Discrete Mathematics 82 (1990) 57–78
North-Holland

REGION DISTRIBUTIONS OF GRAPH EMBEDDINGS AND STIRLING NUMBERS*

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Received 25 February 1988

It is shown that the distribution of the number of regions r in the random orientable embedding of the graph with one vertex and q loops is approximately proportional to the unsigned Stirling numbers of the first kind $s(2q, r)$ where r has different parity from q. This approximation is strong enough to imply that both the limiting mean and variance of this distribution differ from $\ln 2q$ by small known constants. The paper concludes with a result on the unimodality of some recursively defined sequences and also some conjectures regarding region distributions of arbitrary graphs.

0. Introduction

In a previous article [15] the author pointed out that the characters of the symmetric group contained a wealth of information regarding the distributions of the genera of orientable embeddings of certain classes of graphs. This approach was used by Gross, Robbins and Tucker [5] where they applied the recursion on the requisite character sums, obtained by Jackson [8], to show that the genus distribution of the bouquet of circles is strongly unimodal. In this paper we focus on the distribution of the number of regions in these embeddings of the bouquet. Some new bounds for the characters of the symmetric group are derived and these are used to show that the region distribution of the bouquet is asymptotically proportional to the (unsigned) Stirling numbers of the first kind. This approximation is strong enough to yield the results that the limiting mean and variance of the region distribution of the bouquet with q edges are both approximately $\ln 2q$.

In Section 1 we describe the region distribution of the orientable embeddings of all graphs on q edges. The relevance of the Stirling numbers is explained and they are used to obtain the mean and variance of this overall distribution. The region counts of the embeddings of the bouquet are converted into character sums in Section 2. The relevant characters are then bounded and these bounds are used to demonstrate the affinity between the region distribution of the bouquet on q loops and the region distribution of all the graphs on q edges. A variety of lemmas and bounds for the Stirling numbers are derived in Section 3. Finally, some afterthoughts and conjectures are discussed in Section 4.

* This research was supported in part by University of Kansas General Research Allocation #3544-20-0038.

1. Why Stirling numbers?

The word *graph* will be used here to denote what is commonly called a pseudograph; in other words, loops and multiple edges are allowed. All embeddings discussed here are assumed to be 2-cell and orientable. Otherwise, the graph theoretical terminology here agrees with that of [7, 18].

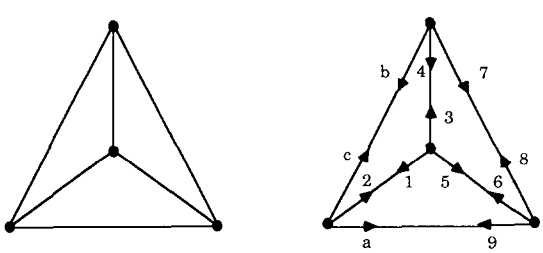
The permutations of the set $\{1, 2, 3, \ldots, n\}$ will usually be described by means of their disjoint cycle decomposition. The set underlying one of these disjoint cycles is called an *orbit*. The number of orbits of the permutation P is denoted by $\|P\|$, and the composition of permutations is to be read from left to right.

It is by now well known to graph theorists that orientable embeddings can be described by a variety of combinatorial structures. The method used here is the one that underlies the work of [2, 14, 15, 17] and others.

Let G be an arbitrary graph. Convert it to a digraph D by replacing each edge of G with a pair of oppositely directed arcs for D. Let S be the set of arcs of D and let Q be the product of all transpositions that associate arcs with their inverses. Then $\|Q\|$ is the number of edges of G. Suppose now that G is embedded on the oriented surface S_g (the sphere with g handles). The orientation of S_g defines a permutation P on S, each of whose cycles consists of the set of arcs emanating from one vertex, cyclically ordered by the orientation of S_g. Again it is clear that $\|P\|$ is the number of vertices of G. It is also easily verified that the regions of the embedding are in a natural one-to-one correspondence with the orbits of the product QP. Consequently $\|QP\|$ equals the number of the regions of the embedding.

Example 1.1. Let $G = K_4$ be embedded in the plane as in *Fig.* 1*a*. The resulting D is shown in *Fig.* 1*b*. Then $Q = (1\,2)(3\,4)(5\,6)(7\,8)(9\,a)(b\,c)$, $P = (2\,c\,a)(1\,5\,3)(6\,9\,8)(b\,4\,7)$, and $QP = (1\,c\,4)(2\,5\,9)(3\,7\,6)(8\,b\,a)$.

If we now consider graphs as edge-labelled with $q = |E(G)|$ distinct labels, we may assume Q to be the fixed point free involution $(1\,2)(3\,4)\cdots(2q-1\,2q)$. As G varies over all embedded edge-labelled graphs, the associated permutation P varies over all the elements of the symmetric group S_{2q}. If a smaller class of



a Fig. 1 b

graphs is to be studied, the range of P should be modified accordingly. For example, the embeddings of cubic graphs correspond to the (conjugacy) class of all permutations in S_{2q} whose orbits consists of 3-cycles.

