The adjoint homology of a family of 2-step nilradicals

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\textbf{A B S T R A C T}

We consider a family of parabolic subalgebras $p$ of a simple Lie algebra of type $A_n$ and give a full description of the adjoint homology of the nilradicals of $p$ as a module over its Levi factor. All the nilradicals considered are 2-step nilpotent.

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\section*{Introduction}

One of the nicest Lie cohomology computations are those describing the cohomology of the nilradicals of parabolic subalgebras of finite-dimensional semisimple complex Lie algebras by Kostant in the fundamental work \cite{Kostant}.

If $p = g_1 \ltimes n$ is the Levi decomposition of a parabolic $p$ of a semisimple Lie algebra $g$ and $V$ is a representation of the nilradical $n$ which is the restriction of a representation $\pi$ of $p$, then the cohomology $H(n, V)$ has a $g_1$-module structure induced by the adjoint action of $g_1$ on $n$ and the representation $\pi$. A large class of cases arise when the representation $V$ is in turn the restriction of a representation $\pi$ of $g$. This is the case covered by Kostant’s result, where the $g_1$-module structure of $H(n, V)$ is precisely described.

Another different case is that of the adjoint representation of $n$, which is the restriction of the adjoint action of $p$ to $n$ but is not the restriction of any representation of $g$. The description of the
$g_1$-module structure of $H(n, n)$ remains open. Additional motivation to understand the adjoint coho-
mology of Lie algebras comes from deformation theory.

The free 2-step nilpotent Lie algebras and the Heisenberg Lie algebras are two families of 2-step
nilradicals of parabolic subalgebras. The adjoint homology of the first family was described in [2] and
that of the second one is contained in [3]. The two cases share a number of similarities.

In this paper we consider one further case, a family of 2-step nilpotent nilradicals of parabolic
subalgebras inside the simple Lie algebras of type $A_n$, with the aim of understanding the general
result for all 2-step nilradicals. In an intermediate step we need the module structure of the trivial
(co)homology of these nilradicals that we derive from Kostant’s result. We point out that the trivial
homology of this family of nilpotent Lie algebras was recomputed years after Kostant’s paper in [1].
However they described only the vector space structure of it, computing the dimensions of the dif-
ferent cohomology groups. In Section 2.3 we show how to compute these dimensions from Kostant’s
result.

1. Preliminaries

1.1. Parabolic subalgebras and nilradicals

Let $g$ be a complex semisimple Lie algebra of finite dimension, $h$ a Cartan subalgebra and $\Delta =
\Delta(g, h)$ the set of roots. A Borel subalgebra of $g$ is a subalgebra $b = h \oplus t$, where $t = \bigoplus_{\alpha \in \Delta^+} g_{\alpha}$ for
some positive system $\Delta^+$ within $\Delta$. Any subalgebra $p$ of $g$ containing a Borel subalgebra is called a
parabolic subalgebra of $g$.

Let $\Pi$ be the simple system determining $\Delta^+$ and $t$. Since $p \supset h$ and roots spaces are 1-dimensional,
$p$ is of the form
\[
p = h \oplus \bigoplus_{\alpha \in \Gamma} g_{\alpha},
\]
for a subset $\Gamma$ of $\Delta$ containing $\Delta^+$. The extreme cases are $p = b$ ($\Gamma = \Delta^+$) and $p = g$ ($\Gamma = \Delta$). For
$\Pi_0 \subset \Pi$ one defines
\[
\Gamma^+ = \Delta^+ \cup \{\alpha \in \Delta: \alpha \in \text{span}(\Pi - \Pi_0)\}.
\]

**Proposition 1.1.** The parabolic subalgebras $p$ containing $b$ are parameterized by the set of subsets of simple
roots; the one corresponding to a subset $\Pi_0$ is of the form (1.1) with $\Gamma$ as in (1.2).

The parabolic subalgebra $p$ decomposes as a semidirect product $p = g_1 \ltimes n$, of its Levi factor which
is reductive and its nilradical, where
\[
g_1 = h \oplus \bigoplus_{\alpha \in \Gamma^+ \setminus \Gamma} g_{\alpha} \quad \text{and} \quad n = \bigoplus_{\alpha \in \Gamma \setminus \Gamma^+} g_{\alpha}.
\]

Let $\Delta(n) = \{\alpha \in \Delta^+: \alpha \notin \text{span}(\Pi - \Pi_0)\}$ and $\Delta_1^+ = \Delta^+ - \Delta(n)$, then
\[
n = \sum_{\alpha \in \Delta(n)} g_{\alpha} \quad \text{and} \quad g_1 = g_1^- \oplus h \oplus g_1^+.
\]

where
\[
g_1^- = \sum_{\alpha \in \Delta_1^+} g_{-\alpha} \quad \text{and} \quad g_1^+ = \sum_{\alpha \in \Delta_1^+} g_{\alpha}.
\]
1.2. The structure of nilradicals

The nilradical \( n \) of a parabolic \( p = g_1 \ltimes n \) has a root space decomposition as \( g_1 \)-module, where roots are defined from the action of the center of \( g_1 \) (see [6]). Each root space is \( g_1 \)-irreducible and the decomposition of \( n \) as \( g_1 \)-module is multiplicity free.

**Proposition 1.2** (Kostant). Let \( p \) be the parabolic subalgebra associated to \( P_0 \) and \( n \) its nilradical. The nilpotency class of \( n \) is the sum of the coefficients of the simple roots in \( P_0 \) in the highest root of \( g \).

1.3. (Co)homology of nilradicals

Given a semisimple Lie algebra \( g \), a parabolic subalgebra \( p = g_1 \ltimes n \) as in Section 1.1 and an irreducible representation \( V_\lambda \) of \( g \) of highest weight \( \lambda \), the \( g_1 \)-module structure of the cohomology of \( n \) with coefficients \( V_\lambda \), \( H^*(n, V_\lambda) \), where \( V_\lambda \) is an \( n \)-module by restriction, was precisely described by Kostant in [5]. The homology case follows as well from this result since, \( H_*(n, V_\mu) \cong H^*(n, V_\mu^*)^* \) as \( g_1 \)-modules.

Recall that if \( (\pi, V) \) is an irreducible finite-dimensional \( g_1 \)-module with highest weight \( \lambda \) and \( (\pi^*, V^*) \) is the dual module, then the highest weight of \( V^* \) is \(-w_0(\lambda)\), where \( w_0 \) is the unique element in the Weyl group of \( g_1 \) such that \( w_0(\Delta_1^+) = -\Delta_1^+ \).

The adjoint representation of \( n \) is the restriction of a representation of \( g \) only if \( n \) is abelian. In fact, assume \( g = \oplus g_i \), with \( g_i \) simple and that \( \text{ad}_n = \pi \mid n \). Let \( n_i = n \cap g_i \) so that \( n = \oplus n_i \) and let \( z_i \) be in the center of \( n_i \). Since \( \pi(z_i) = \text{ad}_{n_i}(z_i) = 0 \), then \( \pi(g_i) = 0 \) and in particular \( \text{ad}(n_i) = 0 \) for all \( i \). Hence \( n \) is abelian.

For our purpose we only need the description of the trivial homology of \( n \). Kostant’s description of the trivial cohomology of \( n \) and that of the trivial homology of \( n \) are as follows.

**Theorem 1.3** (Kostant). Let \( W \) be the Weyl group of \( g \) and let

\[
W^1 = \{ w \in W : w\Delta^- \cap \Delta^+ \subseteq \Delta(n) \} \tag{1.4}
\]

and for each \( w \in W^1 \) let \( \Delta_w \) be the set of roots in \( \Delta(n) \),

\[
\Delta_w = w\Delta^- \cap \Delta^+ = \{ \alpha_1, \ldots, \alpha_l \}. \tag{1.5}
\]

For each \( \alpha \in \Delta(n) \) let \( X^*_{-\alpha} \) be a weight vector in \( n^* \) of weight \(-\alpha \) and let \( X_\alpha \) be a weight vector in \( n \) of weight \( \alpha \). Let in addition \( \rho = \frac{1}{r} \sum_{\alpha \in \Delta^+} \alpha \) and let \( w_0 \) be the unique element in the Weyl group of \( g_1 \) such that \( w_0(\Delta_1^+) = -\Delta_1^+ \). Then

1. The decomposition of the trivial cohomology of \( n \) as a direct sum of irreducible representations of \( g_1 \) is

\[
H^* (n) = \bigoplus_{w \in W^1} H^*_{w(\rho) - \rho}, \tag{1.6}
\]

where \( H^*_{w(\rho) - \rho} \) is of highest weight \( w(\rho) - \rho \). Moreover, \( H^*_{w(\rho) - \rho} \) is of cohomological degree \( l \) and

\[
X^*_{w} = X^*_{-\alpha_1} \wedge \cdots \wedge X^*_{-\alpha_l} \tag{1.7}
\]

is a highest weight vector of \( H^1(n)_{w(\rho) - \rho} \).
2. The decomposition of the trivial homology of $n$ as a direct sum of irreducible representations of $g_1$ is

$$H_\ast(n) = \bigoplus_{w \in W^1} H_\ast(n)_{-w_0(w(\rho) - \rho)},$$

where $H_\ast(n)_{-w_0(w(\rho) - \rho)}$ is of highest weight $-w_0(w(\rho) - \rho)$. Moreover, $H_\ast(n)_{-w_0(w(\rho) - \rho)}$ is of homological degree 1 and

$$X_w = X_{w_0(\alpha_1)} \wedge \cdots \wedge X_{w_0(\alpha_l)}$$

is a highest weight vector of $H_l(n)_{-w_0(w(\rho) - \rho)}$.

