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Row Complete Squares and a Problem of A. Kotzig Concerning P-Quasigroups and Eulerian Circuits

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An $n \times n$ square L on n symbols is called row (column) complete if every ordered pair of the symbols of L occurs just once as an adjacent pair of elements in some row (column) of L. It is called row (column) latin if each symbol occurs exactly once in each row (column) of the square. A square which is both row latin and column latin is called a latin square. All known examples of row complete latin squares can be made column complete as well by suitable re-ordering of their rows and in the present paper we provide a sufficient condition that a given row complete latin square should have this property.

Using row complete and column latin squares as a tool we follow this by showing how to construct code words on n symbols of the maximum possible length $l = \frac{1}{2}n(n-1) + 1$ with the two properties that (*i*) no unordered pair of consecutive symbols is repeated more than once and (ii) no unordered pair of nearly consecutive symbols is repeated more than once. (Two symbols are said to be nearly consecutive if they are separated by a single symbol.) We prove that such code words exist whenever $n = 4r + 3$ with $r \not\equiv 1 \bmod 6$ and $r \not\equiv 2 \bmod 5$. We show that the existence of such a code word for a given value of n guarantees the existence of an Eulerian circuit in the complete undirected n-graph which corresponds to a P-quasigroup, thus answering a question raised by A. Kotzig in the affirmative. (Kotzig has defined a P-groupoid as a groupoid $(G, \cdot)$ having the following three properties: (i) $a \,.\, a = a$ for all $a \in G$; (ii) $a \neq b$ implies $a \neq a \,.\, b$ and $b \neq a \,.\, b$ for all $a, b \in G$; and (iii) $a \,.\, b = c$ implies $c \,.\, b = a$ for all $a, b, c \in G$. Every decomposition of the complete undirected n-graph into disjoint closed circuits defines such a P-groupoid, as is easily seen by defining $a \,.\, b = c$ if and only if a, b, c are consecutive edges of one such closed circuit. A P-groupoid whose multiplication table is a latin square is called a P-quasigroup.)

An $n \times n$ square L on n symbols is called *row* (column) *complete* if every ordered pair of the symbols of L occurs just once as an adjacent pair of elements in some row (column) of L. It is called *row* (column) *latin* if each symbol occurs exactly once in each row (column) of the square. A square which is both row latin and column latin is called a *latin square*.

All known examples of row complete latin squares can be made column complete as well by suitable reordering of their rows. The question has been raised on several occasions whether there exist row complete latin squares which do not have this property. (See, for example, [4] and [8].) Let us call the property, property K. As a contribution to the solution of this problem, we give in Theorem 1 below a sufficient condition for property K to hold and we make the observation that all known examples of row complete latin squares are isotopes of multiplication tables of groups and consequently satisfy our sufficient condition.

Let us remark here that if a loop is isotopic to a group then the loop is a group and the isotopism is an isomorphism (see [1; 2; 7; or 4, Chap. 1]). We note also that all the conjugates of a loop with the inverse property (that is, loops derived from the multiplication table of the given loop by permuting in any of the 3! possible ways the roles of row, column, and element numbers) are isotopic to it (see [14; or 4, Chap. 4]).

THEOREM 1. *Any row complete latin square which represents the multiplication table of a group or of an inverse property loop G which satisfies the identity $(gh)(h^{-1}k) = gk$ for all g, h, $k \in G$ can be made column complete as well as row complete by suitably reordering its rows.*

Proof. Let the given square be the multiplication table of the loop G where $h_1, h_2, \dots, h_n$ and $g_1, g_2, \dots, g_n$ are two orderings of the elements of G, as in Fig. 1.

$$
\begin{array}{c|ccccccc}
 & h_1 & h_2 & \cdots & h_u & \cdots & h_v & \cdots \; h_n \\
\hline
g_1 & g_1h_1 & g_1h_2 & & \vdots & & \vdots & \vdots \\
g_2 & g_2h_1 & g_2h_2 & & \vdots & & \vdots & \vdots \\
\vdots & & & & \vdots & & \vdots & \vdots \\
g_s & \cdots & \cdots & \cdots & g & & \vdots & \vdots \\
\vdots & & & & & & \vdots & \vdots \\
g_t & \cdots & \cdots & \cdots & \cdots & \cdots & g & \vdots \\
\vdots & & & & & & & \vdots \\
g_n & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & g_nh_n
\end{array}
$$

FIGURE 1

Since the square is row complete the elements $h_1^{-1}h_2, h_2^{-1}h_3, \dots, h_{n-1}^{-1}h_n$ are all distinct and are the nonidentity elements of the loop in a new order: for suppose that $h_u^{-1}h_{u+1} = h_v^{-1}h_{v+1} = k$ say. Let the arbitrary element g of G occur in the sth row of column u and in the tth row of column v. Then $g = g_s h_u = g_t h_v$. The entries in the $(u+1)$th column of row s and in the $(v+1)$th column of row t are $g_s h_{u+1} = (g_s h_u)(h_u^{-1}h_{u+1}) = gk$ and $g_t h_{v+1} = (g_t h_v)(h_v^{-1}h_{v+1}) = gk$ respectively. Hence, the ordered pair (g, gk) occurs as adjacent elements in both the sth and the tth rows of the square, contrary to hypothesis.

