relative entropy inequality one has

$$\mu\big(|f|^q U\big) \leqslant \frac{1}{\varepsilon}\mu|f|^q \log \frac{|f|^q}{\mu|f|^q} + \left(\frac{1}{\varepsilon} \log \mu e^{\varepsilon U} \mu\right)\mu|f|^q.$$

Hence, if $\mathbf{LS}_q$ is true, we get

$$\mu\big(|f|^q U\big) \leqslant \frac{c}{\varepsilon} \int |\nabla f|^q \, d\mu + \left(\frac{1}{\varepsilon} \log \mu e^{\varepsilon U}\right)\mu|f|^q.$$

Thus we will be finished if we show $\mu e^{\varepsilon U} < \infty$. This follows from the following result.

Exp-bounds from $\mathbf{LS}_q$

Theorem 4.5. *Assume that a measure μ satisfies $\mathbf{LS}_q$ with some $q \in (1, 2]$. Suppose that for some constants $a, b \in (0, \infty)$, we have*

$$|\nabla f|^q \leqslant af + b.$$

Then the following exp-bound is true

$$\mu e^{tf} < \infty$$

for all $t > 0$ sufficiently small.

Remark. For the case $q = 2$ see [1].

Proof of Theorem 4.5. By our assumption, we have

$$\mu g^q \log \frac{g^q}{\mu g^q} \leqslant c\mu|\nabla g|^q.$$

It is enough to prove the bound under additional assumption that f is bounded. Namely, given $L \in (0, \infty)$, replace f by $F \equiv \chi(|f| \leqslant L)f + L\chi(|f| > L)$. F satisfies our assumptions with the same constants. So we will get the claim letting L go to ∞.

Since now f is bounded, $\exp tf$ is integrable and we have

$$\mu\left(e^{tf} \log \frac{e^{tf}}{\mu e^{tf}}\right) \leqslant cq^{-q}t^q \mu\left(e^{tf}|\nabla f|^q\right).$$

By our assumption $|\nabla f|^q \leqslant af + b$, so we get

$$\mu\left(e^{tf} \log \frac{e^{tf}}{\mu e^{tf}}\right) \leqslant caq^{-q}t^q \mu\left(e^{tf} f\right) + cbq^{-q}t^q \mu\left(e^{tf}\right)$$

which can be rearranged to get

$$\left(1 - caq^{-q}t^{q-1}\right)\mu\left(\frac{e^{tf}}{\mu e^{tf}} \log \frac{e^{tf}}{\mu e^{tf}}\right) \leqslant caq^{-q}t^{q-1} \log \mu\left(e^{tf}\right) + cbq^{-q}t^q.$$

Taking into the account that

$$\mu\left(\frac{e^{tf}}{\mu e^{tf}} \log \frac{e^{tf}}{\mu e^{tf}}\right) = t^2 \frac{d}{dt} \frac{1}{t} \log \mu e^{tf}$$

and setting $G(t) \equiv \frac{1}{t} \log \mu e^{tf}$, after simple transformations we obtain the following differential inequality

$$\frac{d}{dt} G(t) \leqslant \beta t^{q-2} G(t) + \gamma t^{q-2}$$

with $\beta(t) \equiv \frac{caq^{-q}}{(1-caq^{-q}t^{q-1})}$ and $\gamma(t) \equiv \frac{cbq^{-q}}{(1-caq^{-q}t^{q-1})}$ which are well defined for $caq^{-q}t^{q-1} < 1$. Since $G(t) \to \mu f$ as $t \to 0$ and $q \in (1, 2]$, for $caq^{-q}t^{q-1} < \varepsilon < 1$, after integration we get

$$G(t) \leqslant \mu f + \frac{cbq^{-q}}{(q-1)(1-\varepsilon)} t^{q-1} + \frac{caq^{-q}}{(1-\varepsilon)} \int_0^t d\tau\, \tau^{q-2} G(\tau).$$

In our range of $q \in (1, 2]$, this can be solved by iteration. Since $G(t)$ is non-decreasing, in this interval one also has

$$G(t) \leqslant \mu f + \frac{cbq^{-q}}{(q-1)(1-\varepsilon)} t^{q-1} + \frac{caq^{-q}}{(q-1)(1-\varepsilon)} t^{q-1} G(t)$$

which for $\frac{caq^{-q}}{(q-1)(1-\varepsilon)}t^{q-1} \equiv \delta < 1$ yields the following bound

$$\mu e^{(1-\delta)tf} \leqslant \exp\{t\mu f + Ct^q\}$$

with $C \equiv \frac{cbq^{-q}}{(q-1)(1-\varepsilon)}$. One can check that our bound is independent of the cut off L in the given interval of t. □

By the above we have shown the equivalence of the $\mathbf{LS}_q$ and U-bounds in particular in the cases of natural interactions dependent on the metric. Similar considerations can be provided in the subquadratic case for which the exponential bounds are known (see e.g. [25,6]).

5. Weighted U-bounds and coercive inequalities

Let $p \geqslant 2$ and suppose f is a smooth function supported away from the origin. Starting with the identity

$$d^{-\frac{\alpha}{2}}(\nabla f)e^{-\frac{\beta d^p}{2}} = d^{-\frac{\alpha}{2}}\nabla\left(fe^{-\frac{\beta d^p}{2}}\right) + \frac{p\beta}{2}d^{p-\frac{\alpha}{2}-1}(\nabla d)fe^{-\frac{\beta d^p}{2}},$$

squaring and integrating with the measure $d\lambda$, one obtains

$$\int d^{-\alpha}|\nabla f|^2 e^{-\beta d^p}\,d\lambda \geqslant p\beta \int d^{p-\alpha-1}\nabla\left(fe^{-\frac{\beta d^p}{2}}\right)\cdot(\nabla d)fe^{-\frac{\beta d^p}{2}}\,d\lambda + \frac{p^2\beta^2}{4}\int d^{2p-\alpha-2}|\nabla d|^2 f^2 e^{-\beta d^p}\,d\lambda.$$

Hence, after integration by parts in the first term on the right-hand side and simple rearrangements, one arrives at the following bound

$$\int d^{-\alpha}|\nabla f|^2 e^{-\beta d^p}\,d\lambda \geqslant \frac{p^2\beta^2}{4}\int f^2\left(d^{2p-\alpha-2}|\nabla d|^2\right)e^{-\beta d^p}\,d\lambda - \int f^2\left[\frac{p(p-\alpha-1)\beta}{2}d^{p-\alpha-2}|\nabla d|^2 + \frac{p\beta}{2}d^{p-\alpha-1}\Delta d\right]e^{-\beta d^p}\,d\lambda.$$

If we choose $\alpha = p-2$ and assume $|\nabla d| \geqslant \frac{1}{\sigma} > 0$, we obtain

$$\int d^{-\alpha}|\nabla f|^2 e^{-\beta d^p}\,d\lambda \geqslant \frac{p^2\beta^2}{4\sigma^2}\int f^2 d^p e^{-\beta d^p}\,d\lambda - \int f^2\left[\frac{p(p+1)\beta}{2}|\nabla d|^2 + \frac{p\beta}{2}d\Delta d\right]e^{-\beta d^p}\,d\lambda.$$

Finally assuming that there exist constants $K \in (0,\infty)$ and $\delta \in (0, \frac{p^2\beta^2}{4\sigma^2})$, such that

$$\frac{p(p+1)\beta}{2}|\nabla d|^2 + \frac{p\beta}{2}d\Delta d \leqslant K + \delta d^p$$

we arrive at

$$\left(\frac{p^2\beta^2}{4\sigma^2}-\delta\right)\int f^2 d^p e^{-\beta d^p}\,d\lambda \leqslant \int d^{-\alpha}|\nabla f|^2 e^{-\beta d^p}\,d\lambda + K\int f^2 e^{-\beta d^p}\,d\lambda.$$

By adjusting the constant on the right-hand side and replacing $d^{-\alpha}$ by $\langle d\rangle^{-\alpha} \equiv (1+d^2)^{-\frac{\alpha}{2}}$, we conclude with the following result.

Theorem 5.1. *Let $d\mu \equiv e^{-\beta d^p}\,d\lambda/Z$ with $p > 2$. Suppose there are constants $\sigma \in [1,\infty)$ and $K \in (0,\infty)$ and $\delta \in (0, \frac{p^2\beta^2}{4\sigma^2})$ such that $|\nabla d| \geqslant \frac{1}{\sigma}$ and*

$$\frac{p\beta}{2}|\nabla d|^2 + \frac{p\beta}{2} d\Delta d \leqslant K + \delta d^p.$$

Then there are constants $C, D \in (0,\infty)$ such that

$$\mu f^2 d^p \leqslant C\mu\big(\langle d\rangle^{2-p}|\nabla f|^2\big) + D\mu f^2.$$

Using this bound, by similar arguments as in the proof of Poincaré inequality (see Theorem 3.1), we now obtain

Theorem 5.2. *Under the assumptions of Theorem* 4.4 *there is a constant $M \in (0,\infty)$ such that*

$$M\mu(f-\mu f)^2 \leqslant \mu\big(\langle d\rangle^{2-p}|\nabla f|^2\big).$$

Finally following our strategy from the beginning of Section 4 (see proof of Theorem 4.2), with appropriate amendments, we arrive at the following coercive inequality.