The *unsigned Stirling numbers of the first kind* $s(n, k)$ are defined as the coefficients of

$$(x+n-1)_{(n)} = x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} s(n, k)x^k.$$

For the sake of simplicity we shall refer to them as simply the *Stirling numbers*. They are known to satisfy the following equations [1, p. 214].

$$\begin{aligned} &s(n, k) = s(n-1, k-1) + (n-1)s(n-1, k) \\ &s(n, 0) = s(0, k) = 0 \quad \text{except} \quad s(0, 0) = 1. \end{aligned} \tag{1.1}$$

It is known [1, p. 234] that the number of permutations in S_n that have exactly k orbits is $s(n, k)$. This yields the following result.

Proposition 1.2. *The total number of orientable embeddings of graphs on q edges with r regions is $s(2q, r)$.*

Proof. Fix $Q = (1\,2)(3\,4)\cdots(2q-1\,2q)$. Then the map $P \to QP$ is a bijection of S_{2q}. Since in examining all the embeddings of all graphs on q edges P varies over all the elements of S_{2q}, it follows that the corresponding QP also varies over all the elements of S_{2q}. Thus the number of P's for which $\|QP\| = r$ is $s(2q, r)$. □

For each positive integer q let Y_q be the discrete random variable such that

$$\begin{aligned} P_r[Y_q = r] &= \text{proportion of orientable embeddings of graphs on } q \\ &\quad\ \text{edges with } r \text{ regions} \\ &= s(2q, r)/(2q)! \end{aligned}$$

We shall refer to Y_q as the random variable describing the region distribution for all graphs on q edges. For each positive integer n set

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

$$\zeta_n(2) = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Proposition 1.3. *Let Y_q be the random variable describing the region distribution for the embeddings of all graphs on q edges. Then*

$$\mu(Y_q) = H_{2q},$$

$$\sigma^2(Y_q) = H_{2q} - \zeta_{2q}(2).$$

Proof. By Lemma 3.5,

$$\mu(Y_q) = \frac{1}{(2q)!} \sum_{r=1}^{2q} rs(2q, r) = H_{2q}.$$

Similarly, by the same lemma,

$$\sigma^2(Y_q) = \frac{1}{(2q)!} \sum_{r=1}^{2q} r^2 s(2q, r) - \mu^2(Y_q)$$
$$= H_{2q} + H_{2q}^2 - \zeta_{2q}(2) - H_{2q}^2 = H_{2q} - \zeta_{2q}(2). \quad \square$$

2. The region distribution for the bouquet

Let $\underset{\sim}{B}_q$ denote the graph with a single vertex and q loops. We shall refer to it as the *bouquet* on q loops. As was observed in [5], if the embeddings of the bouquet are analyzed in the same manner as was described in Section 1, then we are lead to the consideration of products of the form QP where

$$Q = (1\,2)(3\,4) \cdots (2q-1\,2q),$$

and P is an arbitrary *cyclic* permutation in S_{2q}. The specific choice of Q is clearly immaterial and so the problem of describing $\|QP\|$ for a random cyclic P can be symmetrized by considering all products QP where Q is an arbitrary fixed point free involution and P is an arbitrary cyclic permutation, both in S_{2q}. It is now necessary to digress to a discussion of the conjugacy classes of S_n. This discussion is of necessity incomplete and the reader is referred to [6, 9, 11] for more details.

The conjugacy class of a permutation depends solely on the sizes of its orbits. Specifically, let P be a permutation of n symbols with the disjoint cycle decomposition

$$P = \sigma_1 \sigma_2 \cdots \sigma_k,$$

where $k = \|P\|$ and each σ_i is a cyclic permutation of λ_i symbols. We may assume without loss of generality that $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k$. If P' is another permutation of the same n symbols with the corresponding sequence $\lambda_1' \leqslant \lambda_2' \leqslant \cdots \leqslant \lambda_{k'}'$, then P and P' are conjugate elements of S_n if and only if $k = k'$ and $\lambda_i = \lambda_i'$ for each $1 \leqslant i \leqslant k$. Since each such monotone sequence constitutes a partition of the integer n, it follows that the conjugacy classes of S_n are in a 1-1 correspondence with the partitions of the integer n. We shall henceforth denote a class of S_n by C_λ where $\lambda = (\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k)$ and $\sum_{i=1}^k \lambda_i = n$, and we abbreviate this with the notation $\lambda \vdash n$. There is an alternate description of such partitions λ. Let α_j denote the number of indices i such that $\lambda_i = j$. Then λ is also denoted by $(1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n})$, where those terms whose indices are zero are usually omitted. For example

$$(2 \leqslant 2 \leqslant 3 \leqslant 3 \leqslant 3 \leqslant 5) \quad \text{and} \quad (2^2, 3^3, 5)$$

denote the same conjugacy class of S_{18}. Using this latter notation, if $\lambda = (1^{\alpha_1}, 2^{\alpha_1}, \ldots, n^{\alpha_n})$ we define $\|\lambda\| = \sum_{i=1}^{n} \alpha_i$. If C_λ is the corresponding class then it is clear that

$$\|P\| = \|\lambda\| \quad \text{whenever} \quad P \in C_\lambda.$$

The following formula is attributed to Cauchy:

$$|C_\lambda| = \frac{n!}{1^{\alpha_1}\alpha_1!\, 2^{\alpha_2}\alpha_2! \cdots n^{\alpha_n}\alpha_n!}. \tag{2.1}$$

Of particular significance here are the conjugacy classes of S_{2q} corresponding to the following partitions of $2q$:

- (2^q) — q transpositions,
- $(2q)$ — cyclic permutations,
- $(1^i, 2q-i)$ — i singletons and one cycle of length $2q-i$,
- (1^{2q}) — the identity permutation.