Now let the rows be reordered according to the permutation

$$\begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ h_1^{-1} & h_2^{-1} & \cdots & h_n^{-1} \end{pmatrix}$$

so that the reordered square takes the form shown in Fig. 2. This reordering will not affect the row completeness.

$$\begin{array}{c|cccccccc}
 & h_1 & h_2 & \cdots & h_u & \cdots & h_v & \cdots & h_n \\
\hline
h_1^{-1} & e & h_1^{-1}h_2 & & \vdots & & \vdots & & \vdots \\
h_2^{-1} & h_2^{-1}h_1 & e & & \vdots & & \vdots & & \vdots \\
\vdots & & & & \vdots & & \vdots & & \vdots \\
h_s^{-1} & \cdots & \cdots & \cdots & g & & \vdots & & \vdots \\
\vdots & & & & & & \vdots & & \vdots \\
h_t^{-1} & \cdots & \cdots & \cdots & \cdots & \cdots & g & & \vdots \\
\vdots & & & & & & & & \vdots \\
h_n^{-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & e
\end{array}$$

FIGURE 2

Moreover, in the new square each ordered pair of elements will occur at most once as a pair of adjacent elements in the columns: for, suppose that the entries of the (s, u)th and (t, v)th cells are the same, equal to g say. Then, $h_s^{-1}h_u = g = h_t^{-1}h_v$. The entries of the $(s+1, u)$th and $(t+1, v)$th cells must then be distinct, for $h_{s+1}^{-1}h_u = h_{t+1}^{-1}h_v$ would imply $(h_{s+1}^{-1}h_s)(h_s^{-1}h_u) = (h_{t+1}^{-1}h_t)(h_t^{-1}h_v)$ and so $(h_{s+1}^{-1}h_s)g = (h_{t+1}^{-1}h_t)g$. But then, $h_{s+1}^{-1}h_s = h_{t+1}^{-1}h_t$ whence $(h_{s+1}^{-1}h_s)^{-1} = (h_{t+1}^{-1}h_t)^{-1}$. Now, $h_{s+1}^{-1}h_s$ is the inverse of $h_s^{-1}h_{s+1}$ since $(h_{s+1}^{-1}h_s)(h_s^{-1}h_{s+1}) = h_{s+1}^{-1}h_{s+1} = e$. Thus we would have $h_s^{-1}h_{s+1} = h_t^{-1}h_{t+1}$, which is contrary to hypothesis. This shows that the new square is column complete as well as row complete and so proves the theorem. (See also note 1.)

For the interest of the reader, we summarize briefly the present state of knowledge concerning existence of row complete latin squares. In the first place, we note that there are no row complete latin squares of orders 2, 3, 5, or 7 (This has been shown by Warwick [16] and by Owens [12]). Moreover, all known row complete latin squares are multiplication tables of groups (or quasigroups isotopic to groups).

The multiplication table of a finite group G can be written in the form of a row complete latin square if and only if the group is *sequenceable*: that is, if and only if there exists an ordering of the elements $g_1, g_2, \ldots, g_n$ of G such that the partial products $p_s = \prod_{i=1}^{s} g_i$ for $s = 1, 2, \ldots, n$ are all distinct (see [6]). Evidently, $g_1 = e$ (where e denotes the group identity) is necessary for this condition to hold. Moreover, if the group G is abelian, it is known that $p_n = e$ unless G has a unique element h of order two and that, in the latter case, $p_n = h$ (see [13]). Thus, a finite abelian group can be sequenceable only if it has a unique element of order 2. Gordon [6] has proved that this condition is sufficient as well as necessary. Namely, a finite abelian group is sequenceable if and only if it is the direct product of two groups A and B such that A is a cyclic group of order 2^k, $k > 0$, and B is of odd order.

As regards the sequenceability of groups of odd order, little is known. It is clear from the preceding remarks that an abelian group of odd order cannot be sequenceable. The nonabelian group of smallest odd order is the (unique) nonabelian group of order 21 generated by two elements a and b with the defining relations $a^7 = b^3 = e$, $ab = ba^2$. This group has been shown to be sequenceable by Mendelsohn [11]. The nonabelian group of order 27 generated by two elements a and b with the defining relations $a^p = b^q = e$, $ab = ba^r$, where $p = 9$, $q = 3$, and $r = 4$, has been shown to be sequenceable by the present author [8], and very recently the groups on two generators with similar structure having orders 39 ($p = 13$, $q = 3$, $r = 3$), 55 ($p = 11$, $q = 5$, $r = 3$), and 57 ($p = 19$, $q = 3$, $r = 7$) have been shown to be sequenceable by Wang [15]. The present author has conjectured in [8] that all nonabelian groups of odd order on two generators are sequenceable, and the recent results of Wang lend strength to this conjecture.

Lastly, in regard to nonabelian sequenceable groups of even order, Gordon [6] has shown that the dihedral groups D_3 and D_4 of orders 6 and 8 are not sequenceable, and J. Dénes and E. Török [5] have shown that the dihedral groups D_5, D_6, D_7, and D_8 of orders 10, 12, 14, and 16 are sequenceable but that the remaining nonabelian groups of orders less than or equal to 14 are not sequenceable.