Theorem 5.3. *Under the assumptions of Theorem* 4.4 *there is a constant $c \in (0,\infty)$ such that*

$$\mu\left(f^2\log\frac{f^2}{\mu f^2}\right) \leqslant c\mu\big(\langle d\rangle^{2-p}|\nabla f|^2\big).$$

5.1. Weighted U-bounds and coercive inequalities: Distributions with slow tails on Riemannian manifolds

In this section we consider a non-compact smooth Riemannian manifold $\mathbb{M}$ of dimension $3 \leqslant N < \infty$. In this setup $d(x)$ denotes the Riemannian distance of a point x from a given point $x_0 \in \mathbb{M}$ called later on the origin. By ∇ and Δ we denote the gradient and Laplace–Beltrami operators, respectively.

The aim of this section is to discuss coercive inequalities involving probability measures $d\mu \equiv \rho\,dx$ with density (with respect to the corresponding Riemannian measure $d\lambda$ on $\mathbb{M}$) which is of the form $\rho \equiv e^{-U(d)}/Z$ with leading part of the function U given by a concave function (and therefore also defining a non-Riemannian distance on $\mathbb{M}$). In particular we will consider the following cases:

(i) $U(d) = \beta d^\alpha$, with $\alpha \in (0,\infty)$ and $\beta > 0$,
(ii) $U(d) = \beta\log(1+d)$ with $\beta > 0$.

Before we go on we recall the following Laplacian comparison theorem (cf. [16,24] ([32,39, 40])). For a complete Riemannian manifold $\mathbb{M}$ with $Ric \geqslant (N-1)K$ where $K \in \mathbb{R}$:

($*$) If $K \leqslant 0$, then $\Delta d \leqslant (N-1)d^{-1} + (N-1)\sqrt{|K|}$.
($**$) If $Ric \geqslant 0$, then $\Delta d \leqslant (N-1)d^{-1}$.

By similar computation as we have done in Section 2, for a smooth non-negative function f localised outside a ball $B_\varepsilon \equiv B_\varepsilon(x_0)$ centred at the origin we consider a field

$$(\nabla f)e^{-U} = \nabla\big(fe^{-U}\big) + f(U'\nabla d)e^{-U} \tag{42}$$

to which we will apply a functional

$$\boldsymbol{\alpha}(\mathbf{v}) \equiv \int W(\nabla d \cdot \mathbf{v})\, d\lambda \tag{43}$$

defined with a positive weight function $W \equiv W(d)$ to be specified later. Using the fact that $|\nabla d| = 1$ (for $d \neq 0$), together with arguments involving Hölder inequality and integration by parts one arrives at the following bound

$$\int f\mathcal{V}e^{-U}\, d\lambda \leqslant \int W|\nabla f|e^{-U}\, d\lambda \tag{44}$$

with

$$\mathcal{V} \equiv \chi_{\mathbb{M}\setminus B_\varepsilon}\big(WU' - \operatorname{div}(W\nabla d)\big).$$

Later on we will extend $\mathcal{V}$ to B_ε in a convenient way by adding an arbitrary bounded continuous function. One can handle a function of arbitrary sign replacing f by $|f|$ and using equality $\nabla|f| = \operatorname{sgn}(f)\nabla f$. To include f which are non-zero on a ball centred at the origin we write $f = f_0 + f_1$ where $f_0 = \phi f$, $f_1 = (1-\phi)f$ and $\phi(x) = \min(\varepsilon, \max(2\varepsilon - d(x), 0))$. Then

$$\begin{aligned}\int |f|\mathcal{V}\, d\mu &= \int_{d(x)\leqslant 2\varepsilon} |f|\mathcal{V}\, d\mu + \int_{d(x)>2\varepsilon} |f|\mathcal{V}\, d\mu \\ &\leqslant \sup_{\{d\leqslant 2\varepsilon\}}(\mathcal{V}) \int \phi|f|\, d\mu + \int |f|_1\mathcal{V}\, d\mu.\end{aligned} \tag{45}$$

Next we have

$$|\nabla f_1| \leqslant |\nabla f| + \frac{1}{\varepsilon}\chi_{\{\varepsilon\leqslant d<2\varepsilon\}}|f|, \tag{46}$$

and therefore

$$\int |f_1|\mathcal{V}\, d\mu \leqslant \int W(1-\phi)|\nabla f|\, d\mu + \sup_{\{\varepsilon\leqslant d<2\varepsilon\}}\big(\varepsilon^{-1}W\big) \int_{\varepsilon\leqslant d<2\varepsilon} |f|\, d\mu. \tag{47}$$

Combining (42)–(47) we arrive at the following bound

$$\int |f|\mathcal{V}\,d\mu \leqslant \int W(1-\phi)|\nabla f|\,d\mu + \sup_{\{d\leqslant 2\varepsilon\}}(\mathcal{V})\int \phi|f|\,d\mu + \sup_{\{\varepsilon\leqslant d<2\varepsilon\}}\big(\varepsilon^{-1}W\big)\int_{\varepsilon\leqslant d<2\varepsilon}|f|\,d\mu. \tag{48}$$

Hence with

$$B \equiv \sup_{\{d\leqslant 2\varepsilon\}}(\mathcal{V}) + \sup_{\{\varepsilon\leqslant d<2\varepsilon\}}\big(\varepsilon^{-1}W\big),$$

we have

$$\int |f|\mathcal{V}\,d\mu \leqslant \int W|\nabla f|\,d\mu + B\int |f|\,d\mu. \tag{49}$$

Case (i)

For $U(d)=\beta d^{\alpha}$, with $\alpha\in(0,\infty)$ and $\beta>0$, choosing $W(d)=\alpha^{-1}d^{\kappa}$, with $\kappa\geqslant 1$, we have

$$\mathcal{V}\equiv WU' - \operatorname{div}(W\nabla d) = U - \alpha^{-1}\kappa d^{\kappa-1} - \alpha^{-1}d^{\kappa}\Delta d. \tag{50}$$

Thus if $(*)$ holds, we have

$$\mathcal{V}\geqslant \beta d^{\alpha-1+\kappa} - \chi_{\mathbb{M}\setminus B_\varepsilon}\big(\alpha^{-1}\kappa N d^{\kappa-1} + \alpha^{-1}(N-1)\sqrt{|K|}d^{\kappa}\big). \tag{51}$$

Hence we conclude with the following result

Theorem 5.4. *Let $d\mu\equiv e^{-U}\,d\lambda/Z$ with $U\equiv\beta d^{\alpha}$ where $\alpha\in(0,\infty)$. Suppose Ric $\geqslant (N-1)K$ with $K\leqslant 0$.*

- *If $\alpha>1$, then for any $\kappa\geqslant 1$, there exist constants $c_1, b_1\in(0,\infty)$ such that*

$$\int |f|U\,d\mu \leqslant c_1\int d^{\kappa}|\nabla f|\,d\mu + b_1\int |f|\,d\mu. \tag{52}$$

- *If $\alpha=1$ and $\beta>\alpha^{-1}(N-1)\sqrt{|K|}$, then for any $\kappa\geqslant 1$, there exist constants $c_1, b_1\in(0,\infty)$ such that* (52) *is true.*
- *If $\alpha\in(0,1)$ and Ric $\geqslant 0$, then for any $\kappa\geqslant 1$, there exist constants $c_1, b_1\in(0,\infty)$ such that* (52) *is true.*

Moreover if (52) *holds, then for any $q\in(1,\infty)$, we have*

$$\int |f|^q U\,d\mu \leqslant c_2\int d^{q(\kappa-\frac{\alpha}{p})}|\nabla f|^q\,d\mu + b_2\int |f|^q\,d\mu \tag{53}$$

with $c_2\equiv c_1\lambda q^{q-1}\beta^{\frac{q}{p}}[1-c_1/(p\lambda)]^{-1}$ and $b_2\equiv b_1[1-c_1/(p\lambda)]^{-1}$.

The second part follows from the first by substituting f^q in place of f and using elementary arguments involving Young inequality.

As a consequence, by similar arguments as earlier in this section, we obtain the following result on possible coercive inequalities.

Theorem 5.5. *Let $d\mu \equiv e^{-U}\,d\lambda/Z$ with $U \equiv \beta d^\alpha$ where $\alpha \in (0,\infty)$. Suppose $Ric \geqslant (N-1)K$ with $K \leqslant 0$.*

- *If $\alpha > 1$, then for any $\kappa \geqslant 1$, there exists a constant $c \in (0,\infty)$ such that*

$$\mu|f|^q \log \frac{|f|^q}{\mu|f|^q} \leqslant c \int d^{q(\kappa-\frac{\alpha}{p})} |\nabla f|^q \,d\mu. \tag{54}$$

- *If $\alpha = 1$ and $\beta > \alpha^{-1}(N-1)\sqrt{|K|}$, then for any $\kappa \geqslant 1$, there exists a constant $c \in (0,\infty)$ such that* (54) *is true.*
- *If $\alpha \in (0,1)$ and $Ric \geqslant 0$, then for any $\kappa \geqslant 1$, there exists a constant $c \in (0,\infty)$ such that* (54) *is true.*

As a consequence the following inequality holds

$$M\mu|f-\mu f|^q \leqslant \int d^{q(\kappa-\frac{\alpha}{p})} |\nabla f|^q \,d\mu \tag{55}$$

with some $M \in (0,\infty)$.

