J. Differential Equations 254 (2013) 756–778

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

On a strong solution of the non-stationary Navier–Stokes equations under slip or leak boundary conditions of friction type

Takahito Kashiwabara *

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

ARTICLE INFO

Article history:
Received 8 December 2011
Revised 12 September 2012
Available online 16 October 2012

MSC:
35Q30
35K86

Keywords:
Navier–Stokes equations
Frictional boundary conditions
Variational inequality
Strong solution
Galerkin's method

ABSTRACT

Strong solutions of the non-stationary Navier–Stokes equations under non-linearized slip or leak boundary conditions are investigated. We show that the problems are formulated by a variational inequality of parabolic type, to which uniqueness is established. Using Galerkin's method and deriving a priori estimates, we prove global and local existence for 2D and 3D slip problems respectively. For leak problems, under no-leak assumption at $t = 0$ we prove local existence in 2D and 3D cases. Compatibility conditions for initial states play a significant role in the estimates.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^d$ $(d = 2, 3)$, and fix $T > 0$. We suppose that the boundary $\Gamma = \partial\Omega$ consists of two nonempty open components Γ_0 and Γ_1, that is, $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. We are concerned with the non-stationary incompressible Navier–Stokes equations in Ω:

$$\begin{cases} u' + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f & \text{in } \Omega \times (0, T), \quad (1.1) \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, T), \quad (1.2) \end{cases}$$

* Fax: +81 3 5465 7012.
E-mail address: tkashiwa@ms.u-tokyo.ac.jp.

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.09.015

with the initial condition

$$u = u_0 \quad \text{in } \Omega \times \{0\}. \tag{1.3}$$

Here, ν, u, p, and f denote a viscosity constant, velocity field, pressure, and external force respectively; u' means the time derivative $\frac{\partial u}{\partial t}$.

As for the boundary condition, we impose the adhesive b.c. on Γ_0:

$$u = 0 \quad \text{on } \Gamma_0. \tag{1.4}$$

On the other hand, we consider one of the following nonlinear b.c. on Γ_1:

$$u_n = 0, \qquad |\sigma_\tau| \leqslant g, \qquad \sigma_\tau \cdot u_\tau + g|u_\tau| = 0, \quad \text{on } \Gamma_1, \tag{1.5}$$

which is called the *slip boundary condition of friction type* (SBCF), and

$$u_\tau = 0, \qquad |\sigma_n| \leqslant g, \qquad \sigma_n u_n + g|u_n| = 0, \quad \text{on } \Gamma_1, \tag{1.6}$$

which is called the *leak boundary condition of friction type* (LBCF). Here, n is the outer unit normal vector defined on Γ, and we write $u_n := u \cdot n$ and $u_\tau := u - u_n n$. The stress tensor $\mathbb{T} = (T_{ij})_{i,j=1,\dots,d}$ is given by $T_{ij} = -p\delta_{ij} + \nu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, δ_{ij} being Kronecker delta. We define the stress vector $\sigma = \sigma(u, p)$ as $\sigma = \mathbb{T}n$, and write $\sigma_n := \sigma \cdot n$ and $\sigma_\tau := \sigma - \sigma_n n$. One can easily see that $\sigma_n = \sigma_n(u, p)$ may depend on p, whereas $\sigma_\tau = \sigma_\tau(u)$ does not.

The function g, given on Γ_1 and assumed to be strictly positive, is called a *modulus of friction*. Its physical meaning is the threshold of the tangential (resp. normal) stress. In fact, if $|\sigma_\tau| < g$ (resp. $|\sigma_n| < g$) then (1.5) (resp. (1.6)) implies $u_\tau = 0$ (resp. $u_n = 0$), namely, no slip (resp. leak) occurs; otherwise non-trivial slip (resp. leak) can take place. We notice that if we make $g = 0$ formally, (1.5) and (1.6) reduce to the usual slip and leak b.c. respectively. In summary, SBCF and LBCF are non-linearized slip and leak b.c. obtained from introduction of some friction law on the stress.

It should be also noted that the second and third conditions of (1.5) (resp. (1.6)) are equivalently rewritten, with the notation of subdifferential, as

$$\sigma_\tau \in -g\partial|u_\tau| \quad \bigl(\text{resp. } \sigma_n \in -g\partial|u_n|\bigr).$$

Although we will not pursue this matter further, one can refer to [3,17] for the Navier–Stokes equations with general subdifferential b.c. See also [4], which considers the motion of a Bingham fluid under b.c. with nonlocal friction against slip.

SBCF and LBCF are first introduced in [6,9] for the stationary Stokes and Navier–Stokes equations, where existence and uniqueness of weak solutions are established. Generalized SBCF is considered in [19,20]. The H^2–H^1 regularity for the Stokes equations is proved in [28]. In terms of numerical analysis, [2,13,14,22–25] deal with finite element methods for SBCF or LBCF. Applications of SBCF and LBCF to realistic problems, together with numerical simulations, are found in [15,29].

For non-stationary cases, [7,8] study the time-dependent Stokes equations without external forces under SBCF and LBCF, using a nonlinear semigroup theory. The solvability of nonlinear problems is discussed in [21] for SBCF, and in [1] for a variant of LBCF. They use the Stokes operator associated with the linear slip or leak b.c., and do not take into account a compatibility condition at $t = 0$.

The purpose of this paper is to prove existence and uniqueness of a strong solution for (1.1)–(1.4) with (1.5) or (1.6). We employ the class of solutions of Ladyzhenskaya type (see [18]), searching (u, p) such that

$$\begin{cases} u \in L^\infty\bigl(0, T; H^1(\Omega)^d\bigr), \quad u' \in L^\infty\bigl(0, T; L^2(\Omega)^d\bigr) \cap L^2\bigl(0, T; H^1(\Omega)^d\bigr), \\ p \in L^\infty\bigl(0, T; L^2(\Omega)\bigr). \end{cases}$$

There are several reasons we focus on this strong solution. First, from a viewpoint of numerical analysis, we would like to construct solutions in a class where uniqueness and regularity are assured also for 3D case. Second, we desire an L^∞-estimate with respect to time for p, which may not be obtained for weak solutions of Leray–Hopf type (cf. [30, Proposition III.1.1]). Third, in LBCF, it is not straightforward to deduce a weak solution because of (1.7) below. Similar difficulty already comes up in the linear leak b.c. (see [26]).

The rest of this paper is organized as follows. Basic symbols, notation, and function spaces are given in Section 2.

In Section 3, we investigate the problem with SBCF. The weak formulation is given by a variational inequality, to which we prove uniqueness of solutions. To show existence, we consider a regularized problem, approximate it by Galerkin's method, and derive a priori estimates which allow us to pass on the limit to deduce the desired strong solution. Using the compatibility condition that u_0 must satisfy SBCF, we can adapt u_0 to the regularized problem, which makes an essential point in the estimate.

Section 4 is devoted to a study of the problem with LBCF. There are two major differences from SBCF. First, as was pointed out in the stationary case [6, Remark 3.2], we cannot obtain the uniqueness of an additive constant for p if no leak occurs, namely, $u_n = 0$ on Γ_1. Second, under LBCF, the quantity

$$\int_\Omega \big\{(u\cdot\nabla)v\cdot v\big\}\,dx = \frac{1}{2}\int_\Gamma u_n|v|^2\,ds \quad (\text{if } \operatorname{div} u = 0) \tag{1.7}$$

need not vanish because u_n can be non-zero. This fact affects our a priori estimates badly, and we can extract a solution only when the initial leak $\|u_{0n}\|_{L^2(\Gamma_1)}$ is small enough. Incidentally, if we use the so-called Bernoulli pressure $p + \frac{1}{2}|u|^2$ instead of standard p, the mathematical difficulty arising from (1.7) is resolved; nevertheless the leak b.c. involving the Bernoulli pressure is known to cause an unphysical effect in numerical simulations (see [12, p. 338]). Thereby we employ the usual formulation.

Finally, in Section 5 we conclude this paper with some remarks on higher regularity.

2. Preliminaries

Throughout the present paper, the domain Ω is supposed to be as smooth as required. For the precise regularity of Ω which is sufficient to deduce our main theorems, see Remarks 3.3 and 4.3. We shall denote by C various generic positive constants depending only on Ω, unless otherwise stated. When we need to specify dependence on a particular parameter, we write as $C = C(f, g, u_0)$, and so on.

We use the Lebesgue space $L^p(\Omega)$ $(1 \leqslant p \leqslant \infty)$, and the Sobolev space $H^r(\Omega) = \{\phi \in L^2(\Omega) \mid \|\phi\|^2_{H^r(\Omega)} = \sum_{|\alpha|\leqslant r}\|\partial^\alpha\phi\|^2_{L^2(\Omega)} < \infty\}$ for a nonnegative integer r, where $H^0(\Omega)$ means $L^2(\Omega)$. $H^s(\Omega)$ is also defined for a non-integer $s > 0$ (e.g. [10, Definition 1.2]). We put $L^2_0(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q\,dx = 0\}$. For spaces of vector-valued functions, we write $L^p(\Omega)^d$, and so on.

The Lebesgue and Sobolev spaces on the boundary Γ, Γ_0, or Γ_1, are also used. $H^0(\Gamma_1)$ means $L^2(\Gamma_1)$, and we put $L^2_0(\Gamma_1) = \{\eta \in L^2(\Gamma_1) \mid \int_{\Gamma_1}\eta\,ds = 0\}$, where ds denotes the surface measure. For a positive function g on Γ_1, the weighted Lebesgue spaces $L^1_g(\Gamma_1)$ and $L^\infty_{1/g}(\Gamma_1)$ are defined by the norms

$$\|\eta\|_{L^1_g(\Gamma_1)} = \int_{\Gamma_1} g|\eta|\,ds \quad \text{and} \quad \|\eta\|_{L^\infty_{1/g}(\Gamma_1)} = \operatorname*{ess\,sup}_{\Gamma_1}\frac{|\eta|}{g},$$

respectively. The dual space of $L^1_g(\Gamma_1)$ is $L^\infty_{1/g}(\Gamma_1)$ (see [6, Lemma 2.1]).

The usual trace operator $\phi \mapsto \phi|_\Gamma$ is defined from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$. The restrictions $\phi|_{\Gamma_0}$, $\phi|_{\Gamma_1}$ of $\phi|_\Gamma$, are also considered, and we simply write ϕ to indicate them when there is no fear of confusion. In particular, η_n and η_τ means $(\eta\cdot n)|_\Gamma$ and $(\eta - (\eta\cdot n)n)|_\Gamma$ respectively, for $\eta \in H^{1/2}(\Gamma)^d$. Note that $\|\eta_n\|_{H^{1/2}(\Gamma)} \leqslant C\|\eta\|_{H^{1/2}(\Gamma)^d}$ and $\|\eta_\tau\|_{H^{1/2}(\Gamma)^d} \leqslant C\|\eta\|_{H^{1/2}(\Gamma)^d}$ because n is smooth on Γ.

The inner product of $L^2(\Omega)^d$ is simplified as $(\cdot,\cdot)$, while other inner products and norms are written with clear subscripts, e.g., $(\cdot,\cdot)_{L^2(\Gamma_1)}$ or $\|\cdot\|_{H^1(\Omega)^d}$. For a Banach space X, we denote its dual space by X' and the dual product between X' and X by $\langle\cdot,\cdot\rangle_X$. Moreover, we employ the standard notation of Bochner spaces such as $L^2(0,T;X)$, $H^1(0,T;X)$.