Let us turn now to another problem.

The rows of an $n \times n$ row complete latin square define a decomposition of the complete directed n-graph into n-disjoint Hamiltonian paths, as has been pointed out in [5] and [11]. Each row defines one such path. Equivalently, we may say that each row defines a code word on n symbols of the maximum possible length with the two properties (i) that no symbol is repeated more than once and (ii) that no unordered pair of consecutive symbols is repeated more than once. Analogously, we may ask how to construct a code word (on n symbols as before) of the maximum possible length with the two properties that (i) no unordered pair of consecutive symbols is repeated more than once and (ii) that no unordered pair of nearly consecutive symbols is repeated more than once. (Two symbols are said to be *nearly consecutive* if they are separated by a single symbol. Property (ii) implies, a fortiori, that no unordered triple of consecutive symbols is repeated more than once.) Such a code word has length $\frac{1}{2}n(n-1)+1$ and corresponds to an Eulerian circuit of the complete undirected n-graph which has the additional property that whenever P_hP_i and P_iP_k are consecutive edges of the circuit then P_hP_j and P_jP_k are not consecutive edges for any $j \neq i$. A problem raised by Kotzig [10] suggests the conjecture that such Eulerian circuits (or code words) exist whenever n is an integer congruent to 3 mod 4, as we have pointed out in [9]. By means of Theorems 2, 3, and 4 below, we demonstrate the existence of such code words for most values of n congruent to 3 mod 4 and we then explain the connection with the abovementioned conjecture of Kotzig concerning the existence of P-quasigroups which define Eulerian circuits. In our proof we make use of rectangles which are row complete and column latin. (An $m \times 2m$ rectangle on $2m$ symbols is *row complete* if every unordered pair of the $2m$ symbols occurs just once as an adjacent pair of elements in some row of the rectangle.)

THEOREM 2. *Let U denote the sequence of nonzero integers $u_1, u_2, \ldots, u_r$, where $u_1 = 1$ or 2, $-r \leqslant u_i \leqslant r$, and $|u_j| \neq |u_i|$ unless $j = i$ (so that $|u_1|, |u_2|, \ldots, |u_r|$ is a reordering of the natural numbers $1, 2, \ldots, r$). Also, let $\sigma_s \equiv \sum_{i=1}^{s} u_i$, mod $2r+1$. Then, if for a given positive integer r, such a sequence U exists with the properties* (i) *that the integers ± 1 and the sums $\pm(u_i + u_{i+1})$ for $i = 1, 2, \ldots, r-1$ of adjacent pairs of the sequence U form a complete set of nonzero residues modulo $2r+1$ (implying as necessary conditions that no one of the sums is equal to ± 1 and that $|u_j + u_{j+1}| \neq |u_i + u_{i+1}|$ unless $j = i$);* (ii) *if $u_1 = 1$ then $-3 + \sigma_r$ is prime to $2r+1$; and* (iii) *if $u_1 = 2$ then $-2 + \sigma_r$ is prime to $2r+1$; then there exists a code word on $4r+3$ symbols of length $\frac{1}{2}(4r+3)(4r+2)+1$ in which no pair of consecutive symbols and no pair of nearly consecutive symbols is repeated.*

Proof. We consider first the $(2r+1) \times (4r+2)$ rectangle R whose first row is given below and whose $(h+1)$th row is obtained from the first by adding hk taken modulo $2r+1$ to each suffix, where $k \equiv 2r-2+\sigma_r \bmod 2r+1$ if $u_1 = 1$ and $k \equiv 2r-1+\sigma_r \bmod 2r+1$ if $u_1 = 2$.

The first row is

$$a_0 a_{2r-1} b_0 a_2 a_{2r-3} b_0 a_4 a_{2r-5} b_0 \ldots \ldots b_0 a_{2s} a_{2r-1-2s} b_0 \ldots \ldots$$

$$\ldots \ldots b_0 a_{2r-2} a_1 b_0 a_{2r} b_{2r} b_{2r+\sigma_1} b_{2r+\sigma_2} \ldots \ldots b_{2r+\sigma_{r-1}} b_{2r+\sigma_r}.$$

The rectangle R involves $4r+2$ symbols comprising the $2r+1$ symbols of the set $A = \{a_0, a_1, \ldots, a_{2r}\}$ and the $2r+1$ symbols of the set $B = \{b_0, b_1, \ldots, b_{2r}\}$. Since the suffixes $i, i+k, i+2k, \ldots, \ldots, i+2rk$ are all distinct modulo $2r+1$ because $(k, 2r+1) = 1$, the rectangle R has the entries of each column all different and consequently is a column latin rectangle. We show that it is also row complete.