If C_λ and C_μ, C_ν are three conjugacy classes of S_n then it is known that two elements of C_λ can be expressed as products

$$gh \quad \text{with} \quad g \in C_\mu, \quad h \in C_\nu \tag{2.2}$$

in the same number of ways. This fact can be expressed as the formal sum

$$C_\mu C_\nu = \sum_{\lambda \vdash n} c_{\mu\nu\lambda} C_\lambda,$$

where $c_{\mu\nu\lambda}$ denotes the number of ways an arbitrary element of C_λ can be expressed as a product in (2.2). The coefficient $c_{\mu\nu\lambda}$ is also called the *class multiplication coefficient*. It is known that

$$c_{\mu\nu\lambda} = \frac{1}{n!}|C_\mu| \cdot |C_\nu| \sum_{\tau \vdash n} \frac{\chi_\mu^\tau \chi_\nu^\tau \chi_\lambda^\tau}{\chi_{(1^n)}^\tau},$$

where χ_μ^τ denotes the value that the character of S_n corresponding to C_τ assumes on the class C_μ.

Lemma 2.1. *Let $b(q, r)$ denote the number of embeddings of the bouquet B_q on q edges that have r regions. Then*

$$b(q, r) = \frac{1}{2q} \sum_{\|\lambda\|=r} \left(\sum_{i=0}^{2q-1} \frac{\chi_{(2^q)}^{(1^i,2q-i)} \chi_{(2q)}^{(1^i,2q-i)} \chi_\lambda^{(1^i,2q-i)}}{\chi_{(1^{2q})}^{(1^i,2q-i)}} \right) |C_\lambda|.$$

Proof. Let $Q = (1\,2)(3\,4) \cdots (2q-1\; 2q)$ be the permutation of the arcs of B_q that exchanges each arc with its inverse. The possible rotations of these arcs vary over all the cyclic permutations in the class $C_{(2q)}$. Any such rotation P yields an embedding with r regions exactly when $\|QP\| = r$. Observe [16] that $\chi_{(2q)}^\lambda \neq 0$ only when $\lambda = (1^i, 2q-i)$ for some $i \in \{0, 1, \ldots, 2q-1\}$. Hence, if we set

$$A(i, q, \lambda) = \frac{\chi_{(2^q)}^{(1^i,2q-i)} \chi_{(2q)}^{(1^i,2q-i)} \chi_{\lambda}^{(1^i,2q-i)}}{\chi_{(1^{2q})}^{(1^i,2q-i)}}$$

then

$$\begin{aligned} b(q, r) &= \frac{1}{|C_{(2^q)}|} \sum_{\|\lambda\|=r} c_{(2^q)(2q)\lambda} |C_\lambda| \\ &= \frac{1}{|C_{(2^q)}|} \sum_{\|\lambda\|=r} \frac{1}{(2q)!} |C_{(2^q)}| \cdot |C_{(2q)}| \left(\sum_{i=0}^{2q-1} A(i, q, \lambda) \right) |C_\lambda| \\ &= \frac{1}{2q} \sum_{\|\lambda\|=r} \left(\sum_{i=0}^{2q-1} A(i, q, \lambda) \right) |C_\lambda|. \quad \square \end{aligned}$$

It is known [6, 8, 9, 11] that if $\lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$, then

$$\begin{aligned} &\chi_{(2^q)}^{(1^i,2q-i)} = (-1)^{\{i/2\}} \binom{q-1}{[i/2]} \\ &\chi_{(2q)}^{(1^i,2q-i)} = (-1)^i \\ &\chi_{(1^{2q})}^{(1^i,2q-i)} = \binom{2q-1}{i} \quad i = 0, 1, 2, \ldots, 2q-1 \\ &\chi_{\lambda}^{(1^i,2q-i)} = \sum_{(1^{i_1}, 2^{i_2}, \ldots) \vdash i} \binom{\alpha_1 - 1}{i_1} \binom{\alpha_2}{i_2} \binom{\alpha_3}{i_3} \cdots (-1)^{i_2+i_4+i_6+\cdots} \end{aligned} \tag{2.3}$$

Since $(1^i, 2q-i)$ and $(1^{2q-i-1}, i+1)$ are conjugate classes it follows from a well known relation [6, p. 206] that

$$\chi_{\lambda}^{(1^i,2q-i)} = (-1)^{\|\lambda\|} \chi_{\lambda}^{(1^{2q-i-1},\, i+1)}.$$