2. The family $A_n(1,2)$ and its trivial homology

From now on let $g = \mathfrak{sl}(n + 1)$ and let $\mathfrak{h}$, $\Delta^+$ and $\Pi$ be as usual. That is

$$\Delta = \{e_i - e_j: 1 \leq i, j \leq n + 1, i \neq j\},$$

$$\Delta^+ = \{e_i - e_j: 1 \leq i < j \leq n + 1\}$$

and

$$\Pi = \{e_i - e_{i+1}: 1 \leq i \leq n\}.$$

The Weyl group $W$ is isomorphic to the permutation group $S_{n+1}$, and its action on $\Delta$ is given by

$$\sigma \cdot (e_i - e_j) = e_{\sigma(i)} - e_{\sigma(j)}.$$

The highest root of $g$ is

$$e_1 - e_{n+1} = (e_1 - e_2) + (e_2 - e_3) + \cdots + (e_n - e_{n+1}).$$

Hence by Proposition 1.2, the nilradical of the parabolic subalgebra of $g$ corresponding to $\Pi_0$ is 2-step nilpotent if and only if $\#\Pi_0 = 2$. Let $A_n(i,j)$, for $1 \leq i < j \leq n$ be the nilradical corresponding to $\Pi_0 = \{e_i - e_{i+1}, e_j - e_{j+1}\}$.

2.1. The family $A_n(1,2)$

Let $n \geq 2$, and let us denote $n = A_n(1,2)$. We have that

$$\Delta(n) = \{e_1 - e_2\} \cup \{e_1 - e_j: 3 \leq j \leq n + 1\} \cup \{e_2 - e_j: 3 \leq j \leq n + 1\},$$

$$\Delta^+_1 = \{e_i - e_j: 3 \leq i < j \leq n + 1\}.$$

Hence if we denote $z = E_{1,2}$, $x_i = E_{2,i+2}$ and $y_i = E_{1,i+2}$ for $i = 1, \ldots, n - 1$, $[z, x_1, \ldots, x_n - 1, y_1, \ldots, y_{n-1}]$ is a basis of $n$, and the only nonzero brackets of basis elements are $[z, x_i] = y_i$, for $i = 1, \ldots, n - 1$. The center of $n$ is $Y = \langle [y_1, \ldots, y_{n-1}] \rangle$ and $n$ is generated as a Lie algebra by $V = Z \oplus X$, where $Z = \langle z \rangle$ and $X = \langle [x_1, \ldots, x_{n-1}] \rangle$. 
The parabolic $p = g_1 \times n$ inside $g$ is

$$
\begin{array}{c|cccc}
g_1 & z & y_1 & y_2 & \ldots & y_{n-1} \\
g_1 & x_1 & x_2 & \ldots & x_{n-1} \\
\end{array}
$$

where the reductive part $g_1 \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{sl}(n - 1)$. We fix as a basis of the center of $g_1$ the set $\{H_1, H_2\}$ where

$$
H_1 = E_{1,1} - \sum_{i=3}^{n+1} \frac{1}{n-1} E_{i,i}, \quad H_2 = E_{2,2} - \sum_{i=3}^{n+1} \frac{1}{n-1} E_{i,i}
$$

and define

$$
h_1 = e_1 - \sum_{i=3}^{n+1} \frac{1}{n-1} e_i \quad \text{and} \quad h_2 = e_2 - \sum_{i=3}^{n+1} \frac{1}{n-1} e_i.
$$

As a basis for the dual of the Cartan subalgebra of $g_1$ we take

$$
\mathcal{H} = \{h_1, h_2, e_3, e_4, \ldots, e_n\}. \quad (2.3)
$$

Here $\{e_3, \ldots, e_n\}$ is a basis of $\langle e_3, \ldots, e_n, e_{n+1} \rangle / \langle e_3 + \cdots + e_{n+1} = 0 \rangle$, the dual of the Cartan subalgebra of the simple part of $g_1$ ($\cong \mathfrak{sl}(n - 1)$). At some point it will be convenient to write weights using the basis $\mathcal{H}$.

The subspaces $Z, X$ and $Y$ of $n$ are $g_1$-irreducible, where highest weight vectors are $z, x_{n-1}$ and $y_{n-1}$ respectively. Clearly $z$ is of weight $e_1 - e_2$, $x_{n-1}$ is of weight $e_2 - e_{n+1}$ and $y_{n-1}$ is of weight $e_1 - e_{n+1}$. On the one hand $e_1 - e_2 = h_1 - h_2$. On the other hand $e_2 - e_{n+1} = h_2 + \sum_{i=3}^{n} \frac{1}{n-1} e_i + \left(\frac{1}{n-1} - 1\right) e_{n+1}$ and $e_1 - e_{n+1} = h_1 + \sum_{i=3}^{n} \frac{1}{n-1} e_i + \left(\frac{1}{n-1} - 1\right) e_{n+1}$. Hence, with respect to $\mathcal{H}$, we have that:

$$
\begin{align*}
z & \text{ is of weight } (1, -1, 0, \ldots, 0), \\
x_{n-1} & \text{ is of weight } (0, 1, 1, \ldots, 1), \\
y_{n-1} & \text{ is of weight } (1, 0, 1, \ldots, 1). \quad (2.4)
\end{align*}
$$

Notice that in particular $X$ and $Y$ are equivalent as representations of $\mathfrak{sl}(n - 1)$ and both are equivalent to the last fundamental representation of $\mathfrak{sl}(n - 1)$.
2.2. The trivial homology of $A_n(1,2)$

From now on let us denote $n = A_n(1,2).$ According to Theorem 1.3 to describe the trivial homology of $n$ we need to compute the set

$$ W^1 = \{ \sigma \in S_{n+1}: \sigma \Delta^- \cap \Delta^+ \subseteq \Delta(n) \}. $$

For each $1 \leq a < b \leq n + 1$ let $\sigma_{a,b}$ and $\mu_{a,b}$ be the permutations defined by:

$$ \sigma_{a,b}^{-1}(1) = a, \quad \sigma_{a,b}^{-1}(2) = b, \quad \sigma_{a,b}^{-1}(i) < \sigma_{a,b}^{-1}(j), \quad 3 \leq i < j \leq n + 1; $$

$$ \mu_{a,b}^{-1}(1) = b, \quad \mu_{a,b}^{-1}(2) = a, \quad \mu_{a,b}^{-1}(i) < \mu_{a,b}^{-1}(j), \quad 3 \leq i < j \leq n + 1. $$

It is immediate to check that

$$ \sigma_{a,b}(i) = \begin{cases} 1, & i = a, \\ 2, & i = b, \\ i + 2, & i < a, \\ i + 1, & a < i < b, \\ i, & i > b, \end{cases} $$

$$ \mu_{a,b}(i) = \begin{cases} 1, & i = b, \\ 2, & i = a, \\ i + 2, & i < a, \\ i + 1, & a < i < b, \\ i, & i > b. \end{cases} \quad (2.5) $$

Proposition 2.1. $W^1 = \{ \sigma_{a,b}, \mu_{a,b}: 1 \leq a < b \leq n + 1 \}$ and

$$ \sigma_{a,b} \Delta^- \cap \Delta^+ = \{ e_1 - e_j: 3 \leq j \leq a + 1 \} \cup \{ e_2 - e_j: 3 \leq j \leq b \}, \quad (2.6) $$

$$ \mu_{a,b} \Delta^- \cap \Delta^+ = \{ e_1 - e_j: 2 \leq j \leq b \} \cup \{ e_2 - e_j: 3 \leq j \leq a + 1 \}. \quad (2.7) $$

In particular

$$ \#(\sigma_{a,b} \Delta^- \cap \Delta^+) = a + b - 3, $$

$$ \#(\mu_{a,b} \Delta^- \cap \Delta^+) = a + b - 2. $$

Proof. A permutation $\sigma$ belongs to $W^1$ if and only if $\sigma^{-1}(\Delta^+_1) \subseteq \Delta^+.$ The positive roots in $\Delta^+_1$ are $e_i - e_j$ with $i < j$ and $i \neq 1,2.$ Hence $\sigma \in W^1$ if and only if $\sigma^{-1}(i) < \sigma^{-1}(j)$ for $i \neq 1,2.$ That is $\sigma = \sigma_{a,b}$ or $\sigma = \mu_{a,b}$ with $a = \sigma^{-1}(1)$ and $b = \sigma^{-1}(2)$ or $a = \mu^{-1}(1)$ and $b = \mu^{-1}(2)$ respectively.