The first row of the rectangle contains the ordered pairs of elements $(a_0, a_{2r-1}), (a_2, a_{2r-3}), \ldots, \ldots, (a_{2s}, a_{2r-1-2s}), \ldots, \ldots, (a_{2r-2}, a_1)$ of the set A. The second member of the ordered pair $(a_{2s}, a_{2r-1-2s})$, $s = 0, 1, \ldots, r-1$, is obtained from the first member by adding $-4s-2$, taken modulo $2r+1$, to the suffix of the first member. The first member is obtained from the second by adding $4s+2$, taken modulo $2r+1$, to the second member. The integers $-4s-2$ and $4s+2$, for $s = 0, 1, \ldots, r-1$ form a complete set of nonzero residues modulo $2r+1$: for the integers $-4s-2$, $s = 0, 1, \ldots, r-1$ may be rewritten as the integers $4s+2$, $s = -1, -2, \ldots, -r$. Also $4s+2 \equiv 0 \bmod 2r+1$ implies that $s = r$. Thus $4s+2$, $-r \leqslant s \leqslant r-1$, is a complete set of nonzero residues, as stated.

Because each column of the rectangle R whose first-row element is a member of the set A contains every member of the set A, the element $a_v (0 \leqslant v \leqslant 2r)$ occurs in each such column. It now follows easily that the unordered pair (a_v, a_w) occurs just once in some row of the rectangle R. For, suppose that $w - v \equiv 4s+2 \bmod 2r+1$ for some s such that $0 \leqslant s \leqslant r-1$, then the ordered pair (a_w, a_v) occurs in some row of the submatrix formed by the two adjacent columns whose first-row elements are $a_{2s}, a_{2r-1-2s}$. If, on the other hand, $w - v \equiv 4s+2 \bmod 2r+1$ for some s such that $-r \leqslant s \leqslant -1$, then the ordered pair (a_v, a_w) occurs in some row of the submatrix formed by the two adjacent columns whose first row elements are $a_{2(-1-s)}, a_{2r-1-2(-1-s)}$.

We use a similar argument to show that each unordered pair of distinct elements of the set B occurs just once in some row of the rectangle R.

The first row of the rectangle contains the ordered pairs of elements $(b_{2r}, b_{2r+\sigma_1}), (b_{2r+\sigma_1}, b_{2r+\sigma_2}), \ldots, \ldots, (b_{2r+\sigma_{r-1}}, b_{2r+\sigma_r})$. The second member of the sth ordered pair in this sequence is obtained from the first member by adding u_s, taken modulo $2r + 1$, to the suffix of the first member. The first member of this ordered pair is obtained from the second member by adding $-u_s$, taken modulo $2r + 1$, to the suffix of the second member. Furthermore, by definition of the integers u_s, the elements $\pm u_s$ form a complete set of nonzero residues modulo $2r + 1$.

Because each column of the rectangle R whose first-row element is a member of the set B contains every member of the set B, the element $b_v (0 \leqslant v \leqslant 2r)$ occurs in each such column. It follows easily by a similar argument to that used with respect to the set A that the unordered pair (b_v, b_w) for each choice of v and w $(w \neq v)$ occurs just once in some row of the rectangle R.

To complete the proof that the rectangle R is now complete we have only to show that each unordered pair (b_v, a_w) occurs just once in some row of R. Inspection of the first row of R shows that every pair (b_0, a_w), $w = 0, 1, 2,..., 2r$, occurs just once in it and so, from the rule for constructing subsequent rows of R, it follows easily that every pair (b_{hk}, a_{w+hk}), $w = 0, 1, 2,..., 2r$, occurs just once in the hth row of R. Since hk takes all of the values $0, 1, 2,..., 2r \bmod 2r + 1$ as h varies through the integers $0, 1, 2,..., 2r$, it now follows that R is a row complete and column latin rectangle.

Next, let us adjoin a $(4r + 3)$th column to the rectangle R each element of which is equal to c, where c is an additional symbol, thus forming a $(2r + 1) \times (4r + 3)$ rectangle R'; and let us form a code word of length $(4r + 3)(2r + 1) + 1$ from the extended rectangle by writing its $2r + 1$ rows consecutively in a single row and then adjoining the symbol a_0. (See the example exhibited in Fig. 3.)

In the extended rectangle R', each unordered pair (b_i, c) occurs just once in some row of the submatrix formed by the last two columns of R', and each unordered pair (c, a_i) occurs just once in some row of the submatrix formed by juxtaposing the last and the first columns of R' after first applying the permutation $(2r + 1 \;\; 2r \; \ldots \ldots \; 3 \;\; 2 \;\; 1)$ to the rows of the latter column. It follows immediately that the code word formed as described above by writing the rows of R' consecutively contains every unordered pair of the $4r + 3$ symbols exactly once as a pair of consecutive symbols. To complete the proof of our theorem, it remains to show that the code word contains every unordered pair of the $4r + 3$ symbols at most once as a pair of nearly consecutive symbols.

The first row of the rectangle R contains the following pairs of nearly consecutive symbols belonging to the set A: $(a_{2r-1}, a_2), (a_{2r-3}, a_4),...,$ $..., (a_{-2s}, a_{2s}),..., ..., (a_{-2(r-1)}, a_{2r-2}), (a_1, a_{2r})$. That is, it contains the nearly consecutive pairs (a_{-2s}, a_{2s}) for $s = 1, 2,..., r$, where the suffixes are taken modulo $2r + 1$. The second member of an ordered pair (a_v, a_w) belonging to this set is obtained from the first by adding $4s$ mod $2r + 1$ to the suffix of the first member, the first is obtained from the second by adding $-4s$. Because the integers $\pm 4s$ $(1 \leqslant s \leqslant r)$ form a complete set of nonzero residues modulo $2r + 1$, and because each column of R whose first-row element belongs to the set A contains every member of the set A, it follows that every unordered pair of nearly consecutive symbols taken from the set A occurs just once in some row of R.