For function spaces corresponding to a velocity and pressure, we introduce closed subspaces of $H^1(\Omega)^d$ or $L^2(\Omega)$ as follows:

$$\begin{aligned} V &= \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_0\}, & \mathring{V} &= \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma\}, \\ V_n &= \{v \in V \mid v_n = 0 \text{ on } \Gamma_1\}, & V_\tau &= \{v \in V \mid v_\tau = 0 \text{ on } \Gamma_1\}, \\ Q &= L^2(\Omega), & \mathring{Q} &= L^2_0(\Omega). \end{aligned}$$

To indicate a divergence-free space, we set $H^1_\sigma(\Omega)^d = \{v \in H^1(\Omega)^d \mid \operatorname{div} v = 0\}$. We use the notation $V_\sigma = V \cap H^1_\sigma(\Omega)^d$, $\mathring{V}_\sigma = \mathring{V} \cap H^1_\sigma(\Omega)^d$, $V_{n,\sigma} = V_n \cap H^1_\sigma(\Omega)^d$, and $V_{\tau,\sigma} = V_\tau \cap H^1_\sigma(\Omega)^d$.

Let us define bilinear forms a_0, b, and a trilinear form a_1 by

$$a_0(u,v) = \frac{\nu}{2}\sum_{i,j=1}^{d}\int_\Omega \left(\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)dx \quad \left(u, v \in H^1(\Omega)^d\right),$$

$$a_1(u,v,w) = \int_\Omega \{(u\cdot\nabla)v\}\cdot w\,dx \quad \left(u,v,w \in H^1(\Omega)^d\right),$$

$$b(v,q) = -\int_\Omega \operatorname{div} v\, q\,dx \quad \left(v \in H^1(\Omega)^d, q \in L^2(\Omega)\right).$$

The bilinear forms a_0, b are continuous, and from Korn's inequality [16, Lemma 6.2] there exists a constant $\alpha > 0$ such that

$$a_0(v,v) \geqslant \alpha\|v\|^2_{H^1(\Omega)^d} \quad (\forall v \in V). \tag{2.1}$$

Concerning the trilinear term a_1, we obtain the following two lemmas.

Lemma 2.1.

(i) *When $d = 2$, for all $u, v, w \in H^1(\Omega)^d$ it holds that*

$$\left|a_1(u,v,w)\right| \leqslant C\|u\|^{1/2}_{L^2(\Omega)^d}\|u\|^{1/2}_{H^1(\Omega)^d}\|v\|_{H^1(\Omega)^d}\|w\|^{1/2}_{L^2(\Omega)^d}\|w\|^{1/2}_{H^1(\Omega)^d}. \tag{2.2}$$

(ii) *When $d = 2$ or $d = 3$, for all $u, v, w \in H^1(\Omega)^d$ it holds that*

$$\left|a_1(u,v,w)\right| \leqslant C\|u\|^{1/4}_{L^2(\Omega)^d}\|u\|^{3/4}_{H^1(\Omega)^d}\|v\|_{H^1(\Omega)^d}\|w\|^{1/4}_{L^2(\Omega)^d}\|w\|^{3/4}_{H^1(\Omega)^d}. \tag{2.3}$$

Remark 2.1. In particular, we see from (2.3) that

$$\left|a_1(u,v,w)\right| \leqslant C\|u\|_{H^1(\Omega)^d}\|v\|_{H^1(\Omega)^d}\|w\|_{H^1(\Omega)^d}. \tag{2.4}$$

Proof of Lemma 2.1. These are well-known classical results; see e.g. [18,30]. □

Lemma 2.2.

(i) *For all* $u \in V_{n,\sigma}$ *and* $v \in H^1(\Omega)^d$, $a_1(u, v, v) = 0$.
(ii) *For all* $u \in V_{\tau,\sigma}$ *and* $v \in H^1(\Omega)^d$, $a_1(u, v, v) = \frac{1}{2}\int_{\Gamma_1} u_n |v|^2\,ds$, *and*

$$\left|a_1(u, v, v)\right| \leqslant \gamma_1 \|u_n\|_{L^2(\Gamma_1)} \|v\|^2_{H^1(\Omega)^d}, \tag{2.5}$$

where γ_1 *is a constant depending only on* Ω.

Proof. By integration by parts, we have

$$a_1(u, v, w) + a_1(u, w, v) = -\int_\Omega \operatorname{div} u (v \cdot w)\,dx + \int_\Gamma u_n (v \cdot w)\,ds,$$

from which the conclusion of (i) and the first assertion of (ii) follow. Combining Hölder's inequality with the continuity of the trace operator $H^1(\Omega) \to L^4(\Gamma_1)$ (see [27, Theorem II.6.2]), we obtain (2.5). □

Remark 2.2. Whether γ_1 is small or not, especially when compared to α in (2.1), is a very crucial point in our a priori estimates for LBCF (see Proposition 4.1). This is why we distinguish γ_1 from other constants C and do not combine γ_1 with them. As Lemma 2.2(i) shows, this problem does not happen when we consider SBCF.

The following, which are readily obtainable consequences of standard trace and (solenoidal) extension theorems ([10, Theorems I.1.5–6, Lemma I.2.2], see also [16, Section 5.3]), are frequently used in subsequent arguments.

Lemma 2.3.

(i) *For* $v \in V_n$, *it holds that* $\|v_\tau\|_{H^{1/2}(\Gamma_1)^d} \leqslant C\|v\|_{H^1(\Omega)^d}$.
(ii) *For* $\eta \in H^{1/2}(\Gamma_1)^d$ *satisfying* $\eta_n = 0$ *on* Γ_1, *there exists* $v \in V_{n,\sigma}$ *such that* $v_\tau = \eta$ *on* Γ_1 *and* $\|v\|_{H^1(\Omega)^d} \leqslant C\|\eta\|_{H^{1/2}(\Gamma_1)^d}$.

Lemma 2.4.

(i) *For* $v \in V_\tau$, *it holds that* $\|v_n\|_{H^{1/2}(\Gamma_1)} \leqslant C\|v\|_{H^1(\Omega)^d}$.
(ii) *For* $\eta \in H^{1/2}(\Gamma_1)$ (*resp.* $\eta \in H^{1/2}(\Gamma_1) \cap L^2_0(\Gamma_1)$), *there exists* $v \in V_\tau$ (*resp.* $v \in V_{\tau,\sigma}$) *such that* $v_n = \eta$ *on* Γ_1 *and* $\|v\|_{H^1(\Omega)^d} \leqslant C\|\eta\|_{H^{1/2}(\Gamma_1)}$.

The definition of $\sigma(u, p)$ given in Section 1 becomes ambiguous when (u, p) has only lower regularity, say $u \in H^1(\Omega)^d$, $p \in L^2(\Omega)$. Thus we propose a redefinition of it, based on the following Green formula:

$$(-\nu\Delta u + \nabla p, v) + \int_\Gamma \sigma(u, p) \cdot v\,ds = a_0(u, v) + b(v, p) \quad (\text{if } \operatorname{div} u = 0).$$

Definition 2.1. Let $u(t) \in V_\sigma$, $p(t) \in Q$, $u'(t) \in L^2(\Omega)^d$, $f(t) \in L^2(\Omega)^d$. If (1.1) holds in the distribution sense for a.e. $t \in (0, T)$, that is,

$$\left(u', v\right) + a_0(u, v) + a_1(u, u, v) + b(v, p) = (f, v) \quad (\forall v \in \mathring{V}), \tag{2.6}$$

then we define $\sigma = \sigma(u, p) \in (H^{1/2}(\Gamma_1)^d)'$ by

$$\langle \sigma, v\rangle_{H^{1/2}(\Gamma_1)^d} = a_0(u, v) + b(v, p) - \langle F, v\rangle_V \quad (\forall v \in V), \tag{2.7}$$

where $F(t) \in V'$ is given by $\langle F, v\rangle_V = (f, v) - (u', v) - a_1(u, u, v)$.

The above σ is well-defined by virtue of the trace and extension theorem. It coincides with the previous definition when (u, p) is sufficiently smooth. In addition, by Lemmas 2.3 and 2.4, $\sigma_\tau = \sigma - (\sigma \cdot n)n \in (H^{1/2}(\Gamma_1)^d)'$ and $\sigma_n = \sigma \cdot n \in H^{1/2}(\Gamma_1)'$ are characterized by

$$\begin{cases} \langle \sigma_\tau, \eta n\rangle_{H^{1/2}(\Gamma_1)^d} = 0 & \big(\forall \eta \in H^{1/2}(\Gamma_1)\big), \\ \langle \sigma_\tau, v_\tau\rangle_{H^{1/2}(\Gamma_1)^d} = a_0(u, v) + b(v, p) - \langle F, v\rangle_{V_n} & (\forall v \in V_n), \end{cases}$$

and

$$\langle \sigma_n, v_n\rangle_{H^{1/2}(\Gamma_1)} = a_0(u, v) + b(v, p) - \langle F, v\rangle_{V_\tau} \quad (\forall v \in V_\tau),$$

respectively. By Lemma 2.3(ii), σ_τ actually does not depend on p.

3. Navier–Stokes problem with SBCF

3.1. Weak formulations

Throughout this section, we assume $f \in L^2(\Omega \times (0, T))^d$, $u_0 \in V_{n,\sigma}$, and $g \in L^2(\Gamma_1 \times (0, T))$ with $g > 0$. Further regularity assumptions on these data will be given before Theorem 3.2. In addition, we introduce

$$j_\tau(t; \eta) = \int_{\Gamma_1} g(t)|\eta|\, ds \quad \big(\eta \in L^2(\Gamma_1)^d\big), \tag{3.1}$$

which is just written as $j(\eta)$, to simplify notation, until the end of this section. j is obviously nonnegative, positively homogeneous, and Lipschitz continuous for a.e. $t \in (0, T)$. A primal weak formulation of (1.1)–(1.4) with (1.5) is as follows:

Problem PDE-SBCF. For a.e. $t \in (0, T)$, find $(u(t), p(t)) \in V_n \times \mathring{Q}$ such that $u'(t) \in L^2(\Omega)^d$, $u(0) = u_0$, σ_τ is well-defined in the sense of Definition 2.1, $|\sigma_\tau| \leqslant g$ a.e. on Γ_1, and $\sigma_\tau \cdot u_\tau + g|u_\tau| = 0$ a.e. on Γ_1.

Throughout this section, we refer to Problem PDE-SBCF just as Problem PDE. Similar abbreviation will be made for other problems.

One can easily find that a classical solution of (1.1)–(1.4) with (1.5) solves Problem PDE, and that a sufficiently smooth solution of Problem PDE is a classical solution. As the next theorem shows, Problem PDE is equivalent to the following variational inequality problem.

Problem VI_σ-SBCF. For a.e. $t \in (0, T)$, find $u(t) \in V_{n,\sigma}$ such that $u'(t) \in L^2(\Omega)^d$, $u(0) = u_0$, and

$$\big(u', v - u\big) + a_0(u, v - u) + a_1(u, u, v - u) + j(v_\tau) - j(u_\tau) \geqslant (f, v - u) \tag{3.2}$$

for all $v \in V_{n,\sigma}$. Here $j = j_\tau(t; \cdot)$ is defined in (3.1).

Theorem 3.1. *Problems* PDE *and* VI_σ *are equivalent.*

Proof. Let (u, p) solve Problem PDE. Then, for $v \in V_n$ it follows that

$$\bigl(u', v\bigr) + a_0(u, v) + a_1(u, u, v) + b(v, p) - (\sigma_\tau, v_\tau)_{L^2(\Gamma_1)^d} = (f, v). \tag{3.3}$$

Using this equation together with $|\sigma_\tau| \leqslant g$ and $\sigma_\tau \cdot u_\tau + g|u_\tau| = 0$, we have

$$\begin{aligned}
&\bigl(u', v - u\bigr) + a_0(u, v - u) + a_1(u, u, v - u) + j(v_\tau) - j(u_\tau) - (f, v - u) \\
&= -(\sigma_\tau, v_\tau - u_\tau)_{L^2(\Gamma_1)^d} + j(v_\tau) - j(u_\tau) \\
&= \int_{\Gamma_1} \bigl(g|v_\tau| - \sigma_\tau v_\tau\bigr)\, ds \geqslant 0,
\end{aligned}$$

for all $v \in V_{n,\sigma}$. Hence u is a solution of Problem VI_σ.