Now let $\alpha \in \Delta(n)$ such that $\sigma_{a,b}^{-1}(\alpha) \in \Delta^-.$ If $\alpha = e_1 - e_j, 1 < j,$ and $\sigma_{a,b}^{-1}(e_1 - e_j)$ is negative, then $a > \sigma_{a,b}^{-1}(j)$ and hence $3 \leq j \leq a + 1.$ If $\alpha = e_2 - e_j, 2 < j,$ and $\sigma_{a,b}^{-1}(e_2 - e_j)$ is negative, then $b > \sigma_{a,b}^{-1}(j)$ and hence $3 \leq j \leq b.$ Therefore (2.6) follows and analogously also (2.7). □

The unique $w_0$ in $W_1,$ the Weyl group of $\mathfrak{g}_1,$ such that $w_0(\Delta^+_1) = -\Delta^+_1$ is

$$ w_0 = \begin{pmatrix} 3 & 4 & \cdots & n & n + 1 \\ n + 1 & n & \cdots & 4 & 3 \end{pmatrix}. $$

For convenience let us denote $\sigma_{a,b} = -w_0(\sigma_{a,b}(\rho) - \rho)$ and $\mu_{a,b} = -w_0(\mu_{a,b}(\rho) - \rho).$
Proposition 2.2. For each $1 \leq a < b \leq n + 1$ we have that

$$\sigma_{a,b} = (a - 1, b - 2, 0, \ldots, 0, -1, \ldots, -1, -2, \ldots, -2),$$

and in addition

$$\bar{\mu}_{a,b} = (b - 1, a - 2, 0, \ldots, 0, -1, \ldots, -1, -2, \ldots, -2)$$

and in addition

$$w_0(\sigma_{a,b} \Delta^- \cap \Delta^+) = \{e_1 - e_j : n + 3 - a \leq j \leq n + 1\}$$

$$\cup \{e_2 - e_j : n + 4 - b \leq j \leq n + 1\}.$$

Proof. The first statements follow by direct computation and the second statements are straightforward. □

From Theorem 1.3 we know that for each $1 \leq a < b \leq n + 1$ there are submodules $H^\sigma(a, b)$ and $H^\mu(a, b)$ which are $g_1$-irreducible of highest weight $\sigma_{a,b}$ and $\bar{\mu}_{a,b}$ and of homological degree $a + b - 3$ and $a + b - 2$ respectively.

Let $0 \leq p \leq 2n - 1$. For each $1 \leq a \leq \lfloor \frac{p+2}{2} \rfloor$ let

$$H^\sigma_{p,a} = H^\sigma(a, p + 3 - a)$$

and for each $1 \leq a \leq \lfloor \frac{p+1}{2} \rfloor$ let

$$H^\mu_{p,a} = H^\mu(a, p + 2 - a).$$

Notice that both $H^\sigma_{p,a}$ and $H^\mu_{p,a}$ are submodules of $H_p(n)$. A full description of the trivial homology of $n$ as a module for $g_1$ is as follows.

Theorem 2.3. The $g_1$-module structure of the trivial homology of $n = A_n(1, 2)$, $H_*(n) = \bigoplus_{p=0}^{2n-1} H_p(n)$, is given for each $p$ by:

- $0 \leq p \leq n - 1$,

$$H_p(n) \simeq \sum_{1 \leq a \leq \lfloor \frac{p+2}{2} \rfloor} H^\sigma_{p,a} \oplus \sum_{1 \leq a \leq \lfloor \frac{p+1}{2} \rfloor} H^\mu_{p,a};$$

- $n \leq p \leq 2n - 1$,

$$H_p(n) \simeq \sum_{p-n+2 \leq a \leq \lfloor \frac{p+2}{2} \rfloor} H^\sigma_{p,a} \oplus \sum_{p-n+1 \leq a \leq \lfloor \frac{p+1}{2} \rfloor} H^\mu_{p,a}.$$
The module $H_{0,1}^\sigma$ is trivial and the modules $H_{p,a}^\sigma$ and $H_{p,a}^\mu$ for $p \geq 1$, are $g_1$-irreducible of highest weights

$$\sigma_{a,p+3-a} = (a - 1, p - (a - 1), \underbrace{\frac{2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0}{n-(p+1)+a-1 \ p-2(a-1) \ a-1}})$$

and

$$\mu_{a,p+2-a} = (p - (a - 1), a - 2, \underbrace{\frac{0, \ldots, 0, -1, \ldots, -1, -2, \ldots, -2}{n-p+a-1 \ p-1-2(a-1) \ a-2}})$$

respectively. Moreover, a highest weight vector for $H_{p,a}^\sigma$ is

$$v_{p,a} = x_{n-1} \wedge \cdots \wedge x_{n-p+a-1} \wedge y_{n-1} \wedge \cdots \wedge y_{n-a+1}$$

and a highest weight vector for $H_{p,a}^\mu$ is

$$w_{p,a} = z \wedge x_{n-1} \wedge \cdots \wedge x_{n+1-a} \wedge y_{n-1} \wedge \cdots \wedge y_{n+a-p}.$$

**Remark 2.4.** With respect to the basis $\mathcal{H}$ (see (2.3)) and a given $p \geq 1$ we have that, for $a \geq 2$,

$$\sigma_{a,p+3-a} = \sigma_{1,p+2} = (0, p, 1, \ldots, 1, 0, \ldots, 0),$$

$$\mu_{a,p+2-a} = \mu_{1,p+1}(p, -1, 1, \ldots, 1, 0, \ldots, 0).$$

and for $a = 1$,

$$\sigma_{a,p+3-a} = \sigma_{1,p+2} = (0, p, 1, \ldots, 1, 0, \ldots, 0),$$

$$\mu_{a,p+2-a} = \mu_{1,p+1}(p, -1, 1, \ldots, 1, 0, \ldots, 0).$$

**Remark 2.5.**

1. $H_0(n) = H_{0,1}^\sigma = \mathbb{C}$.
2. $H_1(n) = H_{1,1}^\sigma \oplus H_{1,1}^\mu = \langle v_{1,1} \rangle \oplus \langle w_{1,1} \rangle = \langle x_{n-1} \rangle \oplus \langle z \rangle$.
3. $H_2(n) = H_{2,1}^\sigma \oplus H_{2,2}^\sigma \oplus H_{2,1}^\mu = \langle v_{2,1} \rangle \oplus \langle v_{2,2} \rangle \oplus \langle w_{2,1} \rangle = \langle x_{n-1} \wedge x_{n-2} \rangle \oplus \langle x_{n-1} \wedge y_{n-1} \rangle \oplus \langle z \wedge y_{n-1} \rangle$.

For $p \geq 3$:

4. $H_{p,1}^\sigma = \langle v_{p,1} \rangle = \langle x_{n-1} \wedge \cdots \wedge x_{n-p} \rangle \subseteq A^p X$.
5. $H_{p,1}^\mu = \langle w_{p,1} \rangle = \langle z \wedge y_{n-1} \wedge \cdots \wedge y_{n-p+1} \rangle \subseteq Z \otimes A^{p-1} Y$.
6. For $a \geq 2$, $H_{p,a}^\sigma = \langle v_{p,a} \rangle \subseteq A^{p-(a-1)} X \otimes A^{a-1} Y$, with $p - (a - 1) \geq 1$.
7. For $a \geq 2$, $H_{p,a}^\mu = \langle w_{p,a} \rangle \subseteq Z \otimes A^{a-1} X \otimes A^{p-a} Y$, with $p - a \geq 1$. 
2.3. Dimensions

As we already mentioned the Betti numbers of these Lie algebras were computed in [1]. From the description given above it is almost straightforward to reobtain this calculation.

The dimension of the irreducible representation of \( s(l(n - 1)) \) with highest weight

\[
\dim H^\sigma_{p,a} = \binom{n-1}{n-a} \binom{n}{n+a-p-2} - \binom{n}{n+1-a} \binom{n-1}{n+a-p-3},
\]

\[
\dim H^\mu_{p,a} = \binom{n-1}{n-a} \binom{n}{n+a-p-1} - \binom{n}{n+1-a} \binom{n-1}{n+a-p-2},
\]

**Theorem 2.6 (Armstrong, Cairns and Jessup).** For \( 0 \leq p \leq 2n - 1 \),

\[
\dim H_p(n) = \binom{n}{\left\lfloor \frac{p+1}{2} \right\rfloor} \binom{n-1}{\left\lfloor \frac{n}{2} \right\rfloor}.
\]

**Proof.** By Poincaré duality and the symmetry of the formula it suffices to consider \( 0 \leq p \leq n \),

\[
\dim H_p(n) =\left[\binom{n-1}{p} + \binom{n-1}{p-1}\right] + \sum_{i=2}^{\left\lfloor \frac{p+1}{2} \right\rfloor} \binom{n-1}{n-i} \binom{n}{n+i-p-2} - \sum_{i=2}^{\left\lfloor \frac{p+1}{2} \right\rfloor} \binom{n-1}{n+1-i} \binom{n-1}{n+i-p-3} + \sum_{i=2}^{\left\lfloor \frac{p+1}{2} \right\rfloor} \binom{n-1}{n-i} \binom{n}{n+i-p-1} - \sum_{i=2}^{\left\lfloor \frac{p+1}{2} \right\rfloor} \binom{n-1}{n+1-i} \left[\binom{n-1}{n+i-p-3} + \binom{n-1}{n+i-p-2}\right] - \sum_{i=\left\lfloor \frac{p+1}{2} \right\rfloor}^{\left\lfloor \frac{p+1}{2} \right\rfloor + 1} \binom{n-1}{n+1-i} \binom{n-1}{n+i-p-3}.
\]

\[
\dim H_p(n) = \binom{n}{p} + \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-1}{n-i-1} \binom{n}{n+i-p-1} + \sum_{i=2}^{\left\lfloor \frac{p+1}{2} \right\rfloor} \binom{n-1}{n-i} \binom{n}{n+i-p-1} - \sum_{i=2}^{\left\lfloor \frac{p+1}{2} \right\rfloor} \binom{n-1}{n+1-i} \left[\binom{n-1}{n+i-p-3} + \binom{n-1}{n+i-p-2}\right] - \sum_{i=\left\lfloor \frac{p+1}{2} \right\rfloor}^{\left\lfloor \frac{p+1}{2} \right\rfloor + 1} \binom{n-1}{n+1-i} \binom{n-1}{n+i-p-3},
\]
3. Special setting for 2-step nilpotent Lie algebras