The first row of the rectangle R contains the following pairs of nearly consecutive symbols belonging to the set B: (b_0, b_{2r}), $(b_{2r}, b_{2r+\sigma_2})$, $(b_{2r+\sigma_1}, b_{2r+\sigma_3})$,..., ..., $(b_{2r+\sigma_s}, b_{2r+\sigma_{s+2}})$,..., ..., $(b_{2r+\sigma_{r-2}}, b_{2r+\sigma_r})$. We observe that the second member of each of these pairs except the first is obtained from the first member of the pair by adding $u_i + u_{i+1}$ $(i = 1, 2,..., r - 1)$, taken modulo $2r + 1$, to the suffix of the first member. The first member of the pair is obtained from the second by adding $-(u_i + u_{i+1})$ to the suffix of the second member. For the pair (b_0, b_{2r}), the second member is obtained from the first by adding -1, taken modulo $2r + 1$, to the suffix of the first member. The first member is obtained from the second by adding $+1$, taken modulo $2r + 1$. By condition (i) of our theorem, the integers ± 1 and $\pm(u_i + u_{i+1})$, for $i = 1, 2,..., r - 1$, form a complete set of nonzero residues modulo $2r + 1$. Hence, because each column of the rectangle R whose first-row element belongs to the set B contains every member of the set B, it follows as before that every unordered pair of nearly consecutive symbols taken from the set B occurs just once in some row of R.

From the above two results, we deduce that every nearly consecutive unordered pair of symbols belonging to the set A and every nearly consecutive unordered pair of symbols belonging to the set B occur exactly once in the code word formed from the rows of the rectangle R'.

Next, we see that because the elements of the $(4r + 1)$th column of the rectangle R' are all different and belong to the set B, every unordered pair (b_v, c) occurs exactly once as a nearly consecutive pair in the code word. Also, because the elements of the second column of the rectangle R' (whose first-row element is a_{2r-1}) are all different and belong to the set A, every unordered pair (c, a_w), excepting only the pair (c, a_{2r-1}), occurs exactly once as a nearly consecutive pair in our code word.

It remains to consider the pairs of nearly consecutive symbols of the type (b_v, a_w). Among the first $4r + 4$ symbols in our code word, the following ordered pairs of nearly consecutive symbols of the above type occur: (a_0, b_0), (a_2, b_0),..., ..., (a_{2s}, b_0),..., ..., (a_{2r-2}, b_0), $(a_{2r}, b_{2r+\sigma_1})$, (b_0, a_{2r-3}), (b_0, a_{2r-5}),..., ..., $(b_0, a_{2r-1-2s})$,..., ..., (b_0, a_1), $(b_{2r+\sigma_r}, a_k)$. That is, b_0 occurs with every a_i except a_{2r-1} and a_{2r}. It is easy to see from this and from the structure of the rectangle R' that every pair (b_v, a_w) occurs except possibly pairs (b_v, a_w) for which $w - v \equiv 2r \equiv -1 \bmod 2r + 1$ or $w - v \equiv 2r - 1 \equiv -2 \bmod 2r + 1$. However, since the pairs $(a_{2r}, b_{2r+\sigma_1})$ and $(b_{2r+\sigma_r}, a_k)$ occur, so also do the pairs $(a_{2r+1-\sigma_1}, b_0)$ and $(b_0, a_{k+1-\sigma_r})$. If $u_1 (= \sigma_1) = 1$, we have $k = 2r - 2 + \sigma_r$ and then these two pairs are (a_{2r}, b_0) and (b_0, a_{2r-1}). If $u_1 (= \sigma_1) = 2$, we have $k = 2r - 1 + \sigma_r$ and then the two pairs are (a_{2r-1}, b_0) and (b_0, a_{2r}). This completes the proof of the theorem.

As an illustration of the contruction described in Theorem 2, let us take the case when $r = 3$. A suitable sequence U is then $u_1 = 2$, $u_2 = 3$, $u_3 = 1$. Since $u_1 = 2$, we take $k \equiv 2r - 1 + \sigma_r \equiv -2 + \sigma_r \bmod 2r + 1$. That is, $k \equiv -2 + 6 = 4 \bmod 7$. We note that 4 is prime to 7 as required. The appropriate rectangle R' is as shown in Fig. 3, and it defines a code word of length $\frac{1}{2}(4r + 3)(4r + 2) + 1 = 106$. This also is shown in Fig. 3.