Case (ii)

For $U(d) = \beta \log(1+d)$ with $\beta > 0$, choosing $W(d) = d\log(1+d)$ and setting

$$\begin{aligned} \mathcal{V} &\equiv U + \chi_{\mathbb{M}\setminus B_\varepsilon}\big(W\beta(1+d)^{-1} - \operatorname{div}(W\nabla d)\big) \\ &= U - \chi_{\mathbb{M}\setminus B_\varepsilon}\big[1+\log(1+d)\big] - \chi_{\mathbb{M}\setminus B_\varepsilon} d\log(1+d)\Delta d. \end{aligned} \tag{56}$$

Thus if $(*)$ holds, we have

$$\mathcal{V} \geqslant U - \chi_{\mathbb{M}\setminus B_\varepsilon}\big[1+\log(1+d)\big] - \chi_{\mathbb{M}\setminus B_\varepsilon} d\log(1+d)\big[(N-1)d^{-1} + (N-1)\sqrt{|K|}\big]. \tag{57}$$

Hence we conclude with the following result

Theorem 5.6. *Let $d\mu \equiv (1+d)^{-\beta}\,d\lambda/Z$ with $\alpha \in (0,1)$. Suppose $Ric \geqslant 0$.*

If $\beta > N$, then

$$\int |f| U \,d\mu \leqslant c_1 \int d\log(1+d)|\nabla f|\,d\mu + b_1 \int |f|\,d\mu \tag{58}$$

with

$$c_1 \equiv \beta \cdot [\beta - N]^{-1}$$

and

$$b_1 \equiv \beta \cdot [\beta - N]^{-1} \cdot \Big(N + \sup_{\{d \leqslant 2\varepsilon\}} (\mathcal{V}) + \sup_{\{\varepsilon \leqslant d < 2\varepsilon\}} \big(\varepsilon^{-1} W\big)\Big).$$

Hence, there exist $c_q, b_q \in (0, \infty)$ *such that*

$$\int |f|^q U \, d\mu \leqslant c_q \int d^q \log(1+d) |\nabla f|^q \, d\mu + b_q \int |f|^q \, d\mu. \tag{59}$$

The second part follows from the first by substituting f^q in place of f and using the following Young inequality

$$d\big|\nabla f^q\big| = q\big(|f|^{q-1} \cdot d|\nabla f|\big) \leqslant \lambda^q d^q |\nabla f|^q + \frac{q}{p}\lambda^{-p}|f|^q$$

which implies

$$\begin{aligned} \int d \log(1+d)\big|\nabla f^q\big| \, d\mu &= \int d \log(1+d) q |f|^{q-1} |\nabla f| \, d\mu \\ &\leqslant \lambda^q \int d^q \log(1+d) |\nabla f|^q \, d\mu + \frac{q}{p}\lambda^{-p} \int \log(1+d) q |f|^q \, d\mu. \end{aligned}$$

From this and (58), choosing $c_1 \frac{q}{p}\lambda^{-p} < 1$, one obtains

$$\int |f|^q U \, d\mu \leqslant c_q \int d^q \log(1+d) |\nabla f|^q \, d\mu + b_q \int |f|^q \, d\mu$$

with $c_q \equiv c_1 \lambda^q (1 - c_1 \frac{q}{p}\lambda^{-p})^{-1}$ and $b_q \equiv b_1 (1 - c_1 \frac{q}{p}\lambda^{-p})^{-1}$.

As a consequence of the above theorem, using arguments similar to those of Sections 4.1 and 4.2, we derive the following result on possible coercive inequalities.

Theorem 5.7. *Let* $d\mu \equiv e^{-\beta \log(1+d)} dx/Z$ *with* $\beta > N$. *Suppose Ric* $\geqslant 0$. *Then for any* $q \geqslant 1$, *there are constants* $M_q, c_q \in (0, \infty)$, *such that*

$$M_q \mu |f - \mu f|^q \leqslant \mu (1+d)^q \log(e+d) |\nabla f|^q \tag{60}$$

and

$$\mu |f|^q \log \frac{|f|^q}{\mu |f|^q} \leqslant c_q \mu (1+d)^q \log(e+d) |\nabla f|^q. \tag{61}$$

5.1.1. Weighted inequalities at large β

Let $U \equiv \beta \log(1+d)$, with $\beta > N \equiv \dim(\mathbb{M})$. While the above results are true for any $\beta > N$, we will show that for sufficiently big β and $Ric \geqslant 0$ due to the special nature of the interaction it is possible to improve the weight in the Poincaré and related Log-Sobolev inequalities.

We start from noting that for a non-negative differentiable function supported outside a ball of radius r centred at the origin, one has

$$\int (1+d)|\nabla f| e^{-U}\,dx \geqslant \int (1+d)\nabla d \cdot \nabla f e^{-U}\,dx$$
$$= \int (1+d)\big[\nabla d \cdot \nabla\big(f e^{-U}\big) + f \nabla d \cdot \nabla U e^{-U}\big]\,dx$$

and so, taking into the account that $|\nabla d|^2 = 1$, one gets

$$\int f\big[\beta - 1 - (1+d)\Delta d\big] e^{-U}\,dx \leqslant \int (1+d)|\nabla f| e^{-U}\,dx.$$

When $Ric \geqslant 0$, we have $\Delta d \leqslant (N-1)d^{-1}$ which implies the following bound

$$M_\beta \int f e^{-U}\,dx \leqslant \int (1+d)|\nabla f| e^{-U}\,dx \tag{62}$$

where $M_\beta \equiv [\beta - N - \frac{(N-1)}{r}]$. Since $|\nabla f| \geqslant |\nabla |f||$, this inequality remains true for not necessarily positive function with f replaced by $|f|$ on the right-hand side. Let now consider the following cutoff function

$$\chi(t) \equiv \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant 2r, \\ 1 - \frac{(t-r)}{L} & \text{for } 2r \leqslant t \leqslant R, \\ 0 & \text{for } t \geqslant R, \end{cases}$$

with some $R > 2r$ to be chosen later. Setting $\tilde{f}_1 \equiv (f - \mu f)\chi$ and $\tilde{f}_2 \equiv (f - \mu f)\chi$, we have

$$\mu|f - \mu f| \leqslant \mu|\tilde{f}_1| + \mu|\tilde{f}_2|.$$

As $\tilde{f}_1$ is compactly supported Lipschitz function, there is an $m \equiv m_R \in (0, \infty)$ independent of the function f, such that

$$\mu|\tilde{f}_1| \leqslant m_R^{-1}\mu|\nabla \tilde{f}_1| \leqslant m_R^{-1}\mu\big(|\nabla f|\chi\big) + \frac{1}{m_R(R-2r)}\mu\big(|f - \mu f|\chi(2r < d < R)\big).$$

The second term on the right-hand side can be treated with the help of (62) as follows. Setting $\hat{\chi}$ to be a Lipschitz extension of $\chi(2r < d < R)$ supported outside the ball of radius r, we have

$$\mu\big(|f - \mu f|\chi(2r < d < R)\big) \leqslant \mu\big(|f - \mu f|\hat{\chi}\big)$$
$$\leqslant M_\beta^{-1}\mu(1+d)|\nabla f| + M_\beta^{-1}\sup|\nabla\hat{\chi}|\mu|f - \mu f|.$$

Thus we obtain

$$\begin{aligned}\mu|\tilde{f}_1| &\leqslant m_R^{-1}\mu|\nabla\tilde{f}_1| \\ &\leqslant \left[m_R^{-1} + \frac{1}{m_R(R-2r)}M_\beta^{-1}\right]\mu(1+d)|\nabla f|\hat{\chi} \\ &\quad + \frac{1}{m_R(R-2r)}M_\beta^{-1}\sup|\nabla\hat{\chi}|\mu|f-\mu f|. \end{aligned} \tag{63}$$

On the other hand applying (62) to $\tilde{f}_2$ we obtain

$$\mu|\tilde{f}_2| \leqslant M_\beta^{-1}\mu(1+d)|\nabla f|(1-\chi) + M_\beta^{-1}\frac{1+R}{R-r}\mu\big(|f-\mu f|\chi(r<d<R)\big). \tag{64}$$

Combining (63) and (64) we arrive at

$$\mu|f-\mu f| \leqslant c_0\mu(1+d)|\nabla f| + b_0\mu|f-\mu f| \tag{65}$$

with

$$c_0 \equiv \left[m_R^{-1} + \left(\frac{1}{m_R(R-2r)} + 1\right)M_\beta^{-1}\right]$$

and

$$b_0 \equiv M_\beta^{-1}\left(\frac{1}{m_R(R-2r)}\sup|\nabla\hat{\chi}| + \frac{1+R}{R-r}\right).$$

Since given $R > 2r$, one can choose $\beta > N$ sufficiently large so that $b_0 < 1$, we conclude with the following result

Theorem 5.8. *Suppose $U = \beta\log(1+d)$, with $\beta > N$, and $Ric \geqslant 0$. Then there exists $\beta_0 > N$, such that for any $\beta > \beta_0$, one has*

$$M\mu|f-\mu f| \leqslant \mu(1+d)|\nabla f| \tag{66}$$

with some constant $M \in (0,\infty)$ independent of f. Consequently, we have

$$M_q\mu|f-\mu f|^q \leqslant \mu(1+d)^q|\nabla f|^q \tag{67}$$

with some constant $M_q \in (0,\infty)$.