Let us recall that if \( g \) is a Lie algebra and \( M \) is a \( g \)-module, the homology of \( g \) with coefficients on \( M \) is \( H(g, M) = \ker \partial / \text{im} \partial_1 \), where \( \partial : \Lambda g \otimes M \to \Lambda g \otimes M, \partial = \partial_0 \otimes \text{Id} + \partial_1 \), is defined by

\[
\partial_0 (v_1 \wedge \cdots \wedge v_p) = \sum_{i<j} (-1)^{i+j+1} [v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_p,
\]

\[
\partial_1 (v_1 \wedge \cdots \wedge v_p \otimes m) = \sum_i (-1)^{i+1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p \otimes v_i \cdot m.
\]

Let \( n = V \oplus \mathfrak{z} \) be a 2-step nilpotent Lie algebra, where \( \mathfrak{z} \) is the center of \( n \). Then the short exact sequence of trivial \( n \)-modules (\( V \cong n/\mathfrak{z} \))

\[
\frac{n}{\mathfrak{z}} \to \frac{n}{\mathfrak{z}} \to \frac{n}{\mathfrak{z}}/\mathfrak{z} \to 0.
\]
induces the homology long exact sequence

$$\rightarrow H_{p+1}(n) \otimes V \xrightarrow{\delta_{p+1}} H_p(n) \otimes \mathfrak{z} \rightarrow H_p(n, n) \rightarrow H_p(n) \otimes V \xrightarrow{\delta_p} H_{p-1}(n) \otimes \mathfrak{z} \rightarrow . \quad (3.1)$$

Then $H_p(n, n) \cong \ker \delta_p \oplus \text{coker} \delta_{p+1}$.

**Lemma 3.1.** Let $\delta_p : H_p \otimes V \rightarrow H_{p-1} \otimes \mathfrak{z}$ be the connecting morphism in the above long exact sequence and let $v \in A^p(n) \otimes V$. Then

$$\delta_p([v]) = [\partial_1(v)]. \quad (3.2)$$

**Proof.** Standard. $\square$

**Cancellation property.** We say in general that the long exact sequence of $g_1$-modules (3.1) has the cancellation property if for each of its isotypic subsequences each $\delta_p$ is surjective or it is injective. In our case that sequence is

$$\rightarrow H_{p+1}(n) \otimes (Z \oplus X) \xrightarrow{\delta_{p+1}} H_p(n) \otimes Y \rightarrow H_p(n, n) \rightarrow H_p(n) \otimes (Z \oplus X) \xrightarrow{\delta_p} H_{p-1}(n) \otimes Y \rightarrow .$$

Hence having the cancellation property means that, if for a given type of $g_1$-modules the multiplicity of it on $H_p(n) \otimes (Z \oplus X)$ is bigger than or equal to its multiplicity on $H_{p-1}(n) \otimes Y$, then $\delta_p$ is surjective; and if the multiplicity of a given type on $H_p(n) \otimes (Z \oplus X)$ is smaller than or equal to the its multiplicity on $H_{p-1}(n) \otimes Y$, then $\delta_p$ is injective.

In such a case the description of the $g_1$-module structure of the adjoint homology of $n$, given that of $H(n) \otimes (Z \oplus X)$ and $H(n) \otimes Y$, is straightforward.

In the papers [2,3] two cases of adjoint homology of 2-step nilradicals were computed. Namely the adjoint homology of the free 2-step nilpotent Lie algebras and that of the Heisenberg Lie algebras has been described. In both cases the corresponding long exact sequence (3.1) has the cancellation property. Whether this is or not a general phenomenon is not yet fully clear to us. The computations in this paper add evidence supporting the idea that this is in fact a general phenomenon.

4. The adjoint homology of $A_n(1,2)$

A full description of the structure of $g_1$-module of the adjoint homology of the algebras $n = A_n(1,2)$ follows from the fact that the long exact sequence (3.1) has the cancellation property.

To understand the morphisms $\delta_p$, we study in detail the $g_1$-module structure of the long exact sequence. In particular, in some instances, we construct highest weight vectors for some irreducible submodules and then evaluate the connecting morphisms $\delta_p$.

We find it convenient to represent the irreducible representations of $\mathfrak{s}(n-1)$ using Young diagrams. Such a diagram is an arrangement of boxes representing a highest weight. For instance, for $0 \leq l \leq k \leq n-2$, we will denote by $D_{k,l}$ the two columns diagram with $k$ and $l$ boxes respectively. These diagrams represents irreducible representations with highest weights of the form $(2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$. More precisely

$$D_{k,l} = \begin{array}{c}
\vdots \\
(2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0) \\
\hline
l \\
k-l \\
\hline
n-2-k
\end{array}.$$ 

Diagrams $D_k$ and $D_{k,l,m}$ are defined analogously and have one and three columns respectively.
An irreducible $g_1$-module $W$ will be identified by a triple $(a, b, D)$, where the pair $(a, b)$ determines the action of the center of $g_1$ and $D$ is a Young diagram determining the action of the simple part of $g_1$. We will write

$$ W \simeq (a, b, D). $$

For instance (see Remark 2.4 and (2.4)), for $a \geq 2$,

$$ H^\sigma_{p,a} \simeq (a - 1, p - (a - 1), D_{n-a,n-p+a-2}), \quad (4.1) $$

$$ H^\mu_{p,a} \simeq (p - (a - 1), a - 2, D_{n-a,n-p+a-1}), \quad (4.2) $$

for $a = 1$,

$$ H^\sigma_{p,1} \simeq (0, p, D_{n-(p+1)}), \quad (4.3) $$

$$ H^\mu_{p,1} \simeq (p, -1, D_{n-p}), \quad (4.4) $$

and

$$ Z \simeq (1, -1, D_0), \quad X \simeq (0, 1, D_{n-2}), \quad Y \simeq (1, 0, D_{n-2}). $$

The decomposition, up to isomorphism, of the tensor product of two irreducible representations of $sl(n-1)$ is given by the well known Littlewood–Richardson rule.

### 4.1. Highest weight vectors of the tensor product

In this section we show how to construct highest weight vectors for all the irreducible summands of the tensor product of an arbitrary irreducible representation of $sl(n-1)$ and the last fundamental representation of $sl(n-1)$.

Let us begin by recalling the Littlewood–Richardson rule in terms of Young diagrams (see for instance [4], Appendix A).

Let $V_D$ be an irreducible representation of $sl(n-1)$ represented by the Young diagram $D$ and let $V_{n-2}$ be the last fundamental representation of $sl(n-1)$, which is represented by the Young diagram $D_{n-2}$ (one column with $n-2$ boxes).

**Littlewood–Richardson rule.** The tensor product $V_D \otimes V_{n-2}$ decomposes as the direct sum of the irreducible representations represented by the Young diagram obtained from $D$ by adding one box in each row, plus all Young diagrams that can be constructed from $D$ by removing one box. Notice that there are at least 2 and at most $n-1$ summands in this decomposition.

If $\lambda = (\lambda_1, \ldots, \lambda_{n-2})$ is the highest weight of $V_D$, that is, $\lambda_i$ is the number of boxes in the $i$-row of $D$, then $V_D \otimes V_{n-2}$ has always one component of weight $(\lambda_1 + 1, \ldots, \lambda_{n-2} + 1)$, that we denote by $V_D \otimes V_{n-2}(n-1)$. In addition for each $1 \leq s \leq n-2$ such that $\lambda_s > \lambda_{s+1}$ (let $\lambda_{n-1} = 0$) it has a component of weight $(\lambda_1, \lambda_2, \ldots, \lambda_{s-1}, 1, \ldots, \lambda_{n-2})$ that we denote by $V_D \otimes V_{n-2}(s)$.

**Example.** For $n = 5$, the Littlewood–Richardson rule gives the following for these particular tensor product of representations of $sl(5)$.
The representation $V_{n-2}$. Let $u_{n-1}$ be a highest weight vector of $V_{n-2}$ and let, for $i = n-1, \ldots, 2$, $u_{i-1}$ be defined by $u_{i-1} = -E_{i-1,i}u_i$. Then $\{u_{n-1}, u_{n-2}, \ldots, u_1\}$ is a basis of weight vectors of $V_{n-2}$; $u_{n-1}$ is of weight $(1, \ldots, 1)$ and, for $1 \leq i \leq n-2$, $u_i$ is of weight $(-1, \ldots, -1, 0)$ where $-1$ is in the $i$-th position. Recall that $E_{j,j+1}u_j = -u_{j+1}$ and $E_{j,j+1}u_i = 0$ if $i \neq j$.