$$
\begin{array}{l}
a_0\, a_5\, b_0\, a_2\, a_3\, b_0\, a_4\, a_1\, b_0\, a_6\, b_6\, b_1\, b_4\, b_5\, c \\
a_4\, a_2\, b_4\, a_6\, a_0\, b_4\, a_1\, a_5\, b_4\, a_3\, b_3\, b_5\, b_1\, b_2\, c \\
a_1\, a_6\, b_1\, a_3\, a_4\, b_1\, a_5\, a_2\, b_1\, a_0\, b_0\, b_2\, b_5\, b_6\, c \\
a_5\, a_3\, b_5\, a_0\, a_1\, b_5\, a_2\, a_6\, b_5\, a_4\, b_4\, b_6\, b_2\, b_3\, c \\
a_2\, a_0\, b_2\, a_4\, a_5\, b_2\, a_6\, a_3\, b_2\, a_1\, b_1\, b_3\, b_6\, b_0\, c \\
a_6\, a_4\, b_6\, a_1\, a_2\, b_6\, a_3\, a_0\, b_6\, a_5\, b_5\, b_0\, b_3\, b_4\, c \\
a_3\, a_1\, b_3\, a_5\, a_6\, b_3\, a_0\, a_4\, b_3\, a_2\, b_2\, b_4\, b_0\, b_1\, c
\end{array}
$$

0, 5, 7, 2, 3, 7, 4, 1, 7, 6, 13, 8, 11, 12, 14, 4, 2, 11, 6,
0, 11, 1, 5, 11, 3, 10, 12, 8, 9, 14, 1, 6, 8, 3, 4, 8, 5,
2, 8, 0, 7, 9, 12, 13, 14, 5, 3, 12, 0, 1, 12, 2, 6, 12, 4,
11, 13, 9, 10, 14, 2, 0, 9, 4, 5, 9, 6, 3, 9, 1, 8, 10, 13,
7, 14, 6, 4, 13, 1, 2, 13, 3, 0, 13, 5, 12, 7, 10, 11, 14,
3, 1, 10, 5, 6, 10, 0, 4, 10, 2, 9, 11, 7, 8, 14, 0.

FIGURE 3

THEOREM 3. *A sequence U of nonzero integers $u_1, u_2, \ldots, u_r$ with $u_1 = 1$, $-r \leqslant u_i \leqslant r$ and $|u_j| \neq |u_i|$ unless $j = i$ and which satisfies the conditions* (i) *and* (ii) *of Theorem 2 can be constructed for every positive integer r which satisfies the congruences $r \not\equiv 1 \bmod 6$ and $r \not\equiv 2 \bmod 5$.*

Proof. We make the preliminary observation that a sequence U of nonzero integers $u_1, u_2, \ldots, u_r$ certainly satisfies property (i) of Theorem 2 if it has the stronger property (i)* that the sequence $|u_1 + u_2|, |u_2 + u_3|, \ldots, |u_{r-1} + u_r|$ is a reordering of the natural numbers $2, 3, \ldots, r$.

The proof is completed with the aid of the following two lemmas.

LEMMA A. *If* $r = 2t$ *and* $t \not\equiv 1 \bmod 5$, *a sequence* U *of nonzero integers* $u_1, u_2, \ldots, u_r$ *with* $u_1 = 1$, $-r \leqslant u_i \leqslant r$, *and* $|u_j| \neq |u_i|$ *unless* $j = i$ *and which satisfies the condition* (i)* *stated above and also the condition* (ii) *of Theorem* 2 *always exists.*

Proof. Let us define $u_1 = 1$, $u_2 = -2t$, $u_3 = 2$, $u_4 = -(2t - 1), \ldots, \ldots, u_{2m-1} = m$, $u_{2m} = -(2t - \overline{m - 1})$ for $m \neq t, \ldots, \ldots, u_{2t-1} = t$, $u_{2t} = (2t - \overline{t - 1})$. Then $|u_1|, |u_2|, \ldots, |u_{2t}|$ is a reordering of the natural numbers $1, 2, \ldots, 2t$ since we have $l = u_{2l-1}$ if $1 \leqslant l \leqslant t$, $l = -u_{2(2t+1-l)}$ if $t + 1 < l \leqslant 2t$, and $l = u_{2(l-1)}$ if $l = t + 1$. Thus, $-r \leqslant u_i \leqslant r$ and $|u_j| \neq |u_i|$ unless $j = i$.

Also, $|u_{2m-1} + u_{2m}| = 2t - 2m + 1$ for $m = 1, 2, \ldots, t - 1$, $|u_{2m} + u_{2m+1}| = 2t - 2m$ for $m = 1, 2, \ldots, t - 1$, and $(u_{2t-1} + u_{2t}) = 2t + 1 \equiv -2t \bmod 2r + 1$, whence $|u_{2t-1} + u_{2t}| = 2t$ and so property (i)* holds.

Finally, $\sigma_r = \sum_{i=1}^{2t} u_i = (1 + 2 + \cdots + t) + (t + 1) - [2t + (2t - 1) + \cdots + (t + 3) + (t + 2)] = \frac{1}{2}t(t + 1) + (t + 1) - \frac{1}{2}(t - 1)(3t + 2) = -t^2 + 2t + 2$, and so $-3 + \sigma_r = -(t - 1)^2$. Property (ii) of Theorem 2 holds provided that this is prime to $2r + 1$. Now $2r + 1 = 4t + 1 = 4(t - 1) + 5$. Therefore, $(t - 1, 4t + 1) = 1$ provided that $t - 1 \not\equiv 0 \bmod 5$. This completes the proof of Lemma A.