Next, let u be a solution of Problem VI_σ. Taking $u \pm v$ as a test function in (3.2), with arbitrary $v \in \mathring{V}_\sigma$, we find that

$$\bigl(u', v\bigr) + a_0(u, v) + a_1(u, u, v) = (f, v) \quad (\forall v \in \mathring{V}_\sigma). \tag{3.4}$$

By a standard theory (see [30, Propositions I.1.1 and I.1.2]), there exists unique $p \in \mathring{Q}$ such that (2.6) holds. Therefore, $\sigma_\tau \in (H^{1/2}(\Gamma_1)^d)'$ is well-defined, and thus

$$\bigl(u', v\bigr) + a_0(u, v) + a_1(u, u, v) + b(v, p) - \langle \sigma_\tau, v_\tau \rangle_{H^{1/2}(\Gamma_1)^d} = (f, v) \quad (\forall v \in V_n).$$

Combining this equation with (3.2), we obtain

$$-\langle \sigma_\tau, v_\tau - u_\tau \rangle_{H^{1/2}(\Gamma_1)^d} \leqslant \int_{\Gamma_1} g\bigl(|v_\tau| - |u_\tau|\bigr)\, ds \quad (\forall v \in V_{n,\sigma}), \tag{3.5}$$

and as a result of triangle inequality, $|\langle \sigma_\tau, v_\tau \rangle_{H^{1/2}(\Gamma_1)^d}| \leqslant \int_{\Gamma_1} g|v_\tau|\, ds$ for $v \in V_{n,\sigma}$. In view of Lemma 2.3(ii), this implies that for $\eta \in H^{1/2}(\Gamma_1)^d$

$$\bigl|\langle \sigma_\tau, \eta \rangle_{H^{1/2}(\Gamma_1)^d}\bigr| = \bigl|\langle \sigma_\tau, \eta_\tau \rangle_{H^{1/2}(\Gamma_1)^d}\bigr| \leqslant \|\eta_\tau\|_{L^1_g(\Gamma_1)^d} \leqslant \|\eta\|_{L^1_g(\Gamma_1)^d}.$$

By a density argument, we can extend σ_τ to an element of $(L^1_g(\Gamma)^d)'$ such that

$$\bigl|\langle \sigma_\tau, \eta \rangle_{L^1_g(\Gamma_1)^d}\bigr| \leqslant \|\eta\|_{L^1_g(\Gamma_1)^d} \quad \bigl(\forall \eta \in L^1_g(\Gamma_1)^d\bigr).$$

Since $(L^1_g(\Gamma_1)^d)' = L^\infty_{1/g}(\Gamma_1)^d$, we conclude $|\sigma_\tau| \leqslant g$. Then $\sigma_\tau \cdot u_\tau + g|u_\tau| = 0$ follows from (3.5) with $v = 0$. Hence (u, p) is a solution of Problem PDE. □

3.2. Main theorem. Proof of uniqueness

We are now in a position to state our main theorem. We assume:

(S1) $f \in H^1(0, T; L^2(\Omega)^d)$.
(S2) $g \in H^1(0, T; L^2(\Gamma_1))$ with $g(0) \in H^1(\Gamma_1)$.
(S3) $u_0 \in H^2(\Omega)^d \cap V_{n,\sigma}$, and SBCF is satisfied at $t = 0$, namely,

$$\bigl|\sigma_\tau(u_0)\bigr| \leqslant g(0) \quad \text{and} \quad \sigma_\tau(u_0) \cdot u_{0\tau} + g(0)|u_{0\tau}| = 0 \quad \text{a.e. on } \Gamma_1.$$

Note that $\sigma_\tau(u_0)$ can be defined in a usual sense because $u_0 \in H^2(\Omega)^d$.

Theorem 3.2. *Under* (S1)–(S3)*, when* $d = 2$ *there exists a unique solution* u *of Problem* VI_σ *such that*

$$u \in L^\infty(0, T; V_{n,\sigma}), \qquad u' \in L^\infty\big(0, T; L^2(\Omega)^d\big) \cap L^2(0, T; V_{n,\sigma}).$$

When $d = 3$*, the same conclusion holds on some smaller time interval* $(0, T')$*.*

We call the solution in the above theorem a *strong solution* of Problem VI_σ. First we prove the uniqueness of a strong solution. The existence will be proved in Section 3.4 after some additional preparations.

Proposition 3.1. *If* u_1 *and* u_2 *are strong solutions of Problem* VI_σ*, then* $u_1 = u_2$*.*

Proof. Taking $v = u_2$ and $v = u_1$ in (3.2) for u_1 and that for u_2 respectively, and adding the resulting two inequalities, for a.e. $t \in (0, T)$ we obtain

$$\begin{aligned} &\big(u_1' - u_2', u_1 - u_2\big) + a_0(u_1 - u_2, u_1 - u_2) \\ &\quad\leqslant a_1(u_1, u_1, u_2 - u_1) + a_1(u_2, u_2, u_1 - u_2) \\ &\quad= -a_1(u_1 - u_2, u_2, u_1 - u_2) - a_1(u_2, u_1 - u_2, u_1 - u_2). \end{aligned} \tag{3.6}$$

We deduce from (2.3), together with Young's inequality, that

$$\begin{aligned} \big|a_1(u_1 - u_2, u_2, u_1 - u_2)\big| &\leqslant C\|u_1 - u_2\|^{1/2}_{L^2(\Omega)^d}\|u_1 - u_2\|^{3/2}_{H^1(\Omega)^d}\|u_2\|_{H^1(\Omega)^d} \\ &\leqslant \frac{\alpha}{2}\|u_1 - u_2\|^2_{H^1(\Omega)^d} + C\|u_2\|^2_{H^1(\Omega)^d}\|u_1 - u_2\|^2_{L^2(\Omega)^d}, \\ \big|a_1(u_2, u_1 - u_2, u_1 - u_2)\big| &\leqslant C\|u_2\|_{H^1(\Omega)^d}\|u_1 - u_2\|^{7/4}_{H^1(\Omega)^d}\|u_1 - u_2\|^{1/4}_{L^2(\Omega)^d} \\ &\leqslant \frac{\alpha}{2}\|u_1 - u_2\|^2_{H^1(\Omega)^d} + C\|u_2\|^8_{H^1(\Omega)^d}\|u_1 - u_2\|^2_{L^2(\Omega)^d}. \end{aligned}$$

Combining (2.1) and these estimates with (3.6), we have

$$\frac{d}{dt}\|u_1 - u_2\|^2_{L^2(\Omega)^d} \leqslant C\big(\|u_2\|^2_{H^1(\Omega)^d} + \|u_2\|^8_{H^1(\Omega)^d}\big)\|u_1 - u_2\|^2_{L^2(\Omega)^d}.$$

By Gronwall's inequality, we conclude

$$\big\|u_1(t) - u_2(t)\big\|^2_{L^2(\Omega)^d} \leqslant e^{\int_0^t C(\|u_2\|^2_{H^1(\Omega)^d} + \|u_2\|^8_{H^1(\Omega)^d})\,dt}\big\|u_1(0) - u_2(0)\big\|^2_{L^2(\Omega)^d} = 0,$$

since $u_1(0) = u_2(0) = u_0$. (Note that $\int_0^t(\|u_2\|^2_{H^1(\Omega)^d} + \|u_2\|^8_{H^1(\Omega)^d})\,dt$ remains finite because $u \in L^\infty(0, T; H^1(\Omega)^d)$.) Thus $u_1(t) = u_2(t)$. □

Remark 3.1. In the case of SBCF here, the last term of (3.6) vanishes, according to Lemma 2.2(i). We did not use that fact because we would like to make our proof of uniqueness remain unchanged when we deal with LBCF.

Concerning the associated pressure, we find:

Proposition 3.2. *Under the assumptions of Theorem 3.2, let u be the strong solution of Problem* VI_σ*, and p be the associated pressure obtained in the proof of Theorem 3.1. Then* $p \in L^\infty(0, T; \mathring{Q})$.

Proof. For a.e. $t \in (0, T)$, the well-known inf–sup condition (see [10, I.(5.14)]), together with (3.3), (2.4), and $|\sigma_\tau| \leqslant g$ a.e. on Γ_1, yields

$$\begin{aligned}\|p\|_{L^2(\Omega)} &\leqslant \sup_{v \in \mathring{V}} \frac{b(v,p)}{\|v\|_{H^1(\Omega)^d}} \\ &\leqslant \|u'\|_{L^2(\Omega)^d} + C\|u\|_{H^1(\Omega)^d} + C\|u\|^2_{H^1(\Omega)^d} + C\|g\|_{L^2(\Gamma_1)} + \|f\|_{L^2(\Omega)^d}.\end{aligned}$$

Since RHS is bounded uniformly in t, p is in $L^\infty(0, T; \mathring{Q})$. □

3.3. Regularized problem

To prove the solvability of Problem VI_σ, we consider a regularized variational inequality, which is shown to be equivalent to a variational equation.

Before stating those problems in detail, for fixed $\epsilon > 0$ we introduce

$$j_\epsilon(\eta) = \int_{\Gamma_1} g\rho_\epsilon(\eta)\, ds \quad \bigl(\eta \in L^2(\Gamma_1)^d\bigr),$$

where ρ_ϵ is a regularization of $|\cdot|$ having the following properties:

(a) $\rho_\epsilon \in C^2(\mathbb{R}^d)$ is a nonnegative convex function.
(b) It holds that

$$\bigl|\rho_\epsilon(z) - |z|\bigr| \leqslant \epsilon \quad \bigl(\forall z \in \mathbb{R}^d\bigr). \tag{3.7}$$

(c) If α_ϵ denotes $\nabla\rho_\epsilon$, then

$$\bigl|\alpha_\epsilon(z)\bigr| \leqslant 1 \quad \text{and} \quad \alpha_\epsilon(z)\cdot z \geqslant 0 \quad \bigl(\forall z \in \mathbb{R}^d\bigr). \tag{3.8}$$

In particular, as a result of the convexity, the Hessian of ρ_ϵ, denoted by β_ϵ, is semi-positive definite, that is,

$${}^t y \beta_\epsilon(z) y \geqslant 0 \quad \bigl(\forall y, z \in \mathbb{R}^d\bigr), \tag{3.9}$$

where ${}^t y$ means the transpose of y. Such ρ_ϵ does exist; for example, $\rho_\epsilon(z) = \sqrt{|z|^2 + \epsilon^2}$ enjoys all of (a)–(c) above.

Remark 3.2. One could use the Moreau–Yoshida approximation of $|\cdot|$ as ρ_ϵ, which is considered in [28], but it is only in $C^1(\mathbb{R}^d)$, not in $C^2(\mathbb{R}^d)$.

Since ρ_ϵ is differentiable, the functional j_ϵ is Gâteaux differentiable, with its derivative $Dj_\epsilon(\eta) \in (H^{1/2}(\Gamma_1)^d)'$ computed by

$$\bigl\langle Dj_\epsilon(\eta), \xi\bigr\rangle_{H^{1/2}(\Gamma_1)^d} = \int_{\Gamma_1} g\alpha_\epsilon(\eta)\cdot\xi\, ds \quad \bigl(\eta, \xi \in H^{1/2}(\Gamma_1)^d\bigr).$$

We are ready to state the regularized problems mentioned above.