The second part of the theorem follows by similar arguments as the ones used in the proof of Proposition 2.3 in [11].

Next we study the relative entropy estimate as follows. For a non-negative function f, setting $f_1 \equiv f\chi$ and $f_2 \equiv f(1-\chi)$ with the same Lipschitz cutoff function χ, we have

$$\mu f\log\frac{f}{\mu f} \leqslant \mu f_1\log\frac{f_1}{\mu f_1} + \mu f_2\log\frac{f_2}{\mu f_2}.$$

Since the function f_1 is compactly supported and the density of the measure μ restricted to the ball $B_R(x_0)$ bounded and bounded away from zero (via the arguments involving Sobolev inequality), we get

$$\mu f_1 \log \frac{f_1}{\mu f_1} \leqslant c_1 \mu |\nabla f_1| \leqslant c_1 \mu\big(|\nabla f|\chi\big) + b_1 \sup |\nabla \chi| \mu f \tag{68}$$

with some constants $c_1, b_1 \in (0, \infty)$ independent of f. Next we apply similar arguments based on Sobolev inequality with the function $F \equiv \frac{f_2 e^U}{\int f_2 e^U\, dx}$ and the Riemannian measure dx to get

$$\int F \log \frac{F}{\int F\, dx}\, dx \leqslant a \int |\nabla F|\, dx + b \int F\, dx$$

with some constants $a, b \in (0, \infty)$. Hence we have

$$\mu f_2 \log \frac{f_2}{\mu f_2} \leqslant a\mu |\nabla f|(1-\chi) + \mu f(1-\chi)\big(a|\nabla U| + b - \log Z\big) + \mu f_2 U. \tag{69}$$

In our current setup we have $|\nabla U| \leqslant \beta$. Moreover, by simple relative entropy arguments, we have

$$\begin{aligned} \mu f_2 U &= \frac{1}{\lambda} \mu f_2 \log \frac{e^{\lambda U}}{\mu e^{\lambda U}} + \frac{1}{\lambda} \log \mu e^{\lambda U} \mu f_2 \\ &\leqslant \frac{1}{\lambda} \mu f_2 \log \frac{f_2}{\mu f_2} + \frac{1}{\lambda} \log \mu e^{\lambda U} \mu f_2 \end{aligned}$$

which holds provided that $\beta > N + \lambda$. If we can choose $\lambda > 1$, this together with (69) implies

$$\mu f_2 \log \frac{f_2}{\mu f_2} \leqslant c_2 \mu |\nabla f|(1-\chi) + b_2 \mu f(1-\chi) \tag{70}$$

with

$$c_2 \equiv a\big(1-\lambda^{-1}\big)^{-1}$$

and

$$b_2 \equiv \big(1-\lambda^{-1}\big)^{-1}\left[a\beta + b - \log Z \frac{1}{\lambda} \log \mu e^{\lambda U}\right].$$

Combining (70) and (68) we arrive at the following result

Theorem 5.9. *Suppose $U = \beta \log(1+d)$, with $\beta > N$, and Ric $\geqslant 0$. Then there exists $\beta_0 > N$, such that for any $\beta > \beta_0$, one has*

$$\mu f \log \frac{f}{\mu f} \leqslant \bar{c} \mu (1+d) |\nabla f| + \bar{b} \mu f \tag{71}$$

with some constants $\bar{c}, \bar{b} \in (0, \infty)$ independent of f. Consequently, if the weighted Poincaré inequality (67) *is true for $q > 1$, we have*

$$\mu f^q \log \frac{f^q}{\mu f^q} \leqslant c_q \mu (1+d)^q |\nabla f|^q \tag{$\mathbf{WLS}_q$}$$

with some constant $c_q \in (0, \infty)$.

We remark that (71) implies similar weighted LS_q inequality with f replaced by $|f|^q$ and $|\nabla f|$ by its qth power (which follows simply by substitution and use of Hölder inequality), while the tightening is obtained via Rothaus arguments (see e.g. [11]).

6. Optimal control distance on the Heisenberg group

Heisenberg group H_l as a manifold is isomorphic to $\mathcal{R}^{2l+1} = \mathcal{R}^{2l} \times \mathcal{R}$ with the multiplication given by the formula

$$(x_1, z_1) \circ (x_2, z_2) = \left(x_1 + x_2, z_1 + z_2 + \frac{1}{2} S(x_1, x_2) \right)$$

where $S(x, y)$ is standard symplectic form on $\mathcal{R}^{2l}$:

$$S(x, y) = \sum_{i=1}^{l} (x_i y_{i+l} - x_{i+l} y_i).$$

Vector fields spanning the corresponding Lie algebra are given as follows

$$X_i = \partial_{x_i} + \frac{1}{2} x_{i+l} \partial_z,$$
$$X_{i+l} = \partial_{x_{i+l}} - \frac{1}{2} x_i \partial_z,$$
$$Z = \partial_z$$

where $i = 1, \ldots, l$.

More generally, we say that a Lie algebra $\mathbf{n}$ is a stratified Lie algebra if it can be written as

$$\mathbf{n} = \bigoplus_i^m \mathbf{n}_i,$$
$$[\mathbf{n}_i, \mathbf{n}_j] \subset \mathbf{n}_{i+j}$$

and $\mathbf{n}$ is generated by $\mathbf{n}_1$. Note that stratified Lie algebra is nilpotent.

We say that Lie group N is stratified if it is connected, simply connected and its Lie algebra $\mathbf{n}$ is stratified. Since for stratified groups exponential mapping is a diffeomorphism from $\mathbf{n}$ to N, one can identify N with $\mathbf{n}$.

A Lie algebra is step two if it is stratified with $m = 2$. In other words it can be written in the form

$$\mathbf{n} = \mathbf{v} \oplus \mathbf{z}$$

where $\mathbf{z}$ is the centre (that is $[\mathbf{n}, \mathbf{z}] = 0$) and $[\mathbf{v}, \mathbf{v}] \subset \mathbf{z}$.

On a stratified Lie algebra $\mathbf{n}$ we define dilations by the formula

$$\delta(s)x = s^i x$$

for $x \in \mathbf{n}_i$ (and extend linearly to the whole $\mathbf{n}$). For $s \neq 0$, $\delta(s)$ is an automorphism of $\mathbf{n}$. One can also define dilations on the corresponding group: $\delta(\exp(X)) = \exp(\delta(X))$.

A Lie algebra $\mathbf{n}$ is of H-type (Heisenberg type) if it is step two and there exists an inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{n}$ such that $\mathbf{z}$ is an orthogonal complement to $\mathbf{v}$, and the map $J_Z : \mathbf{v} \mapsto \mathbf{v}$ given by

$$\langle J_Z X, Y\rangle = \big\langle [X, Y], Z\big\rangle$$

for $X, Y \in \mathbf{v}$ and $Z \in \mathbf{z}$ satisfies $J_Z^2 = -|Z|^2 I$ for each $Z \in \mathbf{z}$. Equivalently, for each $v \in \mathbf{v}$ of length 1 the mapping ad_v^* given by

$$\big\langle ad_v^* z, y\big\rangle = \langle z, ad_v y\rangle = \big\langle z, [v, y]\big\rangle$$

is an isometry from $\mathbf{z}^*$ into $\mathbf{v}^*$.

An H-type group is a connected and simply connected Lie group N whose Lie algebra is of H-type. We can identify H-type group N with its Lie algebra $\mathbf{n}$ defining multiplication on $\mathbf{n}$ by the formula:

$$(v_1, z_1) \cdot (v_2, z_2) = \left(v_1 + v_2, z_1 + z_2 + \frac{1}{2}[v_1, v_2]\right)$$

where $v_1, v_2 \in \mathbf{v}$ and $z_1, z_2 \in \mathbf{z}$.

It is easy to see that Heisenberg group is an H-type group. Also H-type group with one-dimensional centre is isomorphic to the Heisenberg group, however there exist H-type groups with centre of arbitrary high dimension [23].