LEMMA B. *If* $r = 2t + 1$ *and* $t \not\equiv 0 \bmod 3$, $t \not\equiv 3 \bmod 5$, *a sequence* U *of nonzero integers* $u_1, u_2, \ldots, u_r$ *with* $u_1 = 1$, $-r \leqslant u_i \leqslant r$, *and* $|u_j| \neq |u_i|$ *unless* $j = i$ *and which satisfies the condition* (i)* *stated above and also the condition* (ii) *of Theorem* 2 *always exists.*

Proof. Let us define $u_1 = 1$, $u_2 = 2t + 1$, $u_3 = -2$, $u_4 = 2t, \ldots, \ldots, u_{2m-1} = -m$ for $m \neq 1$ and $m \neq t + 1$, $u_{2m} = 2t + 2 - m, \ldots, \ldots, u_{2t-1} = -t$, $u_{2t} = t + 2$, $u_{2t+1} = t + 1$. Then $|u_1|, |u_2|, \ldots, |u_{2t}|$ is a reordering of the natural numbers $1, 2, \ldots, 2t + 1$ since we have $l = -u_{2l-1}$ if $2 \leqslant l \leqslant t$, $l = u_{2(2t+2-l)}$ if $t + 2 \leqslant l \leqslant 2t + 1$, and $l = u_{2l-1}$ if $l = 1$ or $l = t + 1$. Thus, $-r \leqslant u_i \leqslant r$ and $|u_j| \neq |u_i|$ unless $j = i$.

Also, $|u_{2m-1} + u_{2m}| = 2t - 2m + 2$ for $m = 2, 3, \ldots, t$, $|u_{2m} + u_{2m+1}| = 2t - 2m + 1$, for $m = 1, 2, \ldots, t - 1$, and $u_1 + u_2 = 2t + 2 \equiv -(2t + 1) \bmod 2r + 1$, $u_{2t} + u_{2t+1} = 2t + 3 \equiv -2t \bmod 2r + 1$. Thence $|u_1 + u_2| = 2t + 1$ and $|u_{2t} + u_{2t+1}| = 2t$, and so property (i)* holds.

Finally, $\sigma_r = \sum_{i=1}^{2t+1} u_i = 1 - (2 + 3 + \cdots + t) + [(2t + 1) + 2t + \cdots + (t + 2) + (t + 1)] = 1 - \frac{1}{2}(t - 1)(t + 2) + \frac{1}{2}(t + 1)(3t + 2) = t^2 + 2t + 3$, and so $-3 + \sigma_r = t(t + 2)$. Property (ii) of Theorem 2 holds provided that this is prime to $2r + 1$. Now, $2r + 1 = 4t + 3 = 4(t + 2) - 5$. Therefore $(t, 4t + 3) = 1$ provided that $t \not\equiv 0 \bmod 3$ and $(t + 2, 4t + 3) = 1$ provided that $t + 2 \not\equiv 0 \bmod 5$. This completes the proof of Lemma B.

We can now deduce easily the truth of Theorem 3. We observe that $r \equiv 1 \bmod 6 \Leftrightarrow r \equiv 1 \bmod 2$ and $r \equiv 1 \bmod 3 \Leftrightarrow r = 2t + 1$ and $t \equiv 0 \bmod 3$. Also $r \equiv 2 \bmod 5 \Leftrightarrow r = 2t$ and $t \equiv 1 \bmod 5$ or $r = 2t + 1$ and $t \equiv 3 \bmod 5$ (since $2t \equiv 1 \bmod 5 \Leftrightarrow 2t - 1 = 5h \Leftrightarrow 2t - 6 = 5(h - 1) \Leftrightarrow t - 3 = 5k$). From these implications, we see at once that $r \not\equiv 1 \bmod 6$ and $r \not\equiv 2 \bmod 5$ together ensure that the construction of Lemma A will give a sequence U if r is even and that the construction of Lemma B will give a sequence U if r is odd.

Next, we explain the connection between Theorem 2 and the conjecture of A. Kotzig which we mentioned earlier.

A groupoid $(V, \cdot)$ is called a *P-groupoid* if (i) $a \,.\, a = a$ for all $a \in V$; (ii) $a \neq b$ implies $a \neq a \,.\, b$ and $b \neq a \,.\, b$ for all $a, b \in V$; and (iii) $a \,.\, b = c$ implies $c \,.\, b = a$ for all $a, b, c \in V$.

It is easy to see that there exists a one-to-one correspondence between P-groupoids of n elements and decompositions of complete undirected graphs of n vertices into disjoint closed paths.

The correspondence is established by labeling the vertices of the graph with the elements of the P-groupoid and prescribing that the edges $(a \,.\, b)$ and $(b \,.\, c)$ shall belong to the same closed path of the graph if and only if $a \,.\, b = c$, $a \neq b$. An example is given in Fig. 4.