Problem VI$^\epsilon_\sigma$-SBCF. For a.e. $t \in (0, T)$, find $u_\epsilon(t) \in V_{n,\sigma}$ such that $u'_\epsilon(t) \in L^2(\Omega)^d$, $u_\epsilon(0) = u^\epsilon_0$ and

$$\begin{aligned}&\left(u'_\epsilon, v - u_\epsilon\right) + a_0(u_\epsilon, v - u_\epsilon) + a_1(u_\epsilon, u_\epsilon, v - u_\epsilon) + j_\epsilon(v_\tau) - j_\epsilon(u_{\epsilon\tau})\\ &\quad \geqslant (f, v - u_\epsilon) \quad (\forall v \in V_{n,\sigma}).\end{aligned} \tag{3.10}$$

Problem VE$^\epsilon_\sigma$-SBCF. For a.e. $t \in (0, T)$, find $u_\epsilon(t) \in V_{n,\sigma}$ such that $u'_\epsilon(t) \in L^2(\Omega)^d$, $u_\epsilon(0) = u^\epsilon_0$ and

$$\left(u'_\epsilon, v\right) + a_0(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + \int_{\Gamma_1} g\alpha_\epsilon(u_{\epsilon\tau}) \cdot v_\tau \, ds = (f, v) \quad (\forall v \in V_{n,\sigma}). \tag{3.11}$$

Here, u^ϵ_0 is a perturbation of the original initial velocity u_0. The way one obtains u^ϵ_0 from u_0 is described later. By an elementary observation (e.g. [5, Section 3.3] or [28, Lemma 3.3]), we see that:

Proposition 3.3. *Problems* $\mathrm{VI}^\epsilon_\sigma$ *and* $\mathrm{VE}^\epsilon_\sigma$ *are equivalent.*

Now we focus on the construction of a perturbed initial velocity u^ϵ_0. Since $u_0 \in H^2(\Omega)^d$ satisfies SBCF by (S3), it follows from the Green formula $a_0(u_0, v) = (-\nu\Delta u_0, v) + \int_{\Gamma_1} \sigma_\tau(u_0) \cdot v_\tau \, ds$, for $v \in V_{n,\sigma}$, that

$$a_0(u_0, v - u_0) + \int_{\Gamma_1} g(0)|v_\tau| \, ds - \int_{\Gamma_1} g(0)|u_{0\tau}| \, ds \geqslant (-\nu\Delta u_0, v - u_0). \tag{3.12}$$

Here we consider the regularized problem: find $u^\epsilon_0 \in V_{n,\sigma}$ such that

$$a_0\left(u^\epsilon_0, v - u^\epsilon_0\right) + \int_{\Gamma_1} g(0)\rho_\epsilon(v_\tau) \, ds - \int_{\Gamma_1} g(0)\rho_\epsilon\left(u^\epsilon_{0\tau}\right) ds \geqslant \left(-\nu\Delta u_0, v - u^\epsilon_0\right) \quad (\forall v \in V_{n,\sigma}), \tag{3.13}$$

which is equivalent to (cf. Proposition 3.3)

$$a_0\left(u^\epsilon_0, v\right) + \int_{\Gamma_1} g(0)\alpha_\epsilon\left(u^\epsilon_{0\tau}\right) \cdot v_\tau \, ds = (-\nu\Delta u_0, v) \quad (\forall v \in V_{n,\sigma}). \tag{3.14}$$

By a standard theory of elliptic variational inequalities [11], (3.13) admits a unique solution u^ϵ_0, which is the perturbation of u_0 in question. With this setting, we find:

Lemma 3.1.

(i) *When* $\epsilon \to 0$, $u^\epsilon_0 \to u_0$ *strongly in* $H^1(\Omega)^d$.
(ii) $u^\epsilon_0 \in H^2(\Omega)^d$ *and*

$$\left\|u^\epsilon_0\right\|_{H^2(\Omega)^d} \leqslant C\left(\|\nu\Delta u_0\| + \left\|g(0)\right\|_{H^1(\Gamma_1)}\right). \tag{3.15}$$

Proof. (i) Taking $v = u_0$ in (3.13) and $v = u^\epsilon_0$ in (3.12), adding the resulting two inequalities, applying Korn's inequality, and using (3.7), we conclude

$$\alpha \left\| u_0^\epsilon - u_0 \right\|^2_{H^1(\Omega)^d} \leqslant \int_{\Gamma_1} g(0)\left(\left|u_0^\epsilon\right| - \rho_\epsilon\left(u_0^\epsilon\right)\right) ds + \int_{\Gamma_1} g(0)\left(\rho_\epsilon(u_0) - |u_0|\right) ds$$
$$\leqslant 2\epsilon \int_{\Gamma_1} g(0)\, ds \to 0 \quad (\epsilon \to 0).$$

(ii) Since $g(0) \in H^1(\Gamma_1)$ by (S2), we can directly apply the regularity result [28, Lemma 5.2] to the elliptic variational inequality (3.13), and obtain (3.15). Though our ρ_ϵ and α_ϵ are different from those of [28], it makes no difference in the proof of that lemma. □

Remark 3.3.

(i) As a result of (i) above, for sufficiently small $\epsilon > 0$ we have

$$\left\| u_0^\epsilon \right\|_{L^2(\Omega)^d} \leqslant 2\|u_0\|_{L^2(\Omega)^d} \quad \text{and} \quad \left\| u_0^\epsilon \right\|_{H^1(\Omega)^d} \leqslant 2\|u_0\|_{H^1(\Omega)^d}. \tag{3.16}$$

(ii) Concerning the regularity of the domain, [28] assumes that Γ_0 and Γ_1 are class of C^2 and C^4 respectively, which is sufficient for our theory as well.

Remark 3.4. In [28], dealing with the stationary problem, the author stated that $g \in H^{1/2}(\Gamma_1)$ was enough to derive $u \in H^2(\Omega)^d$ and $p \in H^1(\Omega)$. However, it turned out that his proof presented there worked only for $g \in H^1(\Gamma_1)$; see the errata by the same author. This is why we have assumed $g(0) \in H^1(\Gamma_1)$ in (S2), not $g(0) \in H^{1/2}(\Gamma_1)$.

3.4. Proof of existence

Due to Proposition 3.3, we concentrate on solving Problem $\mathrm{VE}_\sigma^\epsilon$. In doing so, we construct approximate solutions by Galerkin's method. Since $V_{n,\sigma} \subset H^1(\Omega)^d$ is separable, there exist members $w_1, w_2, \ldots \in V_{n,\sigma}$, linear independent to each other, such that $\bigcup_{m=1}^\infty \operatorname{span}\{w_k\}_{k=1}^m \subset V_{n,\sigma}$ dense in $H^1(\Omega)^d$. Here ϵ is fixed, and thus we may assume $w_1 = u_0^\epsilon$.

Problem $\mathrm{VE}_\sigma^{\epsilon,m}$-SBCF. Find $c_k \in C^2([0,T])$ $(k = 1, \ldots, m)$ such that $u_m \in V_{n,\sigma}$ defined by $u_m = \sum_{k=1}^m c_k(t) w_k$ satisfies $u_m(0) = u_0^\epsilon$ and

$$\left(u_m', w_k\right) + a_0(u_m, w_k) + a_1(u_m, u_m, w_k) + \int_{\Gamma_1} g\alpha_\epsilon(u_{m\tau}) \cdot w_{k\tau}\, ds = (f, w_k) \quad (k = 1, \ldots, m). \tag{3.17}$$

Since $\alpha_\epsilon \in C^1(\mathbb{R}^d)^d$, the system of ordinal differential equations (3.17) admits unique solutions $c_k \in C^2([0,\tilde{T}])$ $(k = 1, \ldots, m)$ for some $\tilde{T} \leqslant T$. The a priori estimate below shows $\tilde{T}$ can be taken as T, so that we write T instead of $\tilde{T}$ from the beginning.

Proposition 3.4. *Let* (S1)–(S3) *be valid and ϵ be small enough so that* (3.16) *holds.*

(i) *When $d = 2$, $u_m \in L^\infty(0,T;V_{n,\sigma})$ and $u_m' \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;V_{n,\sigma})$ are bounded independently of m and ϵ.*
(ii) *When $d = 3$, the same conclusion holds for some smaller interval $(0,T')$, which can be taken independently of m and ϵ.*

Proof. Due to space limitations, we simply write $\|u\|_{L^2}, \|g\|_{L^2}, \|f\|_{L^2}, \ldots$ instead of $\|u\|_{L^2(\Omega)^d}$, $\|g\|_{L^2(\Gamma_1)}, \|f\|_{L^2(\Omega)^d}, \ldots$ and so on.

(i) Multiplying (3.17) by $c_k(t)$, and adding the resulting equations for $k = 1, \ldots, m$, we obtain

$$\left(u_m', u_m\right) + a_0(u_m, u_m) + \int_{\Gamma_1} g\alpha_\epsilon(u_{m\tau}) \cdot u_{m\tau}\, ds = (f, u_m),$$

where we have used Lemma 2.2(i). It follows from (2.1) and (3.8) that

$$\frac{1}{2}\frac{d}{dt}\|u_m\|_{L^2}^2 + \alpha\|u_m\|_{H^1}^2 \leqslant (f, u_m) \leqslant \|f\|_{L^2}\|u_m\|_{H^1} \leqslant \frac{\alpha}{2}\|u_m\|_{H^1}^2 + \frac{1}{2\alpha}\|f\|_{L^2}^2,$$

which gives

$$\frac{d}{dt}\|u_m\|_{L^2}^2 + \alpha\|u_m\|_{H^1}^2 \leqslant C\|f\|_{L^2}^2. \tag{3.18}$$

Consequently, for $0 \leqslant t \leqslant T$,

$$\left\|u_m(t)\right\|_{L^2}^2 + \alpha\int_0^T \|u_m\|_{H^1}^2\, dt \leqslant \left\|u_0^\epsilon\right\|_{L^2}^2 + C\int_0^T \|f\|_{L^2}^2\, dt. \tag{3.19}$$

From (3.16), we find that $\|u_m\|_{L^\infty(0,T;L^2)}$ and $\|u_m\|_{L^2(0,T;V_{n,\sigma})}$ are bounded by $C(f, u_0)$ independently of m and ϵ.

Next, we differentiate (3.17) with respect to t, which is possible because $c_k(t)$'s are in $C^2([0,T])$, to deduce

$$\begin{aligned}&\left(u_m'', w_k\right) + a_0\left(u_m', w_k\right) + a_1\left(u_m', u_m, w_k\right) + a_1\left(u_m, u_m', w_k\right)\\&\quad + \int_{\Gamma_1} g'\alpha_\epsilon(u_{m\tau}) \cdot w_{k\tau}\, ds + \int_{\Gamma_1} g\, {}^t u_{m\tau}' \beta_\epsilon(u_{m\tau}) w_{k\tau}\, ds\\&= \left(f', w_k\right) \quad (k = 1, \ldots, m).\end{aligned}$$

Multiplying this by $c_k'(t)$, and adding the resulting equations, we obtain

$$\begin{aligned}&\left(u_m'', u_m'\right) + a_0\left(u_m', u_m'\right) + a_1\left(u_m', u_m, u_m'\right) + \int_{\Gamma_1} g'\alpha_\epsilon(u_{m\tau}) \cdot u_{m\tau}'\, ds + \int_{\Gamma_1} g\, {}^t u_{m\tau}' \beta_\epsilon(u_{m\tau}) u_{m\tau}'\, ds\\&\quad = \left(f', u_m'\right),\end{aligned} \tag{3.20}$$

where we have again used Lemma 2.2(i). Here,

$$\begin{aligned}a_1\left(u_m', u_m, u_m'\right) &\leqslant C\left\|u_m'\right\|_{L^2}\|u_m\|_{H^1}\left\|u_m'\right\|_{H^1} \quad (\text{by } (2.2))\\&\leqslant \frac{\alpha}{6}\left\|u_m'\right\|_{H^1}^2 + C\|u_m\|_{H^1}^2\left\|u_m'\right\|_{L^2}^2,\end{aligned} \tag{3.21}$$

$$\begin{aligned}\left|\int_{\Gamma_1} g'\alpha_\epsilon(u_{m\tau})\cdot u'_{m\tau}\,ds\right| &\leqslant \|g'\|_{L^2}\|u'_{m\tau}\|_{L^2(\Gamma_1)^d} \quad \text{(by (3.8))}\\ &\leqslant C\|g'\|_{L^2}\|u'_m\|_{H^1} \quad \text{(by Lemma 2.3(i))}\\ &\leqslant \frac{\alpha}{6}\|u'_m\|^2_{H^1} + C\|g'\|^2_{L^2},\end{aligned}$$

$$\int_{\Gamma_1} g\,{}^t u'_{m\tau}\beta_\epsilon(u_{m\tau})u'_{m\tau}\,ds \geqslant 0 \quad \text{(by } g>0 \text{ and (3.9))},$$

$$\left|(f', u'_m)\right| \leqslant \|f'\|_{L^2}\|u'_m\|_{H^1} \leqslant \frac{\alpha}{6}\|u'_m\|^2_{H^1} + C\|f'\|^2_{L^2}.$$