Consequently,

$$A(i, q, \lambda) = (-1)^{\|\lambda\|+q+1} A(2q-i-1, q, \lambda).$$

Substituting this into the expression derived in Lemma 2.1 now yields $b(q, r) = 0$ when r and q have the same parity. On the other hand, if r and q have different parities, then

$$b(q, r) = \frac{1}{q} \sum_{\substack{\|\lambda\|=r \\ \lambda \vdash 2q}} \left(\sum_{i=0}^{q-1} A(i, q, \lambda) \right) |C_\lambda|. \tag{2.4}$$

For every even positive integer n and odd integer m such that $0 < m < n$, set

$$\binom{n}{m}_{\text{odd}} = \frac{(n-1)(n-3)\cdots(n-m)}{1 \cdot 3 \cdots m}.$$

It is easily verified that for odd i

$$\frac{\chi_{(2^q)}^{(1^i,2q-i)}}{\chi_{(1^{2q})}^{(1^i,2q-i)}}=(-1)^{(i+1)/2}\binom{2q}{i}_{\text{odd}}^{-1}=\frac{\chi_{(2^q)}^{(1^{i+1},2q-i-1)}}{\chi_{(1^{2q})}^{(1^{i+1},2q-i-1)}}.$$

Substituting this and the evaluations from (2.3) into (2.4) now yields

$$b(q,r)=\frac{1}{q}\sum_{\substack{\|\lambda\|=r\\ \lambda\vdash 2q}}|C_\lambda|\left[1-\sum_{j=1}^{[q-1/2]}\frac{(-1)^j(\chi_\lambda^{(1^{2j-1},2q-2j+1)}-\chi_\lambda^{(1^{2j},2q-2j)})}{\binom{2q}{2j-1}_{\text{odd}}}+\varepsilon\right] \tag{2.5}$$

where

$$\varepsilon=\begin{cases}0 & \text{if } q \text{ is odd}\\ (-1)^{1+q/2}\binom{2q}{q-1}_{\text{odd}}^{-1}\chi_\lambda^{(1^{q-1},q+1)} & \text{if } q \text{ is even.}\end{cases}$$

We now proceed to obtain bounds on $\chi_\lambda^{(1^i,2n-i)}$ which will in turn yield estimates on $b(q,r)$. The derivation of these bounds relies heavily on the Murnaghan–Nakayama recursion formula [(9, p. 60)]. This formula is restated here in its full generality without an explanation of its terms:

$$\chi_\pi^\alpha=\sum_{\substack{i,j\\ h_{ij}^\alpha=k}}(-1)^{l_{ij}^\alpha}\chi_\rho^{[\alpha]\setminus R_{ij}^\alpha}. \tag{2.6}$$

Theorem 2.2. *For every class* $\lambda=(\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_r)$ *of* S_n*, and* $i=0,1,2,\ldots,n-1$,

$$|\chi_\lambda^{(1^i,n-i)}|\leqslant\sum_{m=0}^{i}\binom{r-1}{m}.$$

Proof. It is convenient at this point to extend the definition of $\chi_\lambda^{(1^i,n-i)}$ to arbitrary integers i. We set, by fiat,

$$\chi_\lambda^{(1^i,n-i)}=0\quad\text{if } i<0 \text{ or } i\geqslant n.$$

If $\|\lambda\|>1$, let $\lambda_{\{1\}}=(\lambda_2\leqslant\lambda_3\leqslant\cdots\leqslant\lambda_r)$. We now show that

$$\chi_\lambda^{(1^i,n-i)}=\chi_{\lambda_{\{1\}}}^{(1^i,n-\lambda_1-i)}-(-1)^{\lambda_1}\chi_{\lambda_{\{1\}}}^{(1^{i-\lambda_1},n-i)}. \tag{2.7}$$

If $\lambda_1\leqslant i,\ n-i-1$, then this follows immediately from the Murnaghan–Nakayama formula. If $\lambda_1\leqslant i$ but $\lambda_1>n-i-1$, then $i\geqslant n-\lambda_1$ and so $\chi_\lambda^{(1^i,n-\lambda_1-i)}=0$. In this case, however, formula (2.6) reduces to

$$\chi_\lambda^{(1^i,n-i)}=(-1)^{\lambda_1-1}\chi_{\lambda_{\{1\}}}^{(1^{i-\lambda_1},n-i)},$$

which agrees with (2.7). If $\lambda_1>i$ but $\lambda_1\leqslant n-i-1$ then $\chi_{\lambda_{\{1\}}}^{(1^{i-\lambda_1},n-i)}=0$. In this case formula (2.6) reduces to

$$\chi_\lambda^{(1^i,n-i)}=\chi_{\lambda_{\{1\}}}^{(1^i,n-\lambda_1-i)}$$

which agrees with (2.7). Finally, if $\lambda_1 > i$, $n-i-1$ then $2\lambda_1 \geqslant n+1$. Since $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_r$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$ it follows that $r = 1$, so that reduction (2.7) does not apply here.