In what follows we construct a highest weight vector for each of the irreducible components $V_D \otimes V_{n-2}(s)$ of $V_D \otimes V_{n-2}$. The construction can be derived from Propositions 2.2 and 2.4 and the proof of Proposition 2.2 in [2]. Instead and for making it easier for the reader we state the result for this case and write the proof.

Let us denote by $[[n-1]]$ the set of integers from $1$ to $n-1$ and for a subset $J \subseteq [[n-1]]$ let us write

$$J_{\geq c} = J \cap \{c, \ldots, n-1\} \quad \text{and} \quad J_{\leq c} = J \cap \{1, \ldots, c\}.$$  

**Definition 4.1.** Given a nonempty set $J = \{j_1, \ldots, j_k\} \subset [[n-1]]$, such that $j_1 < \cdots < j_k$, let $U_J \in \mathcal{U}(\mathfrak{sl}(n-1))$ be defined by

$$U_J = \begin{cases} 1, & k = 1, \\ E_{j_k,j_{k-1}}E_{j_{k-1},j_{k-2}} \cdots E_{j_3,j_2}E_{j_2,j_1}, & k > 1. \end{cases}$$

The commutator $[E_{c,c+1}, U_J]$ in $\mathcal{U}(\mathfrak{sl}(n-1))$ can be computed by using the commutation relations

$$[E_{a,b}, E_{c,d}] = \begin{cases} E_{a,a} - E_{b,b}, & \text{if } a = d, \ b = c, \\ E_{a,d}, & \text{if } a \neq d, \ b = c, \\ -E_{b,c}, & \text{if } a = d, \ b \neq c, \\ 0, & \text{if } a \neq d, \ b \neq c. \end{cases}$$

It is not difficult to see that $[E_{c,c+1}, U_J]$ is given by Table 1 according to whether $c$ and $c+1$ are or not in $J$.

**Theorem 4.2.** Let $v$ be a highest weight vector of $V_D$ and fix $s = n - 1$ or $1 \leq s \leq n - 2$ such that $\lambda_s > \lambda_{s+1}$. Let $A(s) = \{J \subset [[n-1]]: \min(J) = s\}$ and for $s \leq i \leq n-2$ let $\sigma_i = \lambda_s - \lambda_i - (s-i) - 1$ and define $\sigma_J = \prod_{i \in J} \sigma_i$. Then

$$w = \sum_{J \in A(s)} \frac{1}{\sigma_J} U_J v \otimes u_{\max(J)}$$

is a highest weight vector of $V_D \otimes V_{n-2}(s)$. 

**Table 1**

| $c \in J$ | $J_{\geq c+2} \neq \emptyset$ | $U_{J_{\geq c+1}}(E_{c,c} - E_{c+1,c+1}) U_{J_{\leq c}}$ |
| $c + 1 \in J$ | $J_{\geq c+2} = \emptyset$ | $(E_{c,c} - E_{c+1,c+1}) U_{J_{\leq c}}$ |
| $c \in J$ | $J_{\geq c+2} \neq \emptyset$ | $-U_{J_{\geq c+1}J_{\geq c+1}} U_{J_{\leq c}}$ |
| $c + 1 \notin J$ | $J_{\geq c+2} = \emptyset$ | 0 |
| $c \notin J$ | $J_{\leq c-1} \neq \emptyset$ | $U_{J_{\geq c+1}} U_{J_{\leq c-1}}$ |
| $c + 1 \in J$ | $J_{\leq c-1} = \emptyset$ | 0 |
| $c, c+1 \notin J$ | | 0 |
Proof. It is clear that $w$ is a weight vector in $V_D \otimes V_{n-2}(s)$. The summand of $w$ corresponding to the subset $J = \{s\}$ is equal to $-v \otimes u_s$, that is linearly independent with the rest, hence $w \neq 0$. To prove that $w$ is a highest weight vector it suffices to show that $E_{c,c+1}w = 0$ for $1 \leq c \leq n - 2$. We consider separately the cases where $c \leq s - 1$, $c = s$ and $c \geq s + 1$.

For $J \in A(s)$ write $S_J = \frac{1}{\sigma_J} U_J v \otimes u_{\max(J)}$ and let $A^{(d)}_{(s)} = \{J \in A(s), \max(J) = d\}$.

(i) Let $c \leq s - 1$. It follows directly from the commutation relations (6) and (7) in Table 1 and the fact that $E_{c,c+1}u_{\max(J)} = 0$, that $E_{c,c+1}S_J = 0$ for all $J \in A(s)$ and hence $E_{c,c+1}w = 0$.

(ii) Let $c = s$. If $s + 1 \notin J$, let us denote $J^{s+1} = J \cup \{s + 1\}$. We have that

$$A(s) = \{\{s\}, \{s, s + 1\}\} \cup \bigcup_{d > s + 1} \{J, J^{s+1} : J \in A^{(d)}_{(s)}, s + 1 \notin J\},$$

where the union is disjoint. On the one hand

$$E_{s,s+1}(S_{\{s\}} + S_{\{s,s+1\}}) = -E_{s,s+1}(v \otimes u_s) + \frac{-1}{\lambda_s - \lambda_{s+1}} E_{s,s+1}(E_{s+1,s} v \otimes u_{s+1})$$

$$= v \otimes u_{s+1} + \frac{-1}{\lambda_s - \lambda_{s+1}}(\lambda_s - \lambda_{s+1})v \otimes u_{s+1}$$

$$= 0.$$

On the other hand, if $d = \max(J) > s + 1$ and $s + 1 \notin J$, then by the commutation relations (3) and (1) in Table 1

$$E_{s,s+1}(S_J + S_{J^{s+1}}) = \frac{1}{\sigma_J} E_{s,s+1}(U_J v \otimes u_d) + \frac{1}{\sigma_J \sigma_{s+1}} E_{s,s+1}(U_{J^{s+1}} v \otimes u_d)$$

$$= -\frac{1}{\sigma_J} U_{\{s+1\} \cup J \geq s+1} v \otimes u_d + \frac{U_{J^{s+1}} (\lambda_s - \lambda_{s+1})v}{\sigma_J (\lambda_s - \lambda_{s+1})} \otimes u_d$$

$$= 0.$$

(iii) Let $c \geq s + 1$. For each $J \in A(s)$ such that $c, c + 1 \notin J$ let us denote $J^c = J \cup \{c\}$, $J^{c+1} = J \cup \{c + 1\}$ and $J^{c,c+1} = J \cup \{c, c + 1\}$. If $d = \max(J)$, then

$$A_{(s)} = \bigcup_{d} \{J, J^c, J^{c+1}, J^{c,c+1} : J \in A^{(d)}_{(s)} \text{ and } c, c + 1 \notin J\}.$$
The long exact sequence and the cancellation property

Since

Therefore

\[ E_{c,c+1}(S_{j^c} + S_{j^{c+1}} + S_{j^{c,c+1}}) = 0 \]

since \(-c_{c+1} + c_{c} + 1 + \lambda_{c} - \lambda_{c+1} = 0\).

If \(d < c\), by (7), (4), (5) and (2) in Table 1, we have that

\[ E_{c,c+1}(S_{j}) = 0, \]

\[ E_{c,c+1}(S_{j^c}) = \frac{1}{\sigma_j \sigma_c} U_{j^c} v \otimes (-u_{c+1}), \]

\[ E_{c,c+1}(S_{j^{c+1}}) = \frac{1}{\sigma_j \sigma_{c+1}} U_{j^{c+1}} U_{[c] \cup j^{c+1} \leq c} v \otimes u_{c+1}, \]

\[ E_{c,c+1}(S_{j^{c,c+1}}) = \frac{1}{\sigma_j \sigma_{c} \sigma_{c+1}} (E_{c,c} - E_{c+1,c+1}) U_{j^{c,c+1} \leq c} v \otimes u_{c+1} = \frac{1}{\sigma_j \sigma_{c+1}} U_{j^{c,c+1} \leq c} (1 + E_{c,c} - E_{c+1,c+1}) v \otimes u_{c+1} = \frac{1 + \lambda_{c} - \lambda_{c+1}}{\sigma_j \sigma_{c+1}} U_{j^{c,c+1} \leq c} v \otimes u_{c+1}. \]

Therefore

\[ E_{c,c+1}(S_{j^c} + S_{j^{c+1}} + S_{j^{c,c+1}}) = 0 \]

since \(-c_{c+1} + c_{c} + 1 + \lambda_{c} - \lambda_{c+1} = 0\). \(\Box\)

4.2. The long exact sequence and the cancellation property

The long exact sequence (3.1) is in our case, where \(n = A_n(1, 2)\),

\[ \rightarrow H_{p+1}(n) \otimes (Z \oplus X) \xrightarrow{\delta_{p+1}} H_{p}(n) \otimes Y \rightarrow H_{p}(n, n) \rightarrow H_{p}(n) \otimes (Z \oplus X) \xrightarrow{\delta_{p}} H_{p-1}(n) \otimes Y \rightarrow \] (4.5)
where $\delta_{2n} : 0 \to H_{2n-1}(n) \otimes Y$ and $\delta_0 : H_0(n) \otimes (Z \oplus X) \to 0$, are both the zero morphism. Analyzing carefully the morphisms $\delta_p$ we shall prove that the exact sequence above has the cancellation property.