On H-type group we consider vector fields $X_1, \dots, X_n$ which form an orthonormal basis of $\mathbf{v}$ and we introduce the following operators:

Subelliptic gradient:

$$\nabla f = (X_1 f, \dots, X_n f).$$

Kohn Laplacian:

$$\Delta = \sum_{i=1}^{n} X_i^2.$$

On Heisenberg group H_l $n = 2l$ and

$$\Delta = \sum_{i=1}^{2l} \partial_{x_i}^2 + \partial_z \sum_{i=1}^{l} (x_{i+l}\partial_{x_i} - x_i \partial_{x_{i+l}}) + \frac{|x|^2}{4}\partial_z^2.$$

On general H-type group we similar, but more complicated expression:

$$\Delta = \sum_{i=1}^{n} \partial_{v_i}^2 + \sum_{i=1}^{k} \partial_{z_i} \sum J_{\alpha,i} + \frac{|v|^2}{4} \sum_{i=1}^{k} \partial_{z_i}^2$$

where $J_{\alpha,i}$ are vector fields corresponding to rotations.

Length of a curve: smooth $\gamma : [0,1] \mapsto G$ is admissible if $\gamma'(s) = \sum_{i=1}^{n} a_i(s) X_i(\gamma(s))$. If γ is admissible, then $|\gamma| = \int_0^1 (\sum_{i=1}^{n} a_i^2(s))^{1/2}$.

Distance

$$d(g) = \inf |\gamma|$$

where infimum is taken over all admissible γ such that $\gamma(0) = e$ and $\gamma(1) = g$.

d is homogeneous of degree 1 with respect to the dilations $\delta(s)$, namely for $s > 0$

$$d\big(\delta(s)g\big) = sd(g).$$

Lemma 6.1. *On H-type group Z distance $d((v,z))$ depends only on $|v|$ and $|z|$. Moreover if $\bar{v}, \bar{z} \in H_1$, $|v| = |\bar{v}|$, $|z| = |\bar{z}|$, then $d((v,z)) = d((\bar{v}, \bar{z}))$.*

Proof. Fix vectors $V, Z \in N$ such that $|V| = 1$, $|Z| = 1$, $v = |v|V$, $z = |z|Z$. Put $X = J_Z(V)$. Since J_Z is antisymmetric and $J_Z^2 = I$, J_Z is orthogonal, so $|X| = 1$. Also, for any $S \in \mathbf{z}$ of length 1, we have

$$\big|\big\langle [X,Y], S\big\rangle\big| = \big|\langle J_S X, Y\rangle\big| \leqslant |X||Y|$$

so since

$$\big\langle [V,X], Z\big\rangle = \langle J_Z V, X\rangle = \langle X, X\rangle = |X|^2 = 1$$

we have $[V,X] = Z$.

Now, it is easy to see that the subgroup (in fact a subspace) of N generated by V, X, Z is isomorphic to H_1. Consequently, using images of curves from H_1 to join with (v,z) we see that $d((v,z)) \leqslant d(((|v|,0),z))$ where on the right-hand we have distance in H_1.

To get inequality in the opposite direction consider quotient group N/M where $M = \{t \in \mathbf{z}: \langle t, Z\rangle = 0\}$. It is easy to see that N/M is still an H-type group (note that since N/M has one-dimensional centre it is enough to check the defining property just for J_Z). Hence, N/M is isomorphic to the Heisenberg group of appropriate dimension. For Heisenberg group our claim is well known. $\square$

It is known [30] that on Heisenberg group if $g = (x,z)$ and $x \neq 0$, then d is smooth at g and $|\nabla d| = 1$, however when $x = 0$ than d is not differentiable at g.

Lemma 6.2. *Let $A_\epsilon = (r,z) \in \mathcal{R}^2$: $z > 0$, $r > -\epsilon z$. There are $\epsilon > 0$ and a smooth function $\psi(r,z)$ defined on A_ϵ such that on each group N of H-type*

$$d((x,z)) = \psi\big(|x|, |z|\big).$$

Moreover, $\partial_r \psi < 0$ *when* $r = 0$.

Proof. First, by Lemma 6.1 without loss of generality we may assume that $N = H_1$. Also, if $|x_1| = |x_2|$ and $|z_1| = |z_2|$, then $d(x_1, z_1) = d(x_2, z_2)$, so ψ is uniquely defined for $r \geqslant 0$. We need to show that it has smooth extension to A_ϵ. Since d is homogeneous, it is enough to construct smooth extension in a neighbourhood of a single point $g = (0, 1)$.

There exists a smooth geodesic (length minimising curve) γ joining $e = (0,0)$ and g. We use length as a parametrisation of γ, so $\gamma(d(g)) = g$. For $s < s_0 = d(g)$ we have $d(\gamma(s)) = s$.

Let $\gamma(s) = (\gamma_x(s), \gamma_z(s))$. Since square of Euclidean distance is smooth $|\gamma_x|^2$ is smooth. We can write $|\gamma_x|^2(s) = (s - s_0)^2\rho(s)$ where ρ is smooth and $\rho(s_0) = 1$, so $|\gamma_x|^2(s)$ has a square root $\phi(s) = (s_0 - s)\rho^{1/2}(s)$ which is smooth for s close to s_0. Since both ϕ and $|\gamma_x|$ are positive square roots of $|\gamma_x|^2$ for $s_0 - \epsilon < s < s_0$ we have

$$\big|\gamma_x(s)\big| = \phi(s)$$

for $s_0 - \epsilon < s \leqslant s_0$. Put

$$\eta(s,t) = \big(t\phi(s), t^2\gamma_z(s)\big).$$

Since γ is admissible $|\gamma_z|'(s_0) = 0$ so the Jacobi matrix at $(s,t) = (s_0, 1)$ is

$$\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

and by the inverse function theorem η is invertible in a neighbourhood of $(s_0, 1)$. So, there exist f_1, f_2 such that

$$(r,p) = \eta\big(f_1(r,p), f_2(r,p)\big).$$

We claim that $\psi(r,p) = f_1(r,p)f_2(r,p)$ give us extension of ψ to a neighbourhood of g. Consider (x,z) close to g. Let $(s,t) = (f_1(|x|, z), f_2(|x|, z))$. We have

$$|x| = t\phi(s) = t\big|\gamma_x(s)\big| = \big|\big(\delta_t\gamma(s)\big)_x\big|,$$
$$z = t^2\gamma_z(s) = \big(\delta_t\gamma(s)\big)_z$$

so

$$d((x,z)) = d\big(\delta_t\gamma(s)\big) = td\big(\gamma(s)\big) = ts = f_1(r,z)f_2(r,z) = \psi(r,z).$$

Now it remains to find sign $(\partial_r\psi)(0,z)$. Form equality $(r,p) = \eta(f_1(r,p), f_2(r,p))$ we see $I = \eta' \cdot f'$. We substitute $(r,p) = (0,1)$ and note that this corresponds to $(s_0, 1)$. So

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} \partial_r f \\ \partial_p f \end{pmatrix}$$

and using first row we get $1 = -(\partial_r f_1)(0,1)$, $0 = -(\partial_r f_2)(0,1)$ so

$$(\partial_r \psi)(0,1) = (\partial_r f_1)(0,1) f_2(0,1) + f_1(0,1)(\partial_r f_2)(0,1) = (\partial_r f_1)(0,1) = -1. \qquad \square$$

Theorem 6.1. *If N is an H-type group, then there is K such that if $d(g) \geqslant 1$, then*

$$\Delta d \leqslant K$$

where Δ is understood in the sense of distributions.

Proof. Due to homogeneity, it is enough to prove the inequality only for g with $d(g) = 1$ (more precisely, in a small neighbourhood of each such g). Namely, if $s = d(g) > 1$, then

$$\Delta d(g) = s^{-2} \Delta d\big(\delta(s) g\big) = s^{-1} \Delta d(g).$$

Next, $d((x,z))$ is smooth when $x \neq 0$, so it is enough to prove the inequality in a small neighbourhood of $(0, z_0)$ where $z_0 >$ is chosen so that $d((0, z_0)) = 1$.

Below we give computation on Heisenberg group:

$$\partial_{x_i} d((x,z)) = \partial x_i \psi\big(|x|, z\big) = \frac{x_i}{|x|} \partial_r \psi\big(|x|, z\big),$$

$$\partial_{x_i}^2 d((x,z)) = \partial x_i \left(\frac{x_i}{|x|} \partial_r \psi\big(|x|, z\big) \right) = \frac{x_i^2}{|x|^2} \partial_r^2 \psi\big(|x|, z\big) + \frac{1}{|x|} \partial_r \psi\big(|x|, z\big) - \frac{x_i^2}{|x|^3} \partial_r \psi\big(|x|, z\big),$$

$$\sum_{i=1}^{2n} \partial_{x_i}^2 d((x,z)) = \frac{2n-1}{|x|} \partial_r \psi\big(|x|, z\big) + \partial_r^2 \psi\big(|x|, z\big),$$

$$(x_{i+n} \partial_{x_i} - x_i \partial_{x_{i+n}}) d((x,z)) = \left(\frac{x_{i+n} x_i}{|x|} - \frac{x_i x_{i+n}}{|x|} \right) \partial_r \psi\big(|x|, z\big) = 0,$$

$$\Delta d((x,z)) = \frac{2n-1}{|x|} \partial_r \psi\big(|x|, z\big) + \partial_r^2 \psi\big(|x|, z\big) + \frac{|x|^2}{4} \partial_z^2 \psi\big(|x|, z\big).$$

Since ψ is smooth the second term and third term is bounded in a neighbourhood of $(0, z_0)$. Since $\partial_r \psi(0, z_0) < 0$ the first term is unbounded, but negative in a neighbourhood of $(0, z_0)$, which gives the claim on Heisenberg group.