It follows from (2.7) that

$$|\chi_\lambda^{(1^i, n-i)}| \leqslant |\chi_{\lambda_{\{1\}}}^{(1^i, n-\lambda_1-i)}| + |(-1)^{\lambda_1} \chi_{\lambda_{\{1\}}}^{(1^{i-\lambda_1}, n-i)}|.$$

If we now replace λ in the above argument with $\lambda_{\{1\}} = (\lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_r)$ and $\lambda_{\{1\}}$ with $\lambda_{\{1,2\}} = (\lambda_3 \leqslant \lambda_4 \leqslant \cdots \leqslant \lambda_r)$, then we get

$$|\chi_{\lambda_{\{1\}}}^{(1^i, n-\lambda_1-i)}| \leqslant |\chi_{\lambda_{\{1,2\}}}^{(1^i, n-\lambda_1-\lambda_2-i)}| + |\chi_{\lambda_{\{1,2\}}}^{(1^{i-\lambda_2}, n-\lambda_1-i)}|$$

$$|\chi_{\lambda_{\{1\}}}^{(1^{i-\lambda_1}, n-i)}| \leqslant |\chi_{\lambda_{\{1,2\}}}^{(1^{i-\lambda_1}, n-\lambda_2-i)}| + |\chi_{\lambda_{\{1,2\}}}^{(1^{i-\lambda_1-\lambda_2}, n-i)}|,$$

so that

$$\begin{aligned} |\lambda_\lambda^{(1^i, n-i)}| \leqslant {} & |\chi_{\lambda_{\{1,2\}}}^{(1^i, n-\lambda_1-\lambda_2-i)}| + |\chi_{\lambda_{\{1,2\}}}^{(1^{i-\lambda_2}, n-\lambda_1-i)}| \\ & + |\chi_{\lambda_{\{1,2\}}}^{(1^{i-\lambda_1}, n-\lambda_2-i)}| + |\chi_{\lambda_{\{1,2\}}}^{(i-\lambda_1-\lambda_2, n-i)}|. \end{aligned}$$

Let I be an arbitrary subset of $\{1, 2, \ldots, r-1\}$ and let $\bar{I}$ denote its relative complement $\{1, 2, \ldots, r-1\} - I$. Let $\sigma(I) = \sum_{i \in I} \lambda_i$. Then $r-1$ applications of the above process result in

$$|\chi_\lambda^{(1^i, n-i)}| \leqslant \sum_I |\chi_{(\lambda_r)}^{(1^{i-\sigma(I)}, n-\sigma(\bar{I})-i)}|.$$

However,

$$\chi_{(\lambda_r)}^{(1^{i-\sigma(I)}, n-\sigma(\bar{I})-i)} = \begin{cases} (-1)^{i-\sigma(I)} & \text{if } i - \sigma(I) \in \{0, 1, \ldots, n-1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Since $1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_r$ it follows that for each $I \subseteq \{1, 2, \ldots, r-1\}$

$$\sigma(I) \geqslant |I|.$$

Consequently $i - \sigma(I) \geqslant 0$ entails $i \geqslant |I|$ and so there are no more than

$$\sum_{m=0}^{i} \binom{r-1}{m}$$

subsets I such that $i - \sigma(I) \in \{0, 1, \ldots, n-1\}$. The theorem now follows from (2.8). □

The above bound will be used to demonstrate the affinity between counters $b(q, r)$ and the Stirling numbers.

Theorem 2.3. *There exists a function $\rho\colon \mathbb{Z}^+ \to \mathbb{R}$ such that $|\rho(r)| \leqslant r^2$, and*

$$b(q, r) = \frac{s(2q, r)}{q} \left[1 + \frac{\rho(r)}{q}\right]$$

whenever $1 \leqslant r \leqslant (2q)^{1/5}$, and $r \not\equiv q \pmod 2$.

Proof. Let $\beta = [(2q)^{1/5}]$. We first examine the portion $2 \leqslant j \leqslant \beta/4$ of summation (2.5). In these considerations $i = 2j-1$ or $2j$, depending on its parity. By Theorem 2.2, since $i \leqslant \beta/2$,

$$|\chi_\lambda^{(1^i,2q-i)}| \leqslant \sum_{m=0}^{i} \binom{r-1}{m} \leqslant \sum_{m=0}^{i} \binom{\beta}{m} \leqslant i\binom{\beta}{i}.$$

If we now set for odd values of i:

$$\phi_i = i\binom{\beta}{i}\binom{2q}{i}_{\text{odd}}^{-1},$$

then

$$\frac{\phi_{i-2}}{\phi_i} = \frac{(i-1)(i-2)(2q-i)}{(\beta-i+1)(\beta-i+2)i} \geqslant \frac{4(2q-\beta/2)}{\beta^3} > 1.$$

It can be similarly shown that $\phi_{i-2} > \phi_i$ for even i as well. Consequently

$$\left|\sum_{j=2}^{[\beta/4]} \frac{(-1)^j(\chi_\lambda^{(1^{2j-1},2q-2j+1)} - \chi_\lambda^{(1^{2j},2q-2j)})}{\binom{2q}{2j-1}_{\text{odd}}}\right|$$

$$\leqslant \frac{\beta}{4} \cdot \frac{|\chi_\lambda^{(1^3,2q-3)}| + |\chi_\lambda^{(1^4,2q-4)}|}{\frac{(2q-1)(2q-3)}{1\cdot 3}} \leqslant \frac{3\beta}{4} \cdot \frac{3\binom{\beta}{3} + 4\binom{\beta}{4}}{(2q-1)(2q-3)} < \frac{1}{2q}.$$