It turns out that for almost all types of irreducible $g_1$-modules, $H_p(n) \otimes (Z \oplus X)$ and $H_{p-1}(n) \otimes Y$, are multiplicity free. However, for $0 < p < 2n - 1$, $H_p^{\mu} \otimes X$ has a submodule isomorphic to $H_{p, \frac{p+2}{2}}^{\sigma} \otimes Z$. In fact, on the one hand

$$H_{p, \frac{p+2}{2}}^{\sigma} \otimes Z \simeq \left[ \frac{p+2}{2}, \frac{p-1}{2} \right] D_{n-[\frac{p+2}{2}], n-[\frac{p+3}{2}]}$$

and on the other hand, if $p$ is even, $H_p^{\mu} = H_{p, \frac{p+2}{2}}^{\mu}$ and

$$H_{p, \frac{p+2}{2}}^{\mu} \otimes X \simeq \left( \frac{p+2}{2}, \frac{p-2}{2}, D_{n-2, n-[\frac{p+4}{2}]} \right) \oplus \left( \frac{p+2}{2}, \frac{p-2}{2}, D_{n-[\frac{p+3}{2}], n-[\frac{p+4}{2}]} \right) \oplus \left( \frac{p+2}{2}, \frac{p-2}{2}, D_{n-[\frac{p+4}{2}], n-[\frac{p+5}{2}]} \right),$$

while if $p$ is odd, $H_p^{\mu} = H_{p, \frac{p+1}{2}}^{\mu}$ and

$$H_{p, \frac{p+1}{2}}^{\mu} \otimes X \simeq \left( \frac{p+1}{2}, \frac{p-1}{2}, D_{n-2, n-[\frac{p+1}{2}]} \right) \oplus \left( \frac{p+1}{2}, \frac{p-1}{2}, D_{n-[\frac{p}{2}], n-[\frac{p+2}{2}]} \right).$$

In both cases, the last summand is of the same type as that of $H_{p, \frac{p+2}{2}}^{\sigma} \otimes Z$. This is in fact the only exception for being $H_p(n) \otimes (Z \oplus X)$ multiplicity free. That is, for all types, but that of $H_{p, \frac{p+2}{2}}^{\sigma} \otimes Z$, the corresponding isotypic component of $H_p(n) \otimes (Z \oplus X)$ is multiplicity free as we prove now.

The following table describes the action of the center of $g_1$ on all irreducible submodules of $H_p(n) \otimes (Z \oplus X)$ and $H_{p-1}(n) \otimes Y$. This follows directly from Theorem 2.3 and (2,4).

| $w = \sigma$ | $H_w^{\sigma} \otimes X$ | $H_w^{\sigma} \otimes Z$ | $H_w^{\sigma} \otimes Y$ | (4.6) |
|--------------|--------------------------|--------------------------|--------------------------|
| $w = \mu$    | $(a-1, p+2-a)$           | $(a, p-a)$               | $(a', p-a')$             |
| $(p+1-a, a-1)$| $(p+2-a, a-3)$           | $(p+1-a', a'-2)$         |

**Proposition 4.3.** For $0 \leq p \leq 2n - 1$, all isotypic components of $H_p(n) \otimes (Z \oplus X)$, of types different from that of $H_{p, \frac{p+2}{2}}^{\sigma} \otimes Z$, are multiplicity free. In addition, the isotypic component of $H_p(n) \otimes (Z \oplus X)$ containing $H_{p, \frac{p+2}{2}}^{\sigma} \otimes Z$ is of multiplicity 2. Also $H_p(n) \otimes Y$ is multiplicity free.

**Proof.** According to the table above, the center of $g_1$ acts by different weights on the family of modules $\{H_p(a) \otimes X, H_p(a) \otimes Z, H_p(a) \otimes X, H_p(a) \otimes Z\}$ for all values of $a$, with one possible exception. In fact $(p+1-a, a-1) = (a', p-a')$ is satisfied with $a = [\frac{p+1}{2}]$ and $a' = [\frac{p+2}{2}]$. This is the case we analyzed before.

The irreducible components of $H_p(a) \otimes X$ and that of $H_p(a) \otimes X$ are non-isomorphic as $\mathfrak{sl}(n-1)$-modules (recall the Littlewood–Richardson rule) and $H_p^{\mu} \otimes Z$ and $H_p^{\mu} \otimes Z$ are irreducible. Therefore,
for all types but the one already pointed out, the isotypical components of \( H_p(n) \otimes (Z \oplus X) \) are multiplicity free.

By analogous arguments \( H_p(n) \otimes Y \) is also multiplicity free. □

Recall that, for all \( 0 \leq p \leq 2n - 1 \),

\[
H_p(n) = \sum_{a=p-n+2}^{[\frac{p+2}{2}]} H^\sigma_{p,a} \oplus \sum_{a=p-n+1}^{[\frac{p+1}{2}]} H^\mu_{p,a},
\]

where we assume that \( H^\sigma_{p,a} = 0 \) and \( H^\mu_{p,a} = 0 \), if \( a \leq 0 \). The following proposition describes completely the morphisms \( \delta_p \). We already noticed that \( \delta_{2n} = 0 \) and \( \delta_0 = 0 \).

**Lemma 4.4.** The linear map \( \partial_0 : \Lambda^p n \longrightarrow \Lambda^{p-1} n \) satisfies:

(i) \( \text{Im } \partial_0 \subseteq \Lambda^k X \otimes \Lambda^l Y \), with \( l > 0 \).

(ii) For \( l \geq 1 \), \( x_{i_1} \wedge \cdots \wedge x_{i_l} \wedge \cdots \wedge x_{i_m} \wedge y_{i_1} \wedge \cdots \wedge y_{i_l} \notin \text{Im } \partial_0 \).

**Proof.** Direct from the definition of \( \partial_0 \) and the bracket multiplication of \( n \). □

**Proposition 4.5.** The action of the \( g_1 \)-morphism \( \delta_p \), for \( 0 \leq p \leq 2n - 1 \), is given on each irreducible submodule of \( H_p(n) \otimes (Z \oplus X) \) by the following:

(i) \( \delta_p(H^\sigma_{p,a} \otimes X) = 0 \), for all \( p-n+2 \leq a \leq [\frac{p+2}{2}] \);

(ii) \( \delta_p(H^\mu_{p,a} \otimes X) = 0 \), for all \( p-n+1 \leq a \leq [\frac{p}{2}] \);

(iii) If \( p \) is odd, \( 0 \neq \delta_p(H^\mu_{p,\frac{p+1}{2}} \otimes X(s)) \subseteq H^\sigma_{p-1,\frac{p+1}{2}} \otimes Y(s) \), for all suitable \( 1 \leq s \leq n-1 \);

(iv) \( 0 \neq \delta_p(H^\sigma_{p,a} \otimes Z) \subseteq H^\sigma_{p-1,a} \otimes Y \), for all \( p-n+2 \leq a \leq \frac{p+1}{2} \);

(v) If \( p \) is even, \( \delta_p(H^\sigma_{p,\frac{p+2}{2}} \otimes Z) = 0 \);

(vi) \( 0 \neq \delta_p(H^\mu_{p,a} \otimes Z) \subseteq H^\mu_{p-1,a-1} \otimes Y \), for all \( p-n+1 \leq a \leq \frac{p+1}{2} \), \( a \neq 1 \);

(vii) \( \delta_p(H^\mu_{p,1} \otimes Z) = 0 \).

**Remark 4.6.**

1. Notice that if \( p \) is even (iii) is vacuous, and since \( [\frac{p}{2}] = [\frac{p+1}{2}] \) (ii) implies that \( \delta_p(H^\mu_{p,a} \otimes X) = 0 \) for all \( 1 \leq a \leq [\frac{p+1}{2}] \).

2. If \( p \) is odd (v) is vacuous, and since \( [\frac{p+1}{2}] = [\frac{p+2}{2}] \) (iv) implies that \( \delta_p(H^\sigma_{p,a} \otimes Z) \neq 0 \) for all \( 1 \leq a \leq [\frac{p+2}{2}] \).

**Proof of Proposition 4.5.** We have that \( \delta_p(H^\sigma_{p,a} \otimes X) \subseteq H_{p-1}(n) \otimes Y \) and \( \delta_p(H^\mu_{p,a} \otimes X) \subseteq H_{p-1}(n) \otimes Y \).

(i) The center of \( g_1 \) acts by different weights on \( H^\sigma_{p,a} \otimes X \) and on all the irreducible submodules of \( H_{p-1}(n) \otimes Y \).

(ii) The center of \( g_1 \) acts on \( H^\mu_{p,a} \otimes X \) by \( (p + 1 - a, a - 1) \) and acts by a different weight on all the irreducible submodules of \( H_{p-1}(n) \otimes Y \), except on \( H^\sigma_{p-1,a'} \otimes Y \) with \( a' = p + 1 - a \). Since \( a \leq [\frac{p+1}{2}] \), then \( p + 1 - a > a - 1 \), \( a' > p - a' \) and \( a' \geq [\frac{p+2}{2}] \). But since \( a' \leq [\frac{p+1}{2}] \) it follows that \( p \) must be odd and \( a' = [\frac{p+1}{2}] = a \).
(iii) For each (suitable) s, $\delta_p(H^{\mu}_{p, \frac{p+1}{2}} \otimes X(s)) \subseteq H^{\sigma}_{p-1, \frac{p+1}{2}} \otimes Y(s)$.