Collecting these estimates, it follows from (3.20) that for $0 \leqslant t \leqslant T$

$$\frac{d}{dt}\|u'_m\|^2_{L^2} + \alpha\|u'_m\|^2_{H^1} \leqslant C\left(\|f'\|^2_{L^2} + \|g'\|^2_{L^2}\right) + C\|u_m\|^2_{H^1}\|u'_m\|^2_{L^2}. \tag{3.22}$$

If the second term of LHS is neglected, Gronwall's inequality leads to

$$\|u'_m(t)\|^2_{L^2} \leqslant \left(\|u'_m(0)\|^2_{L^2} + C\int_0^T \left(\|f'\|^2_{L^2} + \|g'\|^2_{L^2}\right)dt\right)e^{C\int_0^T \|u_m\|^2_{H^1}\,dt}. \tag{3.23}$$

Provided that $\|u'_m(0)\|^2_{L^2}$ is bounded independently of m and ϵ, estimate (3.23) gives the boundedness of $\|u'_m\|_{L^\infty(0,T;L^2)}$ because we already know that of $\|u_m\|_{L^2(0,T;V_{n,\sigma})}$ due to (3.19). Then, by (3.18) and (3.19) we have

$$\alpha\|u_m(t)\|^2_{H^1} \leqslant C\|f\|^2_{L^2} + \|u'_m\|_{L^2}\|u_m\|_{L^2} \leqslant C(f, g, u_0),$$

which implies $\|u_m\|_{L^\infty(0,T;V_{n,\sigma})}$ is bounded. Finally, integrating (3.22), we see that $\|u'_m\|_{L^2(0,T;V_{n,\sigma})}$ is also bounded.

To show the boundedness of $\|u'_m(0)\|^2_{L^2}$, we multiply (3.17) by $c'_k(t)$, add the resulting equations, and make $t=0$, arriving at

$$\begin{aligned}&\|u'_m(0)\|^2_{L^2} + a_0\left(u_0^\epsilon, u'_m(0)\right) + a_1\left(u_0^\epsilon, u_0^\epsilon, u'_m(0)\right) + \int_{\Gamma_1} g(0)\alpha_\epsilon\left(u^\epsilon_{0\tau}\right)\cdot u'_{m\tau}(0)\,ds\\ &\quad = \left(f(0), u'_m(0)\right).\end{aligned} \tag{3.24}$$

From the construction of u_0^ϵ, especially (3.14), we have

$$\begin{aligned}\left|a_0\left(u_0^\epsilon, u'_m(0)\right) + \int_{\Gamma_1} g(0)\alpha_\epsilon\left(u^\epsilon_{0\tau}\right)\cdot u'_{m\tau}(0)\,ds\right| &= \left|\left(-\nu\Delta u_0, u'_m(0)\right)\right|\\ &\leqslant C\|u_0\|_{H^2}\|u'_m(0)\|_{L^2}.\end{aligned} \tag{3.25}$$

Furthermore, by Schwarz's inequality, Sobolev's inequality and (3.15),

$$\big|a_1\big(u_0^\epsilon, u_0^\epsilon, u_m'(0)\big)\big| \leqslant C\big\|u_0^\epsilon\big\|_{L^\infty}\big\|u_0^\epsilon\big\|_{H^1}\big\|u_m'(0)\big\|_{L^2} \leqslant C\big\|u_0^\epsilon\big\|_{H^2}^2\big\|u_m'(0)\big\|_{L^2}$$
$$\leqslant C\big(\|u_0\|_{H^2} + \big\|g(0)\big\|_{H^1}\big)^2\big\|u_m'(0)\big\|_{L^2}.$$

Combining these estimates with (3.24), we obtain

$$\big\|u_m'(0)\big\|_{L^2} \leqslant \big\|f(0)\big\|_{L^2} + C\|u_0\|_{H^2} + C\big(\|u_0\|_{H^2} + \big\|g(0)\big\|_{H^1}\big)^2,$$

which proves the boundedness of $\|u_m'(0)\|_{L^2}^2$. This completes the proof of (i).

(ii) The discussion before (3.21) and the observation for $\|u_m'(0)\|_{L^2}$ are the same as (i). What changes from the case $d=2$ is that when $d=3$, instead of (3.21), we only have (by (2.3) and Young's inequality)

$$\big|a_1\big(u_m', u_m, u_m'\big)\big| \leqslant C\big\|u_m'\big\|_{L^2}^{1/2}\|u_m\|_{H^1}\big\|u_m'\big\|_{H^1}^{3/2}$$
$$\leqslant \gamma\|u_m\|_{H^1}\big\|u_m'\big\|_{H^1}^2 + C\|u_m\|_{H^1}\big\|u_m'\big\|_{L^2}^2,$$

for a constant $\gamma > 0$ which can be arbitrarily small. We choose γ satisfying $\gamma\|u_0\|_{H^1} \leqslant \frac{\alpha}{24}$, and from (3.16) we obtain $\gamma\|u_0^\epsilon\|_{H^1} \leqslant \frac{\alpha}{12}$. Let $T' > 0$, which may depend on m, ϵ at this stage, be the maximum value of t such that $\gamma\|u_m(t)\|_{H^1} \leqslant \frac{\alpha}{6}$. If $\gamma\|u_m(t)\|_{H^1} < \frac{\alpha}{6}$ for all $0 \leqslant t \leqslant T$, we set $T' = T$. Since $\gamma\|u_m(0)\|_{H^1} < \frac{\alpha}{6}$ and $u_m(t)$ is continuous with respect to t, such T' does exist, and furthermore if $T' < T$ then $\gamma\|u_m(t)\|_{H^1} = \frac{\alpha}{6}$.

Therefore, in place of (3.22) we obtain

$$\frac{d}{dt}\big\|u_m'\big\|_{L^2}^2 + \alpha\big\|u_m'\big\|_{H^1}^2 \leqslant C\big(\big\|f'\big\|_{L^2}^2 + \big\|g'\big\|_{L^2}^2\big) + C\|u_m\|_{H^1}\big\|u_m'\big\|_{L^2}^2 \quad \big(0 \leqslant t \leqslant T'\big),$$

which leads to the boundedness of $\|u_m'\|_{L^2(0,T';V_{n,\sigma})}$ and $\|u_m'\|_{L^\infty(0,T';L^2)}$, together with $\|u_m\|_{L^\infty(0,T';V_{n,\sigma})}$.

Finally, let us prove that T' is bounded from below independently of m and ϵ. In fact, if $T' < T$ then we see that

$$\frac{\alpha}{12\gamma} \leqslant \big\|u_m\big(T'\big)\big\|_{H^1} - \big\|u_m(0)\big\|_{H^1} \leqslant \big\|u_m\big(T'\big) - u_m(0)\big\|_{H^1} = \left\|\int_0^{T'} u_m'(t)\,dt\right\|_{H^1}$$
$$\leqslant \int_0^{T'}\big\|u_m'(t)\big\|_{H^1}\,dt \leqslant \sqrt{T'}\big\|u_m'\big\|_{L^2(0,T';V_{n,\sigma})}.$$

Since we already know $\|u_m'\|_{L^2(0,T';V_{n,\sigma})}$ is bounded, we obtain the lower bound for T'. This completes the proof of Proposition 3.4. □

Remark 3.5.

(i) A naive computation gives, by (3.8),

$$\left|\int_{\Gamma_1} g(0)\alpha_\epsilon\big(u_{0\tau}^\epsilon\big)\cdot u_{m\tau}'(0)\,ds\right| \leqslant \big\|g(0)\big\|_{L^2(\Gamma_1)}\big\|u_{m\tau}'(0)\big\|_{L^2(\Gamma_1)^d},$$

but $\|u_{m\tau}'(0)\|_{L^2(\Gamma_1)^d}$ cannot be bounded by $\|u_m'(0)\|_{L^2(\Omega)^d}$ in general. Therefore, the perturbation of u_0, which is based on the compatibility condition in (S3), is essential in deriving (3.25).

(ii) If $d = 3$ and f, g, u_0 are sufficiently small, we can prove $\gamma \|u_m(t)\|_{H^1(\Omega)^d} \leqslant \frac{\alpha}{6}$ for all $0 \leqslant t \leqslant T$, and consequently the existence of a global solution.

As a final step for our proof of the existence, we discuss passing to the limits $m \to \infty$ and $\epsilon \to 0$. The proof below is valid for both $d = 2, 3$, except that when $d = 3$ we have to replace T with T' given in Proposition 3.4.

Proposition 3.5.

(i) *Under the assumptions of Proposition 3.4, there exists a solution u_ϵ of Problem $\mathrm{VI}_\sigma^\epsilon$ such that all of $\|u_\epsilon\|_{L^\infty(0,T;V_{n,\sigma})}$, $\|u'_\epsilon\|_{L^2(0,T;V_{n,\sigma})}$, and $\|u'_\epsilon\|_{L^\infty(0,T;L^2(\Omega)^d)}$ are bounded independently of ϵ.*
(ii) *There exists a strong solution of Problem VI_σ.*

Proof. (i) As a consequence of Proposition 3.4, we can extract a subsequence of $\{u_m\}_{m=1}^\infty$, denoted by the same symbol, such that

$$u_m \rightharpoonup u_\epsilon \quad \text{weakly-}* \text{ in } L^\infty(0, T; V_{n,\sigma}),$$
$$u'_m \rightharpoonup u'_\epsilon \quad \text{weakly in } L^2(0, T; V_{n,\sigma}) \text{ and weakly-}* \text{ in } L^\infty\big(0, T; L^2(\Omega)^d\big),$$

for some $u_\epsilon \in L^\infty(0, T; V_{n,\sigma})$, $u'_\epsilon \in L^2(0, T; V_{n,\sigma}) \cap L^\infty(0, T; L^2(\Omega)^d)$. The norms of u_ϵ and u'_ϵ in those spaces are uniformly bounded in ϵ.

Let us prove u_ϵ solves Problem $\mathrm{VI}_\sigma^\epsilon$. By Proposition 3.3, it suffices to show u_ϵ solves Problem $\mathrm{VE}_\sigma^\epsilon$. For $\phi \in C_0^\infty(0, T)$, it follows from (3.17) that

$$\int_0^T \phi(t)\left\{\big(u'_m, w_k\big) + a_0(u_m, w_k) + a_1(u_m, u_m, w_k) + \int_{\Gamma_1} g\alpha_\epsilon(u_{m\tau}) \cdot w_{k\tau}\, ds - (f, w_k)\right\} dt = 0$$
$$(k = 1, \ldots, m). \tag{3.26}$$

By standard compactness results (see [30, Theorem III.2.1], [27, Theorem II.6.2]), $u_m \to u_\epsilon$ strongly in $L^2(0, T; L^4(\Omega)^d)$ and $u_{m\tau} \to u_{\epsilon\tau}$ strongly in $L^2(\Gamma_1 \times (0, T))^d$. In particular, $u_{m\tau} \to u_{\epsilon\tau}$ a.e. on $\Gamma_1 \times (0, T)$, and thus the continuity of $\alpha_\epsilon(z)$ yields $\alpha_\epsilon(u_{m\tau}) \to \alpha_\epsilon(u_{\epsilon\tau})$ a.e. From Lebesgue's convergence theorem combined with a density argument, we see that (3.26) holds, with u_m and w_k replaced by u_ϵ and arbitrary $v \in V_{n,\sigma}$ respectively. Hence (3.11) holds for a.e. t, which implies that u_ϵ solves Problem $\mathrm{VE}_\sigma^\epsilon$.