As for the segment of (2.5) corresponding to $j > \beta/4$, we first note that for any i,

$$|\chi_\lambda^{(1^i,2q-i)}| \leqslant 2^{r-1} \leqslant 2^\beta.$$

Let γ denote the odd member of the pair $\{[\beta/2], [\beta/2]+1\}$, and set $\delta = (\gamma+1)/2$. Since in this segment of (2.5), $\gamma \leqslant 2j-1 \leqslant q-1$, it follows that

$$\binom{2q}{q-1}_{\text{odd}} \geqslant \binom{2q}{2j-1}_{\text{odd}} \geqslant \binom{2q}{\gamma}_{\text{odd}} \geqslant (2q-\gamma)^\delta \frac{2^\delta \delta!}{(2\delta)!}.$$

Hence the absolute value of this segment is bounded above, for large q, by

$$\frac{q \cdot 2^\beta \cdot (2\delta)!}{(2q-\gamma)^\delta \cdot 2^\delta \cdot \delta!} < \frac{1}{2q},$$

by Stirling's formula.

To bound the contribution of $j = 1$ we first observe that by (2.3)

$$|\chi_\lambda^{(1,2q-1)}| + |\chi_\lambda^{(1^2,2q-2)}| = |\alpha_1 - 1| + \left|\binom{\alpha_1 - 1}{2} - \alpha_2\right| \leqslant \binom{\alpha_1}{2} + \alpha_2.$$

Now, if $r > 2$, then

$$\sum_{\substack{\|\lambda\|=r\\ \lambda\vdash 2q}} \binom{\alpha_1}{2} |C_\lambda| = \sum_{\substack{\|\lambda\|=r\\ \lambda\vdash 2q}} \binom{\alpha_1}{2} \frac{(2q)!}{1^{\alpha_1}\cdot \alpha_1!\cdot 2^{\alpha_2}\cdot \alpha_2!\cdots}$$

$$= \sum_{\substack{\|\lambda\|=r\\ \lambda\vdash 2q}} \binom{2q}{2} \frac{(2q-2)!}{1^{\alpha_1-2}\cdot(\alpha_1-2)!\cdot 2^{\alpha_2}\cdot \alpha_2!\cdots} = \binom{2q}{2} \sum_{\substack{\|\mu\|=r-2\\ \mu\vdash 2q-2}} |C_\mu|$$

where C_μ varies over classes of S_{2q-2}. Consequently, by Lemma 3.3,

$$\sum_{\|\lambda\|=r} \binom{\alpha_1}{2} |C_\lambda| \leqslant \binom{2q}{2} s(2q-2, r-2) \leqslant \frac{2r^2}{9} s(2q, r).$$

A similar argument allows us to conclude that if C_μ now varies over the classes of S_{2q-2}, then

$$\sum_{\|\lambda\|=r} \alpha_2 |C_\lambda| = \binom{2q}{2} \sum_{\|\mu\|=r-1} |C_\mu| = \binom{2q}{2} s(2q-2, r-1)$$

$$\leqslant \frac{r+3}{4} s(2q, r)$$

by Lemma 3.4. Consequently, bearing in mind that $\sum_{\|\lambda\|=r} |C_\lambda| = s(2q, r)$, the above calculations can be summarized as saying that there are numbers A_r, B_r, C_r such that

$$|A_r| \leqslant \frac{2r^2}{9} + \frac{r+3}{4}, \qquad |B_r|, |C_r| \leqslant \frac{1}{2q}$$

and

$$b(q, r) = \frac{s(2q, r)}{q}\left(1 + \frac{A_r}{2q-1} + B_r + C_r\right).$$

Thus, since

$$\frac{\left(\frac{2r^2}{9} + \frac{r+3}{4}\right)}{2q-1} + \frac{1}{2q} + \frac{1}{2q} < \frac{r^2}{q} \quad \text{for } r \geqslant 2,$$

the theorem is proved for $r \geqslant 2$. When $r = 1$ the theorem is proved directly by noting first that when $\|\lambda\| = 1$ we have

$$\chi^{(1^{2j-1}, 2q-2j+1)} - \chi^{(1^{2j}, 2q-2j)} = -2$$

and so by (2.5)

$$b(q, 1) = \frac{1}{q} \sum_{\substack{\|\lambda\|=1\\ \lambda\vdash 2q}} |C_\lambda| \left[1 + \sum_{j=1}^{[q-1/2]} \frac{(-1)^j 2}{\binom{2q}{2j-1}_{\text{odd}}} + \varepsilon\right]$$

However, the terms of the internal sum above alternate in sign and decrease in absolute values and hence it is easily seen that

$$1-\frac{1}{q}\leqslant 1+\sum_{j=1}^{[q-1/2]}\frac{(-1)^j 2}{\binom{2q}{2j-1}_{\text{odd}}}+\varepsilon\leqslant 1+\frac{1}{q},$$

and the proof of the theorem is concluded. □

The following lemma deals with the values of $b(q, r)$ where $r>(2q)^{1/5}$. First, however, a definition is needed. A sequence u_k of nonnegative real numbers is said to be *strongly unimodal* (logarithmically convex) if

$$u_k^2 \geqslant u_{k-1}u_{k+1} \quad \text{for all } k.$$

It is well known [1, p. 270] that every strongly unimodal sequence is unimodal, that is to say, there exist integers a, b (with a possibly $\pm\infty$) such that

$$\cdots < u_{a-2} < u_{a-1} < u_a = u_{a+1} = \cdots = u_b > u_{b+1} > u_{b+2} > \cdots.$$