On general H-type groups instead of $x_{i+n} \partial_{x_i} - x_i \partial_{x_{i+n}}$ one must handle the $J_{\alpha,i}$ term. However, since $J_{\alpha,i}$ generates rotations in v space and d is rotationally invariant again $J_{\alpha,i} d = 0$. $\square$

6.1. Counterexample for homogeneous norm

On stratified groups N one may introduce a homogeneous norm, that is a continuous function $\phi : N \mapsto [0, \infty)$ such that $\phi(e) = 0$, $\phi(x) > 0$ for $x \neq e$ and $\phi(\delta_s(x)) = s\phi(x)$ for $s > 0$. Homogeneous norms are equivalent to each other, if ϕ_1 and ϕ_2 are two homogeneous norms, then there is C such that

$$C^{-1} \phi_1 \leqslant \phi_2 \leqslant C \phi_1.$$

The optimal control distance d gives one example of homogeneous norm, but there are others. In particular, it is possible to choose homogeneous norm so that it is smooth for $x \neq e$ (we will call such homogeneous norm *smooth*). Smooth homogeneous norms are convenient in many situations. For smooth homogeneous norm ϕ the condition $(\Delta\phi)(x) \leqslant K$ for $\phi(x) \geqslant 1$ is automatically satisfied. However, we are going to prove that for such norm $|\nabla\phi|(x) = 0$ for some $x \neq e$, and consequently Log-Sobolev inequality like the one for optimal control distance cannot hold.

Theorem 6.2. *Let N be a stratified group, and ϕ be a smooth homogeneous norm on N. There exists $x \neq e$ such that $|\nabla\phi|(x) = 0$.*

Proof. Let $X_1, \dots, X_n$ be a basis of $\mathbf{n}_1$. We claim that for $(a_1, \dots, a_n) \in R^n - \{0\}$,

$$\sum a_i (X_i\phi)\Big(\exp\Big(\sum a_i X_i\Big)\Big) > 0. \tag{72}$$

Namely, $\exp(t\sum a_i X_i)$ is a one parameter subgroup of N, so

$$\partial_t\Big(\phi\Big(\exp\Big(t\sum a_i X_i\Big)\Big)\Big) = \sum a_i (X_i\phi)\Big(\exp\Big(t\sum a_i X_i\Big)\Big).$$

However, by homogeneity

$$\partial_t\Big(\phi\Big(\exp\Big(t\sum a_i X_i\Big)\Big)\Big) = \partial_t\Big(t\phi\Big(\exp\Big(\sum a_i X_i\Big)\Big)\Big) = \phi\Big(\exp\Big(\sum a_i X_i\Big)\Big) > 0$$

so (72) holds.

Using the $X_1, \dots, X_n$ basis we identify $\mathbf{n}_1$ with R^n. This identification gives us scalar product on $\mathbf{n}_1$. We extend this scalar product to a scalar product on $\mathbf{n}$ such that $\mathbf{n}_i$ is orthogonal to $\mathbf{n}_j$ for $i \neq j$.

Let S ($\tilde{S}$) be the unit sphere in $\mathbf{n}_1$ (in $\mathbf{n}$ respectively). Define mapping $\eta : S \mapsto S$ by the formula $\eta(x) = \frac{(\nabla\phi)(\exp(x))}{|\nabla\phi|(\exp(x))}$ (note that we use identification $\mathbf{n}_1 = R^n$ here). By (72) on S, $|\nabla\phi|(\exp(x)) > 0$ so η is well defined. Also, η is homotopic with identity. Namely put $\chi(\sum a_i X_i) = (a_1, \dots, a_n)$. If f_t is defined by the formula $f_t(x) = t\eta(x) + (1-t)\chi$, then for $x = \sum a_i X_i$ we have $\langle f_t(x), x\rangle > 0$, so f_t takes values in $R^n - \{0\}$. Consequently $g_t(x) = \frac{f_t(x)}{|f_t(x)|}$ gives homotopy of mappings from S to S.

If $(\nabla\phi)(\exp(x)) \neq 0$ on $\tilde{S}$, then η is homotopic to a constant. Namely, $\tilde{S}$ contains a homeomorphic copy of $(n+1)$-dimensional disc D having S as a boundary and $\frac{(\nabla\phi)\circ\exp}{|(\nabla\phi)\circ\exp|}$ gives required homotopy. However, it is well known that identity of the sphere is not homotopic to a constant — so we reach contradiction with assumption that $(\nabla\phi)(\exp(x)) \neq 0$. $\square$

Lemma 6.3. *If f is smooth function on a stratified group N, d is optimal control metric on N, $x_0 \in N$ is fixed, then*

$$\big|f(x) - f(x_0)\big| \leqslant O\big(d(x, x_0)\big).$$

If additionally $(\nabla f)(x_0) = 0$, then

$$\big|f(x) - f(x_0)\big| \leqslant O\big(d^2(x, x_0)\big).$$

Proof. Let $\gamma : [0,1] \mapsto N$ be an admissible curve joining x_0 and x. We have $\gamma'(s) = \sum a_i(s) X_i(\gamma(s))$, so

$$\left|f(x) - f(x_0)\right| = \int_0^1 \left|(f \circ \gamma)'\right| = \int_0^1 \left|\sum a_i(s)(X_i f) \circ \gamma\right|$$
$$\leqslant \int_0^1 |\gamma'|\left|(\nabla f) \circ \gamma\right| \leqslant |\gamma| \sup_{s \in [0,1]} \left|(\nabla f) \circ \gamma(s)\right|.$$

Put $r = d(x, x_0)$. If $|\gamma| \leqslant r + \varepsilon$, then $\gamma(s) \in B(x, r + \varepsilon)$ and

$$\left|f(x) - f(x_0)\right| \leqslant (r + \varepsilon) \sup_{y \in B(x, r+\varepsilon)} \left|(\nabla f)(y)\right|.$$

Taking $\varepsilon \to 0$ we get

$$\left|f(x) - f(x_0)\right| \leqslant r \sup_{y \in B(x,r)} \left|(\nabla f)(y)\right|.$$

Since f is smooth the supremum is finite which gives the first claim of the lemma. If $(\nabla f)(x_0) = 0$, then we can apply the first part to $X_i f$ and get

$$\sup_{y \in B(x,r)} \left|(\nabla f)(y)\right| \leqslant Cr \sup_{y \in B(x,r)} \left|(\nabla\nabla f)(y)\right|,$$
$$\left|f(x) - f(x_0)\right| \leqslant Cr^2 \sup_{y \in B(x,r)} \left|(\nabla\nabla f)(y)\right|$$

which gives the second claim. □

Theorem 6.3. *Let N be a stratified group and ϕ be a smooth homogeneous norm on N. For $\beta > 0$, $p \geqslant 1$ put $\mu_{\beta,p} = \exp(-\beta\phi^p)/Z\, d\lambda$, where Z is a normalising factor such that $\mu_{\beta,p}$ is a probability measure. The measure $\mu_{\beta,p}$ satisfies no* LS_q *inequality with $q \in (1,2]$.*

Proof. Fix $\beta > 0$, $p \geqslant 1$, $q \in (1,2]$. Suppose that $\mu_{\beta,p}$ satisfies LS_q. We are going to show that this leads to contradiction. Let x_0 be such that $(\nabla\phi)(x_0) = 0$. For $t > 0$ put $r = t^{(-p+1)/2}$ and $f = \max(\min((2 - d(x, tx_0))/r, 1), 0)$. By homogeneity and Lemma 6.3 we have $|\phi(x) - \phi(tx_0)| \leqslant C_1 r^2$ on $B(tx_0, 2r) = \{x\colon d(x, tx_0) \leqslant 2r\}$, so $|\phi(x)^p - \phi(tx_0)^p| \leqslant C_2$. Consequently the exponential factor in $\mu_{\beta,p}$ is comparable to a constant on support of f. Also $|\nabla f| \leqslant r^{-1}$ and

$$\mu_{\beta,p}|f|^q \approx r^Q \exp\bigl(-\beta\phi(tx_0)^p\bigr),$$
$$\log\bigl(\mu_{\beta,p}|f|^q\bigr) \approx -t^p,$$
$$\mu_{\beta,p}|\nabla f| \approx r^{-q} r^Q \exp\bigl(-\beta\phi(tx_0)^p\bigr),$$
$$\mu_{\beta,p}\bigl(|f|^q \log\bigl(|f|^q \mu_{\beta,p}|f|^q\bigr)\bigr) \approx \int_{B(tx_0,r)} |f|^q t^p \, d\mu_{\beta,p} \approx t^p r^Q \exp\bigl(-\beta\phi(tx_0)^p\bigr).$$