By Theorem 2.3,

$$v = z \wedge x_{n-1} \wedge \cdots \wedge x_{n+1-\frac{p+1}{2}} \wedge y_{n-1} \wedge \cdots \wedge y_{n+1-\frac{p+1}{2}}$$

is a highest weight vector of $H^{\mu}_{p, \frac{p+1}{2}}$, and by Theorem 4.2,

$$w_s = -v \otimes x_s + \sum_{j \in \Lambda(s)} \frac{1}{\sigma_j} U_j v \otimes x_{\max(j)}$$

is a highest weight vector of $H^{\mu}_{p, \frac{p+1}{2}} \otimes X(s)$. Now

$$\partial_1(w_s) = \partial_1(-v \otimes x_s) + \partial_1\left( \sum_{j \in \Lambda(s)} \frac{1}{\sigma_j} U_j v \otimes x_{\max(j)} \right)$$

$$= -x_{n-1} \wedge \cdots \wedge x_{n+1-\frac{p+1}{2}} \wedge y_{n-1} \wedge \cdots \wedge y_{n+1-\frac{p+1}{2}} \otimes y_s$$

$$+ \partial_1\left( \sum_{j \in \Lambda(s)} \frac{1}{\sigma_j} U_j v \otimes x_{\max(j)} \right)$$

$$= -x_{n-1} \wedge \cdots \wedge x_{n+1-\frac{p+1}{2}} \wedge y_{n-1} \wedge \cdots \wedge y_{n+1-\frac{p+1}{2}} \otimes y_s$$

$$+ \sum_{j \in \Lambda(s)} \frac{1}{\sigma_j} U_j (x_{n-1} \wedge \cdots \wedge x_{n+1-\frac{p+1}{2}} \wedge y_{n-1} \wedge \cdots \wedge y_{n+1-\frac{p+1}{2}}) \otimes y_{\max(j)}$$

$$= \sum_{j \in \Lambda(s)} \frac{1}{\sigma_j} U_j (x_{n-1} \wedge \cdots \wedge x_{n+1-\frac{p+1}{2}} \wedge y_{n-1} \wedge \cdots \wedge y_{n+1-\frac{p+1}{2}}) \otimes y_{\max(j)}$$

which is a highest weight vector of $H^{\sigma}_{p-1, \frac{p+1}{2}} \otimes Y(s)$.

(iv) The center of $g_1$ acts on $H^{\sigma}_{p,a} \otimes Z$ by $(a, p-a)$ and acts by different weights on all submodules $H^{\sigma}_{p-1,a'} \otimes Y$ and $H^{\mu}_{p-1, a} \otimes Y$. Hence $\delta_p(H^{\sigma}_{p,a} \otimes Z) \subseteq H^{\sigma}_{p-1,a} \otimes Y$.

To see that $\delta_p(H^{\sigma}_{p,a} \otimes Z) \neq 0$, we evaluate $\delta_p$ on a highest weight vector $v_a$ of $H^{\sigma}_{p,a} \otimes Z$. For $a = 1$, let $v_1 = x_{n-1} \wedge \cdots \wedge x_{n-p} \otimes z$ and for $2 \leq a \leq \lceil \frac{p+1}{2} \rceil$ let $v_a = x_{n-1} \wedge \cdots \wedge x_{n+a-p} \wedge y_{n-1} \wedge \cdots \wedge y_{n+a-1} \otimes z$. Then

- $\partial_1(v_1) = \sum_{i=0}^{p} (-1)^i x_{n-i} \wedge \cdots \wedge \hat{x}_{n-i} \wedge \cdots \wedge x_{n-p} \otimes y_{n-i}$;
- $\partial_1(v_a) = \sum_{i=0}^{p-a} (-1)^i x_{n-i} \wedge \cdots \wedge x_{n-i} \wedge \cdots \wedge x_{n+a-p} \wedge y_{n-i} \wedge \cdots \wedge y_{n+a-1} \otimes z$.

The conclusion follows from Lemma 4.4.

(v) If $p$ is even, the center of $g_1$ acts on $H^{\mu}_{p, \frac{p+2}{2}} \otimes Z$ by $(\frac{p+2}{2}, p - \frac{p+2}{2})$ and acts by different weights on all submodules $H^{\sigma}_{p-1,a'} \otimes Y$ and $H^{\mu}_{p-1, a} \otimes Y$.

(vi) The center of $g_1$ acts on $H^{\mu}_{p,a} \otimes Z$ by $(p+2-a, a-3)$ and acts by different weights on all submodules $H^{\sigma}_{p-1,a'} \otimes Y$ and $H^{\mu}_{p-1, a} \otimes Y$ but on $H^{\mu}_{p-1,a-1} \otimes Y$. Hence $\delta_p(H^{\sigma}_{p,a} \otimes Z) \subseteq H^{\mu}_{p-1,a-1} \otimes Y$.

To see that $\delta_p(H^{\mu}_{p,a} \otimes Z) \neq 0$, we evaluate $\delta_p$ on a highest weight vector $v_a$ of $H^{\mu}_{p,a} \otimes Z$. For $2 \leq a \leq \lceil \frac{p+1}{2} \rceil$ let $v_a = z \wedge x_{n-1} \wedge \cdots \wedge x_{n+a-p} \wedge y_{n-1} \wedge \cdots \wedge y_{n+a-1} \otimes z$. Then
\[ \partial_1(v_g) = \sum_{i=1}^{a-1} (-1)^{i+1} Z \times X_{n-i} \times \cdots \times X_{n-1} \times Y_{n-1} \times \cdots \times Y_{n+a-p} \otimes Y_{n-i}. \]

The conclusion follows from Lemma 4.4.

(vii) The center of \( g_1 \) acts on \( H^\mu \times Z \) by \((p+1, -2)\) and acts by different weights on all submodules \( H^\sigma_{p-1, a'} \otimes Y \) and \( H^\mu_{p-1, a'} \otimes Y \). \( \square \)

**Theorem 4.7.** The long exact sequence of \( g_1 \)-modules (4.5) has the cancellation property.

**Proof.** Since \( H_{p-1}(n) \otimes Y \) is multiplicity free, it suffice to show that whenever \( \delta_p(W) = 0 \), for an irreducible submodule \( W \), then \( H_{p-1}(n) \otimes Y \) does not contain any submodule isomorphic to \( W \).

This is in fact the case as we already showed in the proof of Proposition 4.5 items (i), (ii), (iv) and (vii). \( \square \)

4.3. The adjoint homology of \( A_n(1, 2) \)

By virtue of Theorem 4.7 a full description of the structure of \( g_1 \)-module of \( H_p(n, n) = \ker \delta_p \oplus \text{coker} \delta_{p+1} \), for all \( p = 0, 1, \ldots, 2n-1 \), reduces to understand which irreducible \( g_1 \)-submodules occur simultaneously in \( H_{p-1}(n) \otimes (Z \otimes X) \) and \( H_{p-1}(n) \otimes Y \) and which ones do not. Those appearing in both must be cancel out, what remains on the first one is \( \ker \delta_p \), while what remains on the second one is \( \text{coker} \delta_p \). This is what we essentially did in Proposition 4.5.

It follows that if \( p \) is even,

\[ \ker \delta_p = (H_p(n) \otimes X) \oplus (H^\sigma_{p, \frac{p+1}{2}} \otimes Z) \oplus (H^\mu_{p, 1} \otimes Z). \]

If \( p \) is odd, each of the submodules \( H^\mu_{p, \frac{p+1}{2}} \otimes X \) and \( H^\sigma_{p, \frac{p+1}{2}} \otimes Z \) do not intersect \( \ker \delta_p \). However, the irreducible submodule of \( H^\mu_{p, \frac{p+1}{2}} \otimes X \) which is isomorphic to \( H^\sigma_{p, \frac{p+1}{2}} \otimes Z \) and this one itself are both mapped onto the same irreducible submodule of \( H_{p-1}(n) \otimes Y \) (see Proposition 4.5 and recall that \( H_{p-1}(n) \otimes Y \) is multiplicity free). More precisely, both are mapped inside \( H^\sigma_{p-1, \frac{p+1}{2}} \otimes Y \). Hence \( \ker \delta_p \) contains an irreducible submodule \( W \) isomorphic to \( H^\sigma_{p, \frac{p+1}{2}} \otimes Z \) which lies inside the direct sum \( (H^\sigma_{p, \frac{p+1}{2}} \otimes Z) \oplus (H^\mu_{p, \frac{p+1}{2}} \otimes X) \). Finally we get that, if \( p \) is odd,

\[ \ker \delta_p \cong \frac{H_p(n) \otimes X}{H^\mu_{p, \frac{p+1}{2}} \otimes X} \oplus W \oplus (H^\mu_{p, 1} \otimes Z) \]
\[ \cong \frac{H_p(n) \otimes X}{H^\mu_{p, \frac{p+1}{2}} \otimes X} \oplus (H^\sigma_{p, \frac{p+1}{2}} \otimes Z) \oplus (H^\mu_{p, 1} \otimes Z). \]

Putting all together we have that, for all \( 1 \leq p \leq 2n-1 \),

\[ \ker \delta_p \cong \frac{H_p(n) \otimes X}{H^\mu_{p, \frac{p+1}{2}} \otimes X} \oplus H^\sigma_{p, \frac{p+1}{2}} \otimes Z \oplus H^\mu_{p, 1} \otimes Z \oplus H^\sigma_{p, \frac{p+1}{2}} \otimes Z, \]  

assuming that \( H^\sigma_{p, c} = 0 \) if \( c \) is not an integer.