Lemma 2.4. *Let j be a fixed integer and $\beta=[(2q)^{1/5}]$. Then*

$$\lim_{q\to\infty}\frac{(2q)^j}{(2q-1)!}\sum_{r=\beta}^{2q} b(q, r)=0.$$

Proof. We appeal here to the results of Gross et al. [5] which are based on those of D.M. Jackson [8]. The authors of [5] define $g_m(q)$ to be the number of embeddings of B_q in the orientable surface of genus m. It follows from the Euler–Poincaré formula that

$$b(q, r)=g_{(1+q-r)/2}(q).$$

In the same paper it is shown that the sequence $\{g_m(q)\}_{m=1}^{q}$ is strongly unimodal and hence it follows that the sequence $\{b(q, r)\}$, where r has different parity from q, is also strongly unimodal. This also follows from Lemma 4.4 and [8, Lemma 6.1ii].

We next show that the mode of this sequence occurs at some integer $r_q<\beta$. Observe that by Theorem 2.2

$$b(q, \beta)\leqslant\frac{2(2q, \beta)}{q}\left(1+\frac{\beta^2}{q}\right)\leqslant\frac{2s(2q, \beta)}{q}.$$

Let $\gamma=\gamma(q)$ be the unique value of r [3] such that

$$s(2q, \gamma)=\max_r\{s(2q, r)\}.$$

It is known that $|\gamma - \ln 2q| < 1$ and hence, by Theorem 2.2 and Lemma 3.8,

$$b(q, \gamma) \geqslant \frac{s(2q, \gamma)}{q}\left(1 - \frac{\beta^2}{q}\right) \geqslant \frac{s(2q, \gamma)}{2q} \geqslant \frac{(2q-1)!}{q}$$
$$\geqslant b(q, \beta) \quad \text{for large } q.$$

Since the sequence $\{b(q, r)\}_{r=1}^{2q}$ is unimodal for $r \not\equiv q (\bmod 2)$ it follows that

$$b(q, \beta) \geqslant b(q, r) \quad \text{whenever } \beta \leqslant r.$$

Consequently, by Lemma 3.8,

$$0 \leqslant \lim_{q \to \infty} \frac{(2q)^j}{(2q-1)!} \sum_{r > \beta} b(q, r) \leqslant \lim_{q \to \infty} \frac{(2q)^{j+1}}{(2q-1)!} b(q, \beta) = 0. \quad \square$$

For each positive integer q let X_q be the discrete random variable such that

$$Pr[X_q = r] = b(q, r)/(2q-1)!$$

we shall say that X_q *describes the region distribution of the bouquet* B_q. The mean and standard deviation of X_q are denoted by μ_q and σ_q respectively. Recall that $\zeta(2) = \lim_{n \to \infty} \zeta_n(2) = \sum_{k=1}^{\infty} (1/k^2)$.

Theorem 2.5. *Let X_q be the random variable describing the region distribution of the bouquet B_q. Then*

$$\lim_{q \to \infty} (\mu_q - H_{2q}) = 0$$

$$\lim_{q \to \infty} (\sigma_q^2 - H_{2q} + \zeta(2)) = 0.$$

Proof. Let $M_q^{(j)}$ denote the jth moment of X_q, i.e.

$$M_q^{(j)} = \frac{1}{(2q-1)!} \sum_{r=1}^{q+1} r^j b(q, r).$$

Then $\mu_q = M_q^{(1)}$ and $\sigma_q^2 = M_q^{(2)} - \mu_q^2$. Now, for $j = 1, 2$ and $\beta = [(2q)^{1/5}]$

$$\frac{1}{(2q-1)!} \sum_{r=1}^{\beta} r^j b(q, r) = \frac{2}{(2q)!} \sum_{\substack{r=1 \\ r \not\equiv q}}^{\beta} r^j s(2q, r)\left[1 + \frac{\rho(r)}{q}\right].$$

However,

$$\frac{2}{(2q)!} \sum_{\substack{r=1 \\ r \not\equiv q}}^{\beta} r^j s(2q, r) = \frac{2}{(2q)!} \sum_{\substack{r=1 \\ r \not\equiv q}}^{2q} r^j s(2q, r)$$
$$- \frac{2}{(2q)!} \sum_{\substack{r > \beta \\ r \not\equiv q}} r^j s(2q, r),$$

and, by Lemma 3.8,

$$\lim_{q\to\infty} \frac{2}{(2q)!} \sum_{r>\beta} r^j s(2q, r) = 0.$$

Also

$$0 \leqslant \lim_{q\to\infty} \frac{2}{(2q)!} \sum_{\substack{r=1\\ r\neq q}}^{\beta} \left| \frac{r^j \rho(r) s(2q, r)}{q} \right|$$

$$\leqslant \lim_{q\to\infty} \frac{2}{(2q)!q} \sum_{r=1}^{2q} r^{j+2} s(2q, r) \leqslant \lim_{q\to\infty} \frac{4}{q} \ln^{j+2}(2q) = 0.$$