$(\cdot)$	1	2	3	4	5
1	1	5	5	5	3
2	4	2	4	3	4
3	5	4	3	2	1
4	2	3	2	4	2
5	3	1	1	1	5

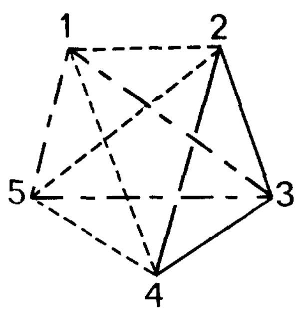

FIGURE 4

Also, in any P-groupoid $(V, \cdot)$ we have that (a) the number of elements is necessarily odd, and (b) the equation $x \,.\, b = c$ is uniquely soluble for x.

The result (a) is deduced by using the correspondence between P-groupoids and graphs just described. Since, for a complete undirected graph which separates into disjoint closed paths the number of edges must clearly be even, any such complete undirected graph must have an odd number of vertices all together.

The result (b) is a consequence of the definition of a groupoid and the fact that $x \,.\, b = c$ implies $c \,.\, b = x$. It follows immediately from the result (b) that the multiplication table of a P-groupoid is a column latin square: that is, each element appears exactly once in each column of the square. If the multiplication table has the further property that each element occurs exactly once in each row also, the multiplication table becomes a latin square and the groupoid is then called a *P-quasigroup.*

The above concepts were introduced in [10]. The following theorem was first given in [9].

THEOREM 4. *A decomposition of the complete undirected graph G_n on n vertices $v_1, v_2, ..., v_n$ into closed paths corresponds to a P-quasigroup $(V, \cdot)$ if and only if, for fixed values of i and k, (v_i, v_j) and (v_j, v_k) are adjacent edges of a closed path for one and only one value of j.*

Proof. If $(V, \cdot)$ is a P-quasigroup, the entry k occurs once and once only in the ith row of the multiplication table of $(V, \cdot)$. Let the column in which this entry occurs be the jth. Then we have $i \,.\, j = k$ and (v_i, v_j), (v_j, v_k) are adjacent edges of a closed path of G_n for this value of j and no other.

One of the problems raised by Kotzig in [10] was that of determining for which odd values of n an Eulerian path exists in G_n which defines a P-quasigroup. Kotzig showed that such an Eulerian path exists when $n = 3$ or 7 but not when $n = 5$, and this led the present author to conjecture that such Eulerian paths exist whenever n is an integer congruent to 3 mod 4. By combining together the results of Theorems 2, 3, and 4 of the present paper, it is quite easy to deduce confirmation of the conjecture for all integers n of the form $n = 4r + 3$ except those values for which $r \equiv 1 \bmod 6$ or $r \equiv 2 \bmod 5$.

Let us suppose that the vertices of G_n are denoted by the symbols $a_0, a_1, ..., a_{2r}, b_0, b_1, ..., b_{2r}, c$. Then the required Eulerian path, if it exists, may be represented by a code word of length $\frac{1}{2}(4r + 3)(4r + 2) + 1$ in which each unordered pair of the symbols occurs exactly once as an adjacent pair, the last symbol of the code word being taken to be the same as the first in order that the path represented should be closed. To see this, it is only necessary to remark that each pair of consecutive symbols represents an edge of the graph joining the vertices represented by those two symbols. By virtue of Theorem 4, the Eulerian path will correspond to a P-quasigroup if and only if no unordered pair of nearly consecutive symbols occurs more than once in the code word. In Theorem 2, sufficient conditions for such a code word to exist are given and, in Theorem 3, it is shown that these conditions are satisfied whenever $n = 4r + 3$ with $r \not\equiv 1 \bmod 6$ and $r \not\equiv 2 \bmod 5$.

Let us end by remarking that code words satisfying the conditions of Theorem 2 can also exist for values of $n \equiv 3 \bmod 4$ which fail to satisfy the special conditions of Theorem 3. In particular, they exist for all $n = 4r + 3$ with $2 \leqslant r \leqslant 7$. We demonstrate this in Fig. 5 by presenting sequences U for each of these values of r which satisfy the conditions of Theorem 2. Theorem 3 provides a special construction for sequences U in which $u_1 = 1$, whereas Theorem 2 permits $u_1 = 1$ or 2. Thus, it remains probable that the author's conjecture is valid universally.

$r = 2; u_1 = 2, u_2 = 1 \pmod 5$

$r = 3; u_1 = 2, u_2 = 3, u_3 = 1 \pmod 7$

$r = 4; u_1 = 2, u_2 = 4, u_3 = 1, u_4 = -3 \pmod 9$

$r = 5; u_1 = 2, u_2 = -5, u_3 = -1, u_4 = 3, u_5 = 4 \pmod{11}$

$r = 6; u_1 = 2, u_2 = -4, u_3 = -5, u_4 = -3, u_5 = 6, u_6 = 1 \pmod{13}$

$r = 7; u_1 = 2, u_2 = -5, u_3 = -6, u_4 = 4, u_5 = 3, u_6 = 7, u_7 = -1 \pmod{15}$

FIGURE 5

Note 1. Since this paper was submitted for publication, V. D. Belousov has pointed out to the author that an inverse property loop which satisfies the identity of Theorem 1 is already a group. To see this, put $h^{-1}k = l$. Then $(gh)l = g(hl)$.

Note 2. In [3], the question of the existence of P-quasigroups which correspond to decompositions of G_n into disjoint Hamiltonian circuits has been discussed.

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