We shall now describe the structure of \( \text{coker} \delta_{p+1} \), for \( 1 \leq p \leq 2n-1 \). First observe that \( H^\sigma_{p, a} \otimes Y \) and \( H^\mu_{p, a-1} \otimes Y \) have in general three irreducible components each. It follows from (4.1) and (4.2) that for \( a \geq 2 \),


\[ H^\sigma_{p,a} \otimes Y = (H^\sigma_{p,a} \otimes Y(n-1)) \oplus (H^\sigma_{p,a} \otimes Y(n-a)) \oplus (H^\sigma_{p,a} \otimes Y(n-p+a-2)) \]

and

\[ H^\mu_{p,a} \otimes Y = (H^\mu_{p,a} \otimes Y(n-1)) \oplus (H^\mu_{p,a} \otimes Y(n-a)) \oplus (H^\mu_{p,a} \otimes Y(n-p+a-1)). \]

For \( a = 1 \) (see (4.3) and (4.4)),

\[ H^\sigma_{p,1} \otimes Y = (H^\sigma_{p,1} \otimes Y(n-1)) \oplus (H^\sigma_{p,1} \otimes Y(n-p-1)), \]
\[ H^\mu_{p,1} \otimes Y = (H^\mu_{p,1} \otimes Y(n-1)) \oplus (H^\mu_{p,1} \otimes Y(n-p)). \]

Since \( H^\sigma_{p+1,a} \otimes Z \) is of the same type as \( H^\sigma_{p,a} \otimes Y(n-p+a-2) \) and \( H^\mu_{p+1,a} \otimes Z \) is of the same type as \( H^\mu_{p,a-1} \otimes Y(n-a+1) \), it follows by Proposition 4.5 that

\[ \delta_{p+1}(H^\sigma_{p+1,a} \otimes Z) = H^\sigma_{p,a} \otimes Y(n-p+a-2), \]
\[ \delta_{p+1}(H^\mu_{p+1,a} \otimes Z) = H^\mu_{p,a-1} \otimes Y(n-a+1). \]

Hence,

\[ \text{coker} \delta_{p+1} = \sum_{a=p+2-n}^{\lfloor \frac{p+1}{2} \rfloor} H^\sigma_{p,a} \otimes Y(n-1) \oplus \sum_{a=p+2-n}^{\lfloor \frac{p+1}{2} \rfloor} H^\sigma_{p,a} \otimes Y(n-a) \]
\[ \oplus \sum_{a=p+1-n}^{\lfloor \frac{p+1}{2} \rfloor} H^\mu_{p,a} \otimes Y(n-1) \oplus \sum_{a=p+2-n}^{\lfloor \frac{p+1}{2} \rfloor} H^\mu_{p,a} \otimes Y(n-p+a-1). \]

**Remark 4.8.** Notice that, in the light of Theorem 4.2, we can exhibit explicit highest weight vectors for all the irreducible submodules of \( \text{ker} \delta_p \) and \( \text{coker} \delta_{p+1} \).

**Theorem 4.9.** The structure of \( g_1 \)-module of the adjoint homology of the Lie algebra \( n = A_n(1, 2) \) is given by the following. For \( 0 \leq p \leq 2n - 1 \),

\[ H_p(n, n) \simeq \frac{H_p(n) \otimes X}{H^\mu_{p+1,a} \otimes X} \oplus (H^\sigma_{p, \frac{p+1}{2}} \otimes Z) \oplus (H^\mu_{p,1} \otimes Z) \oplus (H^\sigma_{p, \frac{p+1}{2}} \otimes Z) \]
\[ \oplus \sum_{a=p+2-n}^{\lfloor \frac{p+1}{2} \rfloor} H^\sigma_{p,a} \otimes Y(n-1) \oplus \sum_{a=p+2-n}^{\lfloor \frac{p+1}{2} \rfloor} H^\sigma_{p,a} \otimes Y(n-a) \]
\[ \oplus \sum_{a=p+1-n}^{\lfloor \frac{p+1}{2} \rfloor} H^\mu_{p,a} \otimes Y(n-1) \oplus \sum_{a=p+2-n}^{\lfloor \frac{p+1}{2} \rfloor} H^\mu_{p,a} \otimes Y(n-p+1+a). \]
We recall that a module $H^2_{p,a}$ exists only for $a \in \mathbb{N}$ and $p + 2 - n \leq a \leq \lfloor \frac{p+2}{2} \rfloor$ and a module $H^2_{p,a}$ exists only for $a \in \mathbb{N}$ and $p + 1 - n \leq a \leq \lfloor \frac{p+1}{2} \rfloor$. Otherwise we assume that $H^2_{p,a}$ and $H^2_{p,a}$ are the null space.

**Corollary 4.10.** The first adjoint homology groups of the Lie algebra $n = A_n(1, 2)$, for $n \geq 4$, as $g_1$-modules are:

$$H_0(n, n) \simeq (H_0(n) \otimes X) \oplus (H^2_{0,1} \otimes Z),$$

$$H_1(n, n) \simeq \frac{H_1(n)}{H^2_{1,1} \otimes X} \oplus (H^2_{1,1} \otimes Z) \oplus (H^2_{1,1} \otimes Z) \oplus (H^2_{1,1} \otimes Y(n-1)) \oplus (H^2_{1,1} \otimes Y(n-1))
\simeq (H^2_{1,1} \otimes X) \oplus (H^2_{1,1} \otimes Z) \oplus (H^2_{1,1} \otimes Z) \oplus (H^2_{1,1} \otimes Y(n-1)) \oplus (H^2_{1,1} \otimes Y(n-1))
\simeq (H^2_{1,1} \otimes X(n-1)) \oplus (H^2_{1,1} \otimes X(n-2)) \oplus (H^2_{1,1} \otimes Z) \oplus (H^2_{1,1} \otimes Z)
\oplus (H^2_{1,1} \otimes Y(n-1)) \oplus (H^2_{1,1} \otimes Y(n-1)),
$$

$$H_2(n, n) \simeq (H_2(n) \otimes X) \oplus (H^2_{2,2} \otimes Z) \oplus (H^2_{2,1} \otimes Z) \oplus (H^2_{2,1} \otimes Y(n-1)) \oplus (H^2_{2,1} \otimes Y(n-2))
\simeq (H^2_{2,1} \otimes X(n-1)) \oplus (H^2_{2,1} \otimes X(n-3)) \oplus (H^2_{2,2} \otimes X(n-1)) \oplus (H^2_{2,2} \otimes X(n-2))
\oplus (H^2_{2,1} \otimes X(n-1)) \oplus (H^2_{2,1} \otimes X(n-2)) \oplus (H^2_{2,2} \otimes Z) \oplus (H^2_{2,1} \otimes Z)
\oplus (H^2_{2,1} \otimes Y(n-1)) \oplus (H^2_{2,1} \otimes Y(n-2)).$$

Moreover, explicit homology highest weight vectors for all the irreducible summands are:

$$H_0(n, n) = (1 \otimes x_{n-1}) \oplus (1 \otimes Z),$$

$$H_1(n, n) = (-x_{n-1} \otimes x_{n-1}) \oplus (-x_{n-1} \otimes x_{n-2} + x_{n-2} \otimes x_{n-1}) \oplus (Z \otimes Z)
\oplus (x_{n-1} Z + Z x_{n-1}),$$

$$H_2(n, n) = (-x_{n-1} \otimes x_{n-2} x_{n-1})
\oplus (-x_{n-1} \otimes x_{n-2} x_{n-3} + x_{n-1} \otimes x_{n-3} x_{n-2} - x_{n-2} \otimes x_{n-3} x_{n-1})
\oplus (-x_{n-1} \otimes y_{n-2} x_{n-1})
\oplus \left(-x_{n-1} \otimes y_{n-1} x_{n-2} + \frac{1}{2} x_{n-1} \otimes x_{n-2} + \frac{1}{2} x_{n-1} \otimes y_{n-1} x_{n-2} x_{n-1}ight)
\oplus (-z x_{n-2} \otimes x_{n-1}) \oplus (-z \otimes y_{n-1} x_{n-2} + z \otimes y_{n-2} x_{n-1})
\oplus (x_{n-1} \otimes y_{n-1} Z) \oplus (Z \otimes y_{n-1} \otimes Z).$$

**Proof.** The decompositions in the first part follow directly from Theorem 4.9, Section 4.1 and (4.1)-(4.4).

The explicit highest weight vectors are constructed according to Theorem 4.2 using those in Theorem 2.3. □

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References