Using LS_q we get

$$t^p r^Q \exp\bigl(-\beta\phi(tx_0)^p\bigr) \leqslant Mr^{-q} r^Q \exp\bigl(-\beta\phi(tx_0)^p\bigr)$$

for large t, so

$$t^p \leqslant Mr^{-q} = Mt^{-q(-p+1)/2}$$

for large t, and $p \leqslant q(p-1)/2$. Since $p \geqslant 1$ and $q \leqslant 2$, this implies $p \leqslant p-1$ which is a contradiction. □

7. Log Sobolev inequalities for heat kernel on the Heisenberg group

The heat kernels bound of the following form

$$\frac{1}{C|B(e,t^{1/2})|} e^{-\sigma d^2(x)t} \leqslant p(x,t) \leqslant \frac{C}{|B(e,t^{1/2})|} e^{-\frac{1}{\sigma}d^2(x)t}$$

were well known since a few decades, see e.g. [17,38] and references therein. While the measures corresponding to the densities on the left and right have nice properties and in particular satisfy Poincaré and logarithmic Sobolev inequality, this kind of sandwich bound does not imply similar properties for the measure corresponding to the density in the middle. Namely on a stratified groups one can write

$$C^{-1} p(x,t/\sigma) \leqslant \frac{1}{|B(e,t^{1/2})|} \exp\bigl(-\phi^2(x)/t\bigr) \leqslant Cp(x,\sigma t)$$

where $C, \sigma \geqslant 1$ are constants and ϕ is a smooth homogeneous norm. In Theorem 6.3 we proved that the density in the middle does not satisfy logarithmic Sobolev inequality. We give another example in Appendix A.

In [26] it was observed that asymptotics from [22] imply the following precise bound (extending [7]) on the heat kernel p (at time $t=1$) on the three-dimensional Heisenberg group H_1:

- (HK)
  There exists a constant $L \in (0,\infty)$ such that for any $x \equiv (\mathbf{x}, z) \in H_1$

$$L^{-1}\bigl(1+\|\mathbf{x}\|d(x)\bigr)^{-\frac{1}{2}} e^{-\frac{d^2(x)}{4}} \leqslant p(x) \leqslant L\bigl(1+\|\mathbf{x}\|d(x)\bigr)^{-\frac{1}{2}} e^{-\frac{d^2(x)}{4}}.$$

Let $d\nu_0 \equiv \rho_0\, d\lambda \equiv e^{-\frac{d^2(x)}{4}}\, d\lambda/Z$ and set $d\mu = p\, d\lambda$.

Theorem 7.1. *There exist constants $C_1, C_2, D_1, D_2 \in (0,\infty)$ such that*

$$\mu\bigl(f^2 d^2\bigr) \leqslant C_2\mu|\nabla f|^2 + D_2\mu f^2$$

and

$$\mu\bigl(|f|d\bigr) \leqslant C_1\mu|\nabla f| + D_1\mu|f|.$$

Proof. Put $W = \frac{-1}{2}\log(1+\varepsilon\|x\|d)$ for some $\varepsilon \in (0,1)$ to be chosen later. We have

$$|\nabla W|^2 = \varepsilon^2 \frac{|d\nabla\|x\| + \|x\|\nabla d|^2}{(1+\varepsilon\|x\|d)^2} \leqslant \varepsilon^2 \frac{d^2+\|x\|^2}{(1+\varepsilon\|x\|d)^2} \leqslant \varepsilon^2 d^2 + 1$$

so, if ε is small enough W satisfies assumptions of Theorem 2.5.

Now we observe that for $\varepsilon \in (0,1)$, we have

$$\left(1+\|x\|d\right)^{-\frac{1}{2}} \leqslant \left(1+\varepsilon\|x\|d\right)^{-\frac{1}{2}} \leqslant \frac{1}{\varepsilon}\left(1+\|x\|d\right)^{-\frac{1}{2}}.$$

This together with (HK) imply we can write $\mu = \exp(-W-V)\mu_0$ and apply Theorem 2.5 to get the first claim. We get the second claim using Theorem 2.2. □

By similar arguments as in Section 3 we obtain the following result

Theorem 7.2. *Let $d\mu \equiv p\,d\lambda$. There exists a constant $M \in (0,\infty)$ such that*

$$M\mu(f-\mu f)^2 \leqslant \mu|\nabla f|^2. \tag{73}$$

We are now ready to prove the Log-Sobolev inequality for the heat kernel measure.

Theorem 7.3. *There exists a constant $c \in (0,\infty)$ such that on Heisenberg group H_n we have*

$$\mu\left(f^2 \log \frac{f^2}{\mu f^2}\right) \leqslant c\mu|\nabla f|^2.$$

Remark. The case of H_1 is proven in [26]. While our proof uses heat kernel estimates from [26], in [26] large part is devoted to proof of estimate (1) for heat kernel measure on H_1 — using our methods we could give different proof for this part, but instead we work directly with Log-Sobolev inequality.

Proof of Theorem 7.3. First consider H_1. In the proof of Theorem 7.1 we wrote $\mu = e^{-W-V}\mu_0$. Consider now $\mu_1 = e^{-W}\mu = e^{-U}\,d\lambda$. μ_1 satisfies Log-Sobolev inequality as a consequence of Theorem 4.1. The result for H_1 follows, since μ is equivalent to μ_1.

Now, write $H_n = G/N$, where $G = \prod_{i=1}^n H_1$, $N = \{((0,z_1),\ldots,(0,z_n))\colon \sum z_i = 0\}$ and let π be the canonical homomorphism from G to H_n. Since heat kernel on H_n is an image of product of heat kernels on $G = \prod_{i=1}^n H_1$, and since Log-Sobolev inequality holds on product, we have

$$\mu_{H_n}\left(f^2\log\frac{f^2}{\mu_{H_n} f^2}\right) = \mu_G\left((f\circ\pi)^2\log\frac{(f\circ\pi)^2}{\mu_G(f\circ\pi)^2}\right) \leqslant c\mu_G\left|\nabla(f\circ\pi)\right|^2 = c\mu_G\left|(\nabla f)\circ\pi\right|^2 = c\mu_{H_n}|\nabla f|^2. \qquad \square$$

Appendix A. Examples of no spectral gap

In case of measures on real line the following necessary and sufficient condition for Poincaré inequality to hold was provided by Muckenhoupt [31] ([2]) which in the special case of a measure $d\mu \equiv \rho\, dx$ can be stated as follows: Given $q \in [1, \infty)$ and $\frac{1}{q} + \frac{1}{p} = 1$

$$\exists C \in (0,\infty) \quad \mu|f - \mu f|^q \leqslant C\mu|f'|^q \quad \Leftrightarrow \quad B_\pm \equiv \sup_{r\in\mathbb{R}^\pm} B_\pm(r) < \infty \tag{74}$$

where

$$B_\pm(r) \equiv \big(\mu\big([r, \pm\infty)\big)\big)^{\frac{1}{q}} \cdot \Bigg(\int\limits_{[0,\pm r]} \rho^{-\frac{p}{q}} \Bigg)^{\frac{1}{p}}.$$

Consider $\rho \equiv e^{-U}\, dx/Z$ with $U \equiv \beta|x|^p(1 + \varepsilon \cos x)$, defined $\varepsilon \in (0, 1)$ and some $\beta \in (0, \infty)$. Then, with $r = 2n\pi + \frac{\pi}{2}$, we have

$$\begin{aligned}
B_+(r) &> \Bigg(\int\limits_{2n\pi+\frac{4}{3}\pi}^{2n\pi+\frac{8}{3}\pi} e^{-\beta|x|^p(1-\frac{\varepsilon}{2})}\, dx \Bigg)^{\frac{1}{q}} \cdot \Bigg(\int\limits_{2n\pi-\frac{2}{3}\pi}^{2n\pi+\frac{2}{3}\pi} e^{+\frac{p}{q}\beta|x|^p(1+\frac{\varepsilon}{2})}\, dx \Bigg)^{\frac{1}{p}} \\
&> e^{-\beta\frac{1}{q}|2n\pi+\frac{8}{3}\pi|^p(1-\frac{\varepsilon}{2})} \bigg(\frac{4}{3}\pi\bigg)^{\frac{1}{q}} \cdot e^{+\frac{1}{q}\beta|2n\pi-\frac{2}{3}\pi|^p(1+\frac{\varepsilon}{2})} \bigg(\frac{4}{3}\pi\bigg)^{\frac{1}{p}} \\
&= \frac{4}{3}\pi \exp\bigg\{ \frac{\beta(2n\pi)^p}{q} \bigg[\bigg|1 - \frac{1}{3n}\bigg|^p \bigg(1 + \frac{\varepsilon}{2}\bigg) - \bigg|1 + \frac{4}{3n}\bigg|^p \bigg(1 - \frac{\varepsilon}{2}\bigg) \bigg] \bigg\} \\
&\sim \frac{4}{3}\pi \exp\bigg\{ \frac{\beta(2n\pi)^p}{q} \bigg(\varepsilon + o\bigg(\frac{1}{n}\bigg) \bigg) \bigg\} \to \infty \quad \text{as } n \to \infty.
\end{aligned}$$