(ii) As a result of (i), we can extract a subsequence of $\{u_\epsilon\}_{\epsilon\downarrow 0}$, denoted by the same symbol, such that

$$u_\epsilon \rightharpoonup u \quad \text{weakly-}* \text{ in } L^\infty(0, T; V_{n,\sigma}),$$
$$u'_\epsilon \rightharpoonup u' \quad \text{weakly in } L^2(0, T; V_{n,\sigma}) \text{ and weakly-}* \text{ in } L^\infty\big(0, T; L^2(\Omega)^d\big),$$

for some $u \in L^\infty(0, T; V_{n,\sigma})$, $u' \in L^2(0, T; V_{n,\sigma}) \cap L^\infty(0, T; L^2(\Omega)^d)$. As before, one sees that $u_\epsilon \to u$ strongly in $L^2(0, T; L^4(\Omega)^d)$ and $u_{\epsilon\tau} \to u_\tau$ strongly in $L^2(\Gamma_1 \times (0, T))$. In addition, $u_\epsilon \rightharpoonup u$ weakly in $L^2(0, T; V_{n,\sigma})$, and thus it follows that $\int_0^T a_0(u, u)\, dt \leqslant \underline{\lim}_{\epsilon\to 0} \int_0^T a_0(u_\epsilon, u_\epsilon)\, dt$.

Let $\tilde{v} \in L^2(0, T; V_{n,\sigma})$ be arbitrary. We take $v = \tilde{v}(t)$ in (3.10) and integrate the resulting equation over $(0, T)$ to deduce

$$\int_0^T \big\{ \big(u'_\epsilon, \tilde{v} - u_\epsilon\big) + a_0(u_\epsilon, \tilde{v} - u_\epsilon) + a_1(u_\epsilon, u_\epsilon, \tilde{v} - u_\epsilon) \\ + j_\epsilon(\tilde{v}_\tau) - j_\epsilon(u_{\epsilon\tau}) - (f, \tilde{v} - u_\epsilon)\big\}\, dt \geqslant 0. \tag{3.27}$$

In view of (3.7), together with triangle inequality and Lipschitz continuity of j, we have $\int_0^T j_\epsilon(\tilde{v}_\tau)\, dt \to \int_0^T j(\tilde{v}_\tau)\, dt$ and $\int_0^T j_\epsilon(u_{\epsilon\tau})\, dt \to \int_0^T j(u_\tau)\, dt$ when $\epsilon \to 0$. Therefore, taking the lower limit, we see that (3.27), with u_ϵ replaced by u, holds. Then, a technique using the Lebesgue differentiation theorem (see [5, p. 57]) enables us to conclude that u satisfies (3.2) at a.e. t.

For the initial condition, Lemma 3.1(i) leads to $u(0) = \lim_{\epsilon\to 0} u_\epsilon(0) = \lim_{\epsilon \to 0} u_0^\epsilon = u_0$. Hence u is a strong solution of Problem VI$_\sigma$. □

Propositions 3.1 and 3.5(ii) complete the proof of Theorem 3.2.

4. Navier–Stokes problem with LBCF

4.1. Weak formulations

Throughout this section, we assume $f \in L^2(\Omega \times (0,T))^d$, $u_0 \in V_{\tau,\sigma}$, and $g \in L^2(\Gamma_1 \times (0,T))$ with $g > 0$. Further regularity assumptions on these data will be given before Theorem 4.2. As in SBCF, we introduce

$$j_n(t;\eta) = \int_{\Gamma_1} g(t)|\eta|\, ds \quad \big(\eta \in L^2(\Gamma_1)\big), \tag{4.1}$$

which is simply written as $j(\eta)$ until the end of this section (note that η is scalar). A primal weak formulation of (1.1)–(1.4) with (1.6) is as follows:

Problem PDE-LBCF. For a.e. $t \in (0,T)$, find $(u(t), p(t)) \in V_\tau \times Q$ such that $u'(t) \in L^2(\Omega)^d$, $u(0) = u_0$, σ_n is well-defined in the sense of Definition 2.1, $|\sigma_n| \leqslant g$ a.e. on Γ_1, and $\sigma_n u_n + g|u_n| = 0$ a.e. on Γ_1.

Throughout this section, we refer to Problem PDE-LBCF just as Problem PDE. Similar abbreviation will be made for other problems. Next, as in SBCF, we propose a variational inequality problem:

Problem VI$_\sigma$-LBCF. For a.e. $t \in (0,T)$, find $u(t) \in V_{\tau,\sigma}$ such that $u'(t) \in L^2(\Omega)^d$, $u(0) = u_0$ and

$$\big(u', v - u\big) + a_0(u, v-u) + a_1(u,u,v-u) + j(v_n) - j(u_n) \geqslant (f, v-u) \tag{4.2}$$

for all $v \in V_{\tau,\sigma}$. Here $j = j_n(t;\cdot)$ is defined in (4.1).

Unlike the case of SBCF, Problem VI$_\sigma$ is not exactly equivalent to Problem PDE, as is shown in the following theorem.

Theorem 4.1.

(i) *If (u,p) solves Problem* PDE, *then u solves Problem* VI$_\sigma$.
(ii) *If u solves Problem* VI$_\sigma$, *then there exists at least one p such that (u,p) solves Problem* PDE. *If another p^* satisfies the same condition, then for a.e. $t \in (0,T)$ there exists a unique $\delta(t) \in \mathbb{R}$ such that*

$$p(t) = p^*(t) + \delta(t) \quad \textit{and} \quad \sigma_n\big(u(t), p(t)\big) = \sigma_n\big(u(t), p^*(t)\big) - \delta(t). \tag{4.3}$$

(iii) *In* (ii)*, if we assume furthermore* $u_n(t) \neq 0$*, then* $\delta(t) = 0$*. Namely, the associated pressure is uniquely determined.*

Proof. (i) This can be proved by the same way as Theorem 3.1.

(ii) For a.e. $t \in (0, T)$ and $v \in \mathring{V}_\sigma$, it follows from (4.2) that $(u', v) + a_0(u, v) + a_1(u, u, v) = (f, v)$, and thus there exists unique $\mathring{p} \in \mathring{Q}$ such that

$$\big(u', v\big) + a_0(u, v) + a_1(u, u, v) + b(v, \mathring{p}) = (f, v) \quad (\forall v \in \mathring{V}).$$

According to Definition 2.1, $\mathring{\sigma}_n = \sigma_n(u, \mathring{p})$ is well-defined, so that

$$\big(u', v\big) + a_0(u, v) + b(v, \mathring{p}) + a_1(u, u, v) - \langle \mathring{\sigma}_n, v_n \rangle_{H^{1/2}(\Gamma_1)} = (f, v) \quad (\forall v \in V_\tau).$$

Substituting this equation into (4.2), we obtain $-\langle \mathring{\sigma}_n, v_n - u_n \rangle_{H^{1/2}(\Gamma_1)} \leqslant j(v_n) - j(u_n)$ for all $v \in V_{\tau,\sigma}$. It follows from Lemma 2.4(ii) that

$$\big|\langle \mathring{\sigma}_n, \eta \rangle_{H^{1/2}(\Gamma_1)}\big| \leqslant \int_{\Gamma_1} g|\eta|\, ds \quad \big(\forall \eta \in H^{1/2}(\Gamma_1) \cap L^2_0(\Gamma_1)\big).$$

The Hahn–Banach theorem allows us to extend $\mathring{\sigma}_n$ to a linear functional $\sigma_n : L^1_g(\Gamma_1) \to \mathbb{R}$ satisfying the same inequality as above for all $\eta \in L^1_g(\Gamma_1)$. Therefore, $\sigma_n \in L^\infty_{1/g}(\Gamma_1)$ and $|\sigma_n| \leqslant g$. In addition, $\sigma_n u_n + g|u_n| = 0$ follows.

Since $\mathring{\sigma}_n - \sigma_n$ vanishes on $H^{1/2}(\Gamma_1) \cap L^2_0(\Gamma_1)$, there exists a constant $\delta(t)$ such that $\mathring{\sigma}_n - \sigma_n = \delta(t)$. Now, by setting $p(t) = \mathring{p}(t) + \delta(t)$, it follows that σ_n given above actually equals $\sigma_n(u(t), p(t))$ and that $(u(t), p(t))$ solves Problem PDE. Relation (4.3) can be verified by a similar argument.

(iii) Since $\int_{\Gamma_1} u_n\, ds = \int_\Omega \operatorname{div} u\, dx = 0$, the assumption $u_n(t) \neq 0$ implies that there exist subsets A_+, A_- of Γ_1 with positive $d-1$ dimensional Lebesgue measure satisfying $u_n(t) > 0$ on A_+ and $u_n(t) < 0$ on A_-. Because $|\sigma_n| \leqslant g$ and $\sigma_n u_n + g|u_n| = 0$ on Γ_1, $\sigma_n = -g(t)$ on A_+ and $\sigma_n = g(t)$ on A_-. Hence $\delta(t)$ in (4.3) cannot be other than zero. □

Remark 4.1. Since $|\sigma_n| \leqslant g$, $\delta(t)$ is no more than $2g(t)$ nor less than $-2g(t)$.

4.2. Main theorem

Let us state our main theorems for the case of LBCF. As in SBCF, some compatibility condition is necessary; it is rather complicated because normal stress at $t = 0$ involves a pressure at $t = 0$, which is not given as a data. The precise description is as follows: we say that LBCF is satisfied at $t = 0$ if $u_0 \in H^2(\Omega)^d \cap V_{\tau,\sigma}$ and there exists $p_0 \in H^1(\Omega)$ such that

$$\big|\sigma_n(u_0, p_0)\big| \leqslant g(0) \quad \text{and} \quad \sigma_n(u_0, p_0)u_{0n} + g(0)|u_{0n}| = 0 \quad \text{a.e. on } \Gamma_1. \tag{4.4}$$

We remark that a similar compatibility condition appears in nonlinear semigroup approaches (see [7,8]).

Furthermore, in order to overcome a difficulty arising from (1.7), we need no-leak condition at $t = 0$, that is, $u_{0n} = 0$ on Γ_1. In view of (4.4), this is automatically satisfied if $|\sigma_n(u_0, p_0)| < g(0)$ on Γ_1. Examining our proof of the a priori estimates carefully, one finds that this assumption can be weaken to the condition that $\|u_{0n}\|_{L^2(\Gamma_1)}$ is sufficiently small.

Including what we have discussed above, we assume the following:

(L1) $f \in H^1(0, T; L^2(\Omega)^d)$.
(L2) $g \in H^1(0, T; L^2(\Gamma_1))$ with $g(0) \in H^1(\Gamma_1)$.

(L3) $u_0 \in H^2(\Omega)^d \cap V_{\tau,\sigma}$, and LBCF is satisfied at $t = 0$.
(L4) $u_{0n} = 0$ a.e. on Γ_1.

Theorem 4.2. *Under* (L1)–(L4) *above, there exists a unique solution u of Problem* VI_σ *on some interval* $(0, T')$, *with* $T' \leqslant T$, *such that*

$$u \in L^\infty\big(0, T'; V_{\tau,\sigma}\big), \quad u' \in L^\infty\big(0, T'; L^2(\Omega)^d\big) \cap L^2\big(0, T'; V_{\tau,\sigma}\big).$$

The uniqueness can be proved by the same way as Proposition 3.1. We can also obtain $p \in L^\infty(0, T'; L^2(\Omega))$ by a similar manner to Proposition 3.2, using the rather infamous inf–sup condition (see [28, Lemma 2.2])

$$C\|p\|_{L^2(\Omega)} \leqslant \sup_{v \in V_\tau} \frac{b(v, p)}{\|v\|_{H^1(\Omega)^d}} \quad \big(\forall p \in L^2(\Omega)\big).$$

The rest of this section is devoted to the proof of the existence. To state regularized problems, for fixed $\epsilon > 0$ we introduce

$$j_\epsilon(\eta) = \int_{\Gamma_1} g\rho_\epsilon(\eta)\, ds \quad \big(\eta \in L^2(\Gamma_1)\big),$$

where ρ_ϵ is a function satisfying properties (a)–(c) for the case $d = 1$, considered at the beginning of Section 3.3. We use the notation introduced there such as $\alpha_\epsilon = d\rho/dz$ and $\beta_\epsilon = d^2\rho/dz^2$.