Alternatively one can study lower bound asymptotic for $B_\pm$ thinking of $U = V + \delta V$ as a perturbation of $V \equiv \beta|x|^p$ as follows. We notice that by Jensen inequality

$$B_+(r, U) \geqslant B_+(r, V) \exp\bigg\{ -\frac{1}{q}\beta \frac{\int_r^\infty \delta V e^{-V}\, dx}{\int_r^\infty e^{-V}\, dx} + \frac{p}{q}\beta\varepsilon \frac{\int_0^r \delta V e^{+V}\, dx}{\int_0^r e^{+V}\, dx} \bigg\}.$$

Hence one can use a procedure based essentially on integration by parts to study the integrals in the exponential. For example in case $p = 2$ one gets the following an asymptotic lower bound

$$B_+(r, U) \geqslant B_+(r, V) \exp\big\{ -\beta\varepsilon r \cos r + O(1) \big\}.$$

We summarise our considerations in the above as follows

Proposition A.1. *Suppose* $p \geqslant 1$. *In any neighbourhood*

$$\frac{1}{C} e^{-(1+\delta)\beta|x|^p} \leqslant \rho \leqslant C e^{-\frac{1}{1+\delta}\beta|x|^p}$$

with arbitrary $\delta \in (0,1)$ and some $C \in (1,\infty)$, of a measure $d\mu_0 \equiv \frac{e^{-\beta|x|^p}dx}{Z}$ satisfying the Poincaré inequality there is a measure $d\mu \equiv \rho\, dx$ for which this inequality fails.

The example provided above illustrates similar phenomenon for other coercive inequalities.

Note added in proof

For the benefit of the reader we would like to mention the following two recent works [5] and [15] containing certain related results in Euclidean setup as well as some results concerning isoperimetry.

We would like to thank Michel Ledoux and Patrick Cattiaux for this information.

References

[1] S. Aida, D.W. Stroock, Moment estimates derived from Poincaré and logarithmic Sobolev inequalities, Math. Res. Lett. 1 (1994) 75–86.
[2] C. Ané, S. Blachère, D. Chafai, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer, Sur les inégalités de Sobolev logarithmiques, Panor. Synthèses, vol. 10, S.M.F., Paris, 2000.
[3] A.Val. Antonjuk, A.Vict. Antonjuk, Weighted spectral gap and logarithmic Sobolev inequalities and their applications, Preprint 93.33, Acad. Sci. of Ukraine, Kiev, 1993.
[4] D. Bakry, L'hypercontractivité et son utilisation en théorie des semigroupes, in: Lectures on Probability Theory (École d'été de probabilités de St-Flour 1992), in: Lecture Notes in Math., vol. 1581, Springer, Berlin, 1994, pp. 1–114.
[5] D. Bakry, P. Cattiaux, A. Guillin, Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré, J. Funct. Anal. 254 (2008) 727–759.
[6] F. Barthe, P. Cattiaux, C. Roberto, Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry, Rev. Mat. Iberoamericana 22 (2006) 993–1067.
[7] R. Beals, B. Gaveau, P.C. Greiner, Hamiltonian–Jacobi theory and heat kernel on Heisenberg groups, J. Math. Pures Appl. 79 (2000) 633–689.
[8] S.G. Bobkov, F. Götze, Exponential integrability and transportation cost related to logarithmic Sobolev inequalities, J. Funct. Anal. 163 (1999) 1–28.
[9] S. Bobkov, M. Ledoux, From Brunn–Minkowski to Brascamp–Lieb and to logarithmic Sobolev inequalities, Geom. Funct. Anal. 10 (2000) 1028–1052.
[10] S. Bobkov, M. Ledoux, Weighted Poincaré-type inequalities for Cauchy and other convex measures, Ann. Probab. 37 (2) (2009) 403–427.
[11] S. Bobkov, B. Zegarliński, Entropy bounds and isoperimetry, Mem. Amer. Math. Soc. 176 (829) (2005), x+69 pp.
[12] Th. Bodineau, B. Helffer, The log-Sobolev inequalities for unbounded spin systems, J. Funct. Anal. 166 (1999) 168–178.
[13] H.J. Brascamp, E. Lieb, On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for Logconcave functions, and with an application to diffusion equation, J. Funct. Anal. 22 (1976) 366–389.
[14] E.A. Carlen, D.W. Stroock, An application of the Bakry–Émery criterion to infinite dimensional diffusions, in: Séminaire de probabilités de Strasbourg, vol. 20, 1986, pp. 341–348.
[15] P. Cattiaux, N. Gozlan, A. Guillin, C. Roberto, Functional inequalities for heavy tails distributions and application to isoperimetry, preprint, 2008.
[16] B. Chow, Lu Peng, Ni Lei, Hamilton's Ricci Flow, Grad. Stud. Math., Amer. Math. Soc., 2006.
[17] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge Univ. Press, 1989.
[18] E.B. Davies, L. Gross, B. Simon, Hypercontractivity: A bibliographic review, in: Ideas and Methods in Quantum and Statistical Physics, Oslo, 1988, Cambridge Univ. Press, Cambridge, 1992, pp. 370–389.
[19] B.K. Driver, T. Melcher, Hypoelliptic heat kernel inequalities on the Heisenberg group, J. Funct. Anal. 221 (2005) 340–365.
[20] I. Gentil, C. Roberto, Spectral gaps for spin systems: Some non-convex phase examples, J. Funct. Anal. 180 (2001) 66–84.
[21] A. Guionnet, B. Zegarliński, Lectures on logarithmic Sobolev inequalities, in: Séminaire de Probabilités, XXXVI, in: Lecture Notes in Math., vol. 1801, Springer, Berlin, 2003, pp. 1–134.

[22] H. Hueber, D. Müller, Asymptotics for some Green kernels on the Heisenberg group, Math. Ann. 283 (1989) 97–119.
[23] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms, Trans. Amer. Math. Soc. 258 (1980) 147–153.
[24] A. Kasue, A Laplacian comparison theorem and function theoretic properties of a complete Riemannian manifold, Japan J. Math. 8 (1982) 309–341.
[25] R. Latała, K. Oleszkiewicz, Between Sobolev and Poincaré, Lecture Notes in Math., vol. 1745, 2000, pp. 147–168.
[26] H.-Q. Li, Estimation optimale du gradient du semi-groupe de la chaleur sur le groupe de Heisenberg, J. Funct. Anal. 236 (2006) 369–394.
[27] H.-Q. Li, Estimations asymptotiques du noyau de la chaleur sur les groupes de Heisenberg, C. R. Math. Acad. Sci. 344 (2007) 497–502.
[28] P. Ługiewicz, B. Zegarliński, Coercive inequalities for Hörmander type generators in infinite dimensions, J. Funct. Anal. 247 (2007) 438–476.
[29] V. Mazya, Sobolev Spaces, Springer, 1985.
[30] R. Monti, Some properties of Carnot–Carathéodory balls in the Heisenberg group, Rend. Mat. Acc. Lincei 11 (2000) 155–167.
[31] B. Muckenhoupt, Hardy's inequality with weights, Studia Math. 44 (1972) 31–32, collection of articles honoring the completion by Antoni Zygmunt of 50 years of scientific activity.
[32] Ding Qing, A new Laplacian comparison theorem and the estimate of eigenvalues, Chinese Ann. Math. Ser. B 15 (1994) 35–42.
[33] C. Roberto, B. Zegarliński, Orlicz–Sobolev inequalities for sub-Gaussian measures and ergodicity of Markov semi-groups, J. Funct. Anal. 243 (2006) 28–66.
[34] J. Rosen, Sobolev inequalities for weight spaces and supercontractivity, Trans. Amer. Math. Soc. 222 (1976) 367–376.
[35] B. Simon, Schrödinger semigroups, Bull. Amer. Math. Soc. (N.S.) 7 (3) (1982).
[36] D.W. Stroock, B. Zegarliński, The logarithmic Sobolev inequality for continuous spin systems on a lattice, J. Funct. Anal. 104 (1992) 299–326.
[37] D.W. Stroock, B. Zegarliński, The equivalence of the logarithmic Sobolev inequality and the Dobrushin–Shlosman mixing condition, Comm. Math. Phys. 144 (2) (1992) 303–323.
[38] N.Th. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and Geometry on Groups, Cambridge Univ. Press, 1992.
[39] F.-Y. Wang, Logarithmic Sobolev inequalities on noncompact Riemannian manifolds, Probab. Theory Related Fields 109 (1997) 417–424.
[40] F.-Y. Wang, Functional Inequalities, Markov Processes and Spectral Theory, Science Press, Beijing, 2005.
[41] N. Yosida, The log-Sobolev inequality for weakly coupled lattice fields, Probab. Theory Related Fields 115 (1999) 1–40.
[42] B. Zegarliński, Entropy bounds for Gibbs measures with non-Gaussian tails, J. Funct. Anal. 187 (2001) 368–395.