Now let us state the regularized problems.

Problem $\mathrm{VI}_\sigma^\epsilon$-LBCF. For a.e. $t \in (0, T)$, find $u_\epsilon(t) \in V_{\tau,\sigma}$ such that $u'_\epsilon(t) \in L^2(\Omega)^d$, $u_\epsilon(0) = u_0^\epsilon$ and

$$\begin{aligned}&\big(u'_\epsilon, v - u_\epsilon\big) + a_0(u_\epsilon, v - u_\epsilon) + a_1(u_\epsilon, u_\epsilon, v - u_\epsilon) + j_\epsilon(v_n) - j_\epsilon(u_{\epsilon n})\\ &\quad \geqslant (f, v - u_\epsilon) \quad (\forall v \in V_{\tau,\sigma}).\end{aligned}$$

Problem $\mathrm{VE}_\sigma^\epsilon$-LBCF. For a.e. $t \in (0, T)$, find $u_\epsilon(t) \in V_{\tau,\sigma}$ such that $u'_\epsilon(t) \in L^2(\Omega)^d$, $u_\epsilon(0) = u_0^\epsilon$ and

$$\big(u'_\epsilon, v\big) + a_0(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + \int_{\Gamma_1} g\alpha_\epsilon(u_{\epsilon n})v_n\, ds = (f, v) \quad (\forall v \in V_{\tau,\sigma}).$$

As in Proposition 3.3, Problems $\mathrm{VI}_\sigma^\epsilon$ and $\mathrm{VE}_\sigma^\epsilon$ are equivalent. The construction of the perturbed initial velocity u_0^ϵ is similar to that of SBCF. In fact, since LBCF holds at $t = 0$ by (L3), the Green formula leads to

$$a_0(u_0, v - u_0) + \int_{\Gamma_1} g(0)|v_n|\, ds - \int_{\Gamma_1} g(0)|u_{0n}|\, ds \geqslant (-\nu\Delta u_0 + \nabla p_0, v - u_0),$$

for $v \in V_{\tau,\sigma}$. Consider the regularized problem: find $u_0^\epsilon \in V_{\tau,\sigma}$ such that

$$\begin{aligned}&a_0\big(u_0^\epsilon, v - u_0^\epsilon\big) + \int_{\Gamma_1} g(0)\rho_\epsilon(v_n)\, ds - \int_{\Gamma_1} g(0)\rho_\epsilon\big(u_{0n}^\epsilon\big)\, ds\\ &\quad \geqslant \big(-\nu\Delta u_0 + \nabla p_0, v - u_0^\epsilon\big) \quad (\forall v \in V_{\tau,\sigma}),\end{aligned} \tag{4.5}$$

which is equivalent to (cf. Proposition 3.3)

$$a_0\big(u_0^\epsilon, v\big) + \int_{\Gamma_1} g(0)\alpha_\epsilon\big(u_{0n}^\epsilon\big) v_n \, ds = (-\nu\Delta u_0 + \nabla p_0, v) \quad (\forall v \in V_{\tau,\sigma}). \tag{4.6}$$

The elliptic variational inequality (4.5) admits a unique solution u_0^ϵ, which is the perturbation of u_0 in question. With this setting, we find:

Lemma 4.1.

(i) *When $\epsilon \to 0$, $u_0^\epsilon \to u_0$ strongly in $H^1(\Omega)^d$. In particular, it follows that $u_0^\epsilon \to 0$ in $L^2(\Gamma_1)$.*
(ii) *$u_0^\epsilon \in H^2(\Omega)^d$ and*

$$\big\|u_0^\epsilon\big\|_{H^2(\Omega)^d} \leqslant C\big(\|\nu\Delta u_0 + \nabla p_0\|_{L^2(\Omega)^d} + \big\|g(0)\big\|_{H^1(\Gamma_1)}\big). \tag{4.7}$$

Proof. (i) is proved by the same way as Lemma 3.1(i). Since $g(0) \in H^1(\Gamma_1)$ by (L3), (ii) is a direct consequence of [28, Lemma 4.1]. $\square$

Remark 4.2. By (i) and (L4), for sufficiently small $\epsilon > 0$ we have

$$\big\|u_0^\epsilon\big\|_{L^2(\Omega)^d} \leqslant 2\|u_0\|_{L^2(\Omega)^d}, \qquad \big\|u_0^\epsilon\big\|_{H^1(\Omega)^d} \leqslant 2\|u_0\|_{H^1(\Omega)^d}, \qquad \big\|u_{0n}^\epsilon\big\|_{L^2(\Gamma_1)} \leqslant \frac{\alpha}{8\gamma_1}, \tag{4.8}$$

where α and γ_1 are the constants in (2.1) and (2.5) respectively.

Remark 4.3. As in SBCF, if Γ_0 is C^2 and Γ_1 is C^4, then we can apply Lemma 4.1 of [28]. On the other hand, $g(0) \in H^{1/2}(\Gamma_1)$, stated in [28], is actually insufficient to deduce the H^2–H^1 regularity (see the errata of [28]).

To solve Problem $\mathrm{VE}_\sigma^\epsilon$, we construct approximate solutions by Galerkin's method. Since $V_{\tau,\sigma} \subset H^1(\Omega)^d$ is separable, there exist $w_1, w_2, \ldots \in V_{\tau,\sigma}$, linear independent to each other, such that $\bigcup_{m=1}^\infty \operatorname{span}\{w_k\}_{k=1}^m \subset V_{\tau,\sigma}$ dense in $H^1(\Omega)^d$. Here we may assume $w_1 = u_0^\epsilon$.

Problem $\mathbf{VE}_\sigma^{\epsilon,m}$-LBCF. Find $c_k \in C^2([0,T])$ $(k = 1, \ldots, m)$ such that $u_m \in V_{\tau,\sigma}$ defined by $u_m = \sum_{k=1}^m c_k(t) w_k$ satisfies $u_m(0) = u_0^\epsilon$ and

$$\big(u_m', w_k\big) + a_0(u_m, w_k) + a_1(u_m, u_m, w_k) + \int_{\Gamma_1} g\alpha_\epsilon(u_{mn}) w_{kn} \, ds = (f, w_k) \quad (k = 1, \ldots, m). \tag{4.9}$$

Since $\alpha_\epsilon \in C^1(\mathbb{R})$, there exist unique solutions $c_k \in C^2([0,\tilde{T}])$ $(k = 1, \ldots, m)$ for some $\tilde{T}$, which may depend on m and ϵ at this stage.

Proposition 4.1. *Assume* (L1)–(L4), *and let $\epsilon > 0$ be sufficiently small so that* (4.8) *holds. Then there exists some interval $(0, T')$ such that $u_m \in L^\infty(0,T'; V_{\tau,\sigma})$ and $u_m' \in L^\infty(0,T'; L^2(\Omega)^d) \cap L^2(0,T'; V_{\tau,\sigma})$ are uniformly bounded with respect to m and ϵ. Here, T' is independent of m and ϵ.*

Proof. Due to space limitations, we sometimes simply write $\|u\|_{L^2}, \|g\|_{L^2}, \ldots,$ instead of $\|u\|_{L^2(\Omega)^2}$, $\|g\|_{L^2(\Gamma_1)}, \ldots,$ when there is no fear of confusion.

First we consider the case $d = 2$. Multiplying (4.9) by $c_k(t)$ for $k = 1, \dots, m$, adding them, and using (2.1), (2.5) and (3.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m\|_{L^2}^2 + \big(\alpha - \gamma_1\|u_{mn}\|_{L^2(\Gamma_1)}\big)\|u_m\|_{H^1}^2 \leqslant (f, u_m). \tag{4.10}$$

Since $\|u_{mn}(t)\|_{L^2(\Gamma_1)}$ is continuous with respect to t and (4.8) holds, there exists a maximum value $T_1 \in (0, \tilde{T}]$ of t such that $\gamma_1\|u_{mn}(t)\|_{L^2(\Gamma_1)} \leqslant \frac{\alpha}{4}$. If this inequality holds for all $0 \leqslant t \leqslant \tilde{T}$, we take $T_1 = \tilde{T}$. Noting $|(f, u_m)| \leqslant \frac{\alpha}{4}\|u_m\|_{H^1}^2 + \frac{1}{\alpha}\|f\|_{L^2}^2$, we find from (4.10) that

$$\frac{d}{dt}\|u_m\|_{L^2}^2 + \alpha\|u_m\|_{H^1}^2 \leqslant C\|f\|_{L^2}^2 \quad (0 \leqslant t \leqslant T_1).$$

Hence $u_m \in L^\infty(0, T_1; L^2) \cap L^2(0, T_1; V_{\tau,\sigma})$ is uniformly bounded in m and ϵ.

Next, differentiating (4.9), multiplying the resulting equation by $c_k'(t)$, and adding them, we obtain

$$\begin{aligned}
&\big(u_m'', u_m'\big) + a_0\big(u_m', u_m'\big) + a_1\big(u_m', u_m, u_m'\big) + a_1\big(u_m, u_m', u_m'\big) \\
&\quad + \int_{\Gamma_1} g'\alpha_\epsilon(u_{mn})u_{mn}'\,ds + \int_{\Gamma_1} g\beta_\epsilon(u_{mn})\big|u_{mn}'\big|^2\,ds \\
&= \big(f', u_m'\big).
\end{aligned} \tag{4.11}$$

Here, we estimate each term in (4.11) as follows:

$$\begin{aligned}
\big|a_1\big(u_m', u_m, u_m'\big)\big| &\leqslant C\big\|u_m'\big\|_{L^2}\|u_m\|_{H^1}\big\|u_m'\big\|_{L^2} \\
&\leqslant \frac{\alpha}{12}\big\|u_m'\big\|_{H^1}^2 + C\|u_m\|_{H^1}^2\big\|u_m'\big\|_{L^2},
\end{aligned} \tag{4.12}$$

$$\big|a_1\big(u_m, u_m', u_m'\big)\big| \leqslant \gamma_1\|u_{mn}\|_{L^2(\Gamma_1)}\big\|u_m'\big\|_{H^1}^2 \leqslant \frac{\alpha}{4}\big\|u_m'\big\|_{H^1}^2,$$

$$\left|\int_{\Gamma_1} g'\alpha_\epsilon(u_{mn})u_{mn}'\,ds\right| \leqslant C\big\|g'\big\|_{L^2}\big\|u_m'\big\|_{H^1} \leqslant \frac{\alpha}{12}\big\|u_m'\big\|_{H^1}^2 + C\big\|g'\big\|_{L^2}^2,$$

$$\int_{\Gamma_1} g\beta_\epsilon(u_{mn})\big|u_{mn}'\big|^2\,ds \geqslant 0,$$

$$\big|\big(f', u_m'\big)\big| \leqslant \frac{\alpha}{12}\big\|u_m'\big\|_{H^1}^2 + C\big\|f'\big\|_{L^2}^2.$$

Collecting these estimates, we derive from (4.11) that for $0 \leqslant t \leqslant T_1$

$$\frac{d}{dt}\big\|u_m'\big\|_{L^2} + \alpha\big\|u_m'\big\|_{H^1}^2 \leqslant C\big(\big\|f'\big\|_{L^2}^2 + \big\|g'\big\|_{L^2}^2\big) + C\|u_m\|_{H^1}^2\big\|u_m'\big\|_{L^2}^2. \tag{4.13}$$

Combining the technique used in Proposition 3.4 with (4.6) and (4.7), we observe that $\|u_m'\|_{L^\infty(0,T_1;L^2)}$, $\|u_m'\|_{L^2(0,T_1;V_{\tau,\sigma})}$, and $\|u_m\|_{L^\infty(0,T_1;V_{\tau,\sigma})}$ are bounded by $C(f, g, u_0, p_0)$.

It remains to show that T_1 is bounded from below independently of m, ϵ. If $\gamma_1\|u_{mn}(T_1)\|_{L^2(\Gamma_1)} < \alpha/4$ and thus $T_1 = \tilde{T}$, we can extend $u_m(t)$ beyond $t = \tilde{T}$ and repeat the above discussion until we reach either

$$\max_{0\leqslant t\leqslant T} \gamma_1 \left\|u_{mn}(t)\right\|_{L^2(\Gamma_1)} \leqslant \alpha/4 \quad \text{or} \quad \gamma_1 \left\|u_{mn}(T_1)\right\|_{L^2(\Gamma_1)} = \alpha/4.$$

In the former case $T_1 = T$. In the latter case, we have

$$\begin{aligned} \frac{\alpha}{8\gamma_1} &\leqslant \left\|u_{mn}(T_1)\right\|_{L^2(\Gamma_1)} - \left\|u_{mn}(0)\right\|_{L^2(\Gamma_1)} \leqslant \left\|u_{mn}(T_1) - u_{mn}(0)\right\|_{L^2(\Gamma_1)} \\ &\leqslant \int_0^{T_1} \left\|u'_{mn}(t)\right\|_{L^2(\Gamma_1)} dt \leqslant C \int_0^{T_1} \left\|u'_m\right\|_{H^1(\Omega)^d} dt \leqslant C\sqrt{T_1}\left\|u'_m\right\|_{L^2(0,T_1;V_{\tau,\sigma})}. \end{aligned}$$

Hence T_1 is bounded from below, and we complete the proof for $d = 2$.

Second let us consider the case $d = 3$. What changes from $d = 2$ is that (4.12) is replaced with

$$\begin{aligned} \left|a_1\left(u'_m, u_m, u'_m\right)\right| &\leqslant C\left\|u'_m\right\|_{L^2}^{1/2} \|u_m\|_{H^1} \left\|u'_m\right\|_{H^1}^{3/2} \\ &\leqslant \gamma_2 \|u_m\|_{H^1} \left\|u'_m\right\|_{H^1}^2 + C\|u_m\|_{H^1}\left\|u'_m\right\|_{L^2}^2, \end{aligned}$$

where γ_2 can be arbitrarily small. We choose γ_2 satisfying $\gamma_2\|u_0\|_{H^1} \leqslant \frac{\alpha}{48}$, so that $\gamma_2\|u_0^\epsilon\|_{H^1} \leqslant \frac{\alpha}{24}$ by virtue (4.8). Let T_2 be the maximum value of $t \in (0, \tilde{T}]$ such that $\gamma_2\|u_m(t)\|_{H^1} \leqslant \frac{\alpha}{12}$. If this inequality holds for all $t \in (0, \tilde{T}]$, we set $T_2 = \tilde{T}$. Such T_2 does exist, and if $T_2 < \tilde{T}$ then $\gamma_2\|u_m(T_2)\|_{H^1} = \frac{\alpha}{12}$.

Therefore, setting $T' = \min(T_1, T_2)$, instead of (4.13) we get

$$\frac{d}{dt}\left\|u'_m\right\|_{L^2} + \alpha\left\|u'_m\right\|_{H^1}^2 \leqslant C\left(\left\|f'\right\|_{L^2}^2 + \left\|g'\right\|_{L^2}^2\right) + C\|u_m\|_{H^1}\left\|u'_m\right\|_{L^2}^2 \quad \left(0 \leqslant t \leqslant T'\right).$$

As a consequence, we observe that $\|u'_m\|_{L^2(0,T';V_{\tau,\sigma})}$, $\|u'_m\|_{L^\infty(0,T';L^2)}$, and $\|u_m\|_{L^\infty(0,T';V_{\tau,\sigma})}$ are bounded by $C(f, g, u_0, p_0)$.

Now, if $T_1 < \tilde{T}$ or $T_2 < \tilde{T}$ then T' are bounded from below as follows:

$$\begin{aligned} \frac{\alpha}{12\gamma_1} &\leqslant \left\|u_{mn}\left(T'\right)\right\|_{L^2(\Gamma_1)} - \left\|u_{mn}(0)\right\|_{L^2(\Gamma_1)} \leqslant \int_0^{T'} \left\|u'_{mn}\right\|_{L^2(\Gamma_1)} dt \\ &\leqslant C\int_0^{T'} \left\|u'_m\right\|_{H^1} dt \leqslant C\sqrt{T_1}\left\|u'_m\right\|_{L^2(0,T';V_{\tau,\sigma})}, \\ \frac{\alpha}{24\gamma_2} &\leqslant \left\|u_m\left(T'\right)\right\|_{H^1} - \left\|u_m(0)\right\|_{H^1} \leqslant \int_0^{T'} \left\|u'_m\right\|_{H^1} dt \leqslant \sqrt{T'}\left\|u'_m\right\|_{L^2(0,T';V_{\tau,\sigma})}. \end{aligned}$$

When $T_1 = \tilde{T}$ and $T_2 = \tilde{T}$, we can extend $u_m(t)$ beyond $t = \tilde{T}$ and repeat the above discussion. This completes the proof of Proposition 4.1. □

The last step of the proof, i.e. passing to the limits $m \to \infty$ and $\epsilon \to 0$, proceeds in the same way as Proposition 3.5, with n replaced by τ and vice versa. This proves that a solution of Problem VI_σ exists, which, combined with the uniqueness result, completes the proof of Theorem 4.2.

Remark 4.4. At first glance one may think Theorem 4.2, where we get only a time-local solution in spite of a smallness assumption on u_0 even if $d = 2$, is too poor. However, in view of the fact that we obtain only time-local solutions in 2D case under the linear leak b.c. (see [12, Theorem 6] or [26]), such limitations cannot be avoided to some extent.

Remark 4.5. Under additional smallness assumptions on the data f, g, u_0, p_0, we can derive global existence results for both $d = 2$ and $d = 3$.

5. Concluding remarks

By the discussion presented above, we have established the existence and uniqueness, while we did not get in touch with higher regularity, such as $u \in L^\infty(0, T; H^2(\Omega)^d)$, $p \in L^\infty(0, T; H^1(\Omega))$. This is because some regularity results for the elliptic cases are not available. For instance, Problem VI$_\sigma$-SBCF is rewritten as

$$a_0(u, v-u) + j_\tau(t; v_\tau) - j_\tau(t; u_\tau) \geqslant (f, v-u) - \left(u', v-u\right) - a_1(u, u, v-u)$$
$$=: \left\langle F(t), v-u \right\rangle_{V_{n,\sigma}} \quad (\forall v \in V_{n,\sigma}),$$

with $F(t) \in L^p(\Omega)^d$ for some $p < 2$. If we prove this elliptic variational inequality has a unique solution in $W^{2,p}(\Omega)^d$ when $p < 2$, then a technique similar to [30, Theorems III.3.6 and III.3.8] allows us to deduce $u(t) \in H^2(\Omega)^d$. Thereby, we need to extend the regularity theory of [28] to cases $p \neq 2$.

Acknowledgments

The author would like to thank for Professor Norikazu Saito for encouraging him through valuable discussions. This work was supported by Grant-in-Aid for JSPS Fellows and CREST, JST.

References

[1] R. An, Y. Li, K. Li, Solvability of Navier–Stokes equations with leak boundary conditions, Acta Math. Appl. Sin. Engl. Ser. 25 (2009) 225–234.
[2] M. Ayadi, M.K. Gdoura, T. Sassi, Mixed formulation for Stokes problem with Tresca friction, C. R. Math. Acad. Sci. Paris 348 (2010) 1069–1072.
[3] A.Y. Chebotarev, Modeling of steady flows in a channel by Navier–Stokes variational inequalities, J. Appl. Mech. Tech. Phys. 44 (2003) 852–857.
[4] L. Consiglieri, Existence for a class of non-newtonian fluids with a nonlocal friction boundary condition, Acta Math. Sin. (Engl. Ser.) 22 (2006) 523–534.
[5] G. Duvaut, J.L. Lions, Les inéquations en Méchanique et en Physique, Dunod, 1972.
[6] H. Fujita, A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions, RIMS Kôkyûroku 888 (1994) 199–216.
[7] H. Fujita, Non-stationary Stokes flows under leak boundary conditions of friction type, J. Comput. Math. 19 (2001) 1–8.
[8] H. Fujita, A coherent analysis of Stokes flows under boundary conditions of friction type, J. Comput. Appl. Math. 149 (2002) 57–69.
[9] H. Fujita, H. Kawarada, A. Sasamoto, Analytical and numerical approaches to stationary flow problems with leak and slip boundary conditions, Lect. Notes Numer. Appl. Anal. 14 (1995) 17–31.
[10] V. Girault, P.A. Raviart, Finite Element Methods for Navier–Stokes Equations, Springer-Verlag, 1986.
[11] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer-Verlag, 1984.
[12] J.G. Heywood, R. Rannacher, R. Turek, Artificial boundaries and flux and pressure conditions for the incompressible Navier–Stokes equations, Internat. J. Numer. Methods Fluids 22 (1996) 325–352.
[13] T. Kashiwabara, On a finite element approximation of Stokes equations under slip boundary condition of friction type, submitted for publication.
[14] T. Kashiwabara, On a finite element approximation of Stokes equations under leak boundary condition of friction type, in preparation.
[15] H. Kawarada, H. Fujita, H. Suito, Wave motion breaking upon the shore, GAKUTO Internat. Ser. Math. Sci. Appl. 11 (1998) 145–159.
[16] N. Kikuchi, J.T. Oden, Contact Problems in Elasticity, SIAM, Philadelphia, 1988.
[17] D.S. Konovalova, Subdifferential boundary value problems for Navier–Stokes evolution equations, Differ. Equ. 36 (2000) 878–885.

[18] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, 1969.
[19] C. Le Roux, Steady Stokes flows with threshold slip boundary conditions, Math. Models Methods Appl. Sci. 15 (2005) 1141–1168.
[20] C. Le Roux, A. Tani, Steady solutions of the Navier–Stokes equations with threshold slip boundary conditions, Math. Methods Appl. Sci. 30 (2007) 595–624.
[21] Y. Li, R. An, Semi-discrete stabilized finite element methods for Navier–Stokes equations with nonlinear slip boundary conditions based on regularization procedure, Numer. Math. 117 (2011) 1–36.
[22] Y. Li, R. An, Two-level pressure projection finite element methods for Navier–Stokes equations with nonlinear slip boundary conditions, Appl. Numer. Math. 61 (2011) 285–297.
[23] Y. Li, K. Li, Penalty finite element method for Stokes problem with nonlinear slip boundary conditions, Appl. Math. Comput. 204 (2008) 216–226.
[24] Y. Li, K. Li, Locally stabilized finite element method for Stokes problem with nonlinear slip boundary conditions, J. Comput. Math. 28 (2010) 826–836.
[25] Y. Li, K. Li, Pressure projection stabilized finite element method for Navier–Stokes equations with nonlinear slip boundary conditions, Computing 87 (2010) 113–133.
[26] S. Marušić, On the Navier–Stokes system with pressure boundary condition, Ann. Univ. Ferrara Sez. VII Sci. Mat. 53 (2007) 319–331.
[27] J. Nečas, Direct Methods in the Theory of Elliptic Equations, Springer, 2012.
[28] N. Saito, On the Stokes equation with the leak or slip boundary conditions of friction type: regularity of solutions, Publ. Res. Inst. Math. Sci. 40 (2004) 345–383.
[29] H. Suito, H. Kawarada, Numerical simulation of spilled oil by fictitious domain method, Japan J. Indust. Appl. Math. 21 (2004) 219–236.
[30] R. Temam, Navier–Stokes Equations, North-Holland, 1977.