Consequently

$$\lim_{q\to\infty} \left| \frac{1}{(2q-1)!} \sum_{r=1}^{\beta} r^j b(q, r) - \frac{2}{(2q)!} \sum_{\substack{r=1\\ r\neq q}}^{2q} r^j s(2q, r) \right| = 0.$$

As for the tail sum, Lemma 4.4 yields

$$0 \leqslant \frac{1}{(2q-1)!} \sum_{r>\beta} r^j b(q, r) \leqslant \frac{(2q)^j}{(2q-1)!} \sum_{r>\beta} b(q, r) \to 0.$$

Thus we may now conclude that

$$\lim_{q\to\infty} \left| M_q^{(j)} - \frac{2}{(2q)!} \sum_{\substack{r=1\\ r\neq q}}^{2q} r^j s(2q, r) \right| = 0.$$

In view of Lemma 3.6, case $j = 1$ yields

$$\lim_{q\to\infty} (\mu_q - H_{2q}) = 0,$$

and case $j = 2$ yields

$$\lim_{q\to\infty} (M_q^{(2)} - H_{2q} - H_{2q}^2 + \zeta(2)) = 0.$$

Since $\sigma_q^2 = M_q^{(2)} - \mu_q^2$ it follows that

$$\lim_{q\to\infty} (\sigma_q^2 - H_{2q} + \zeta(2)) = 0. \quad \square$$

3. About Stirling numbers

As the title implies, this chapter consists of a variety of propositions about the unsigned Stirling numbers of the first kind.

Lemma 3.1. $s^2(n, k) \geqslant s(n, k-1)s(n, k+1)k(n-k+1)/(k-1)(n-k)$ *for* $1 < k < n$.

Proof. See [1, pp. 270–271]. □

Lemma 3.2. $\dfrac{s(n,k)}{s(n,k-1)} \geqslant \dfrac{2(n-k+1)}{n(k-1)}$.

Proof. By descending induction on k. Since $s(n, n-1) = \binom{n}{2}$, the lemma holds for $k = n$. Assuming the lemma to hold when k is replaced by $k+1$, note that by Lemma 3.1

$$\frac{s(n,k)}{s(n,k-1)} \geqslant \frac{s(n,k+1)}{s(n,k)} \cdot \frac{k(n-k+1)}{(k-1)(n-k)} \geqslant \frac{2[n-(k+1)+1]}{n[(k+1)-1]} \cdot \frac{k(n-k+1)}{(k-1)(n-k)}$$
$$= \frac{2(n-k+1)}{n(k-1)}. \quad \square$$

Lemma 3.3. $\binom{n}{2} s(n-2,\ k-2) \leqslant (2k^2/9)\ s(n,k)$ *for* $n \geqslant 4$, $s \leqslant k \leqslant n/2+1$.

Proof. We first settle the extreme case $k = n/2+1$, i.e., $n = 2k-2$. Two applications of (1.1) yield

$$s(n,k) = s(n-2,k-2) + (2n-3)s(n-2,k-1) + (n-1)(n-2)s(n-2,k), \tag{3.1}$$

and hence

$$\frac{s(2k-2,k)}{\binom{2k-2}{2} s(2k-4,k-2)} = \frac{1}{\binom{2k-2}{2}} \left[1 + (4k-7)\frac{s(2k-4,k-1)}{s(2k-4,k-2)} + (2k-3)(2k-4)\frac{s(2k-4,k)}{s(2k-4,k-2)}\right].$$

But, by Lemma 3.2,

$$\frac{s(2k-4,k-1)}{s(2k-4,k-2)} \geqslant \frac{2}{2k-4} \cdot \frac{2k-4-k+1+1}{k-1-1} = \frac{1}{k-2}$$

and

$$\frac{s(2k-4,k)}{s(2k-4,k-2)} = \frac{s(2k-4,k)}{s(2k-4,k-1)} \cdot \frac{s(2k-4,k-1)}{s(2k-4,k-2)}$$
$$\geqslant \frac{2}{2k-4} \cdot \frac{2k-4-k+1}{k-1} \cdot \frac{1}{k-2} = \frac{k-3}{(k-1)(k-2)^2}.$$

Hence,

$$\frac{s(2k-2,k)}{\binom{2k-2}{2}s(2k-4,k-2)} \geq \frac{2}{(2k-2)(2k-3)}$$

$$\times\left[1+\frac{4k-7}{k-2}+(2k-3)(2k-4)\frac{k-3}{(k-1)(k-2)^2}\right]$$

$$\geq \frac{1}{(k-1)(2k-3)}\left[1+4+\frac{4(k-3)}{k-1}\right] \geq \frac{9}{2k^2}.$$

The other extreme case $k=3$ follows from two applications of Lemma 3.2:

$$\frac{2\cdot 3^2}{9}s(n,3)=2\cdot\frac{s(n,3)}{s(n,2)}\cdot\frac{s(n,2)}{s(n,1)}\cdot s(n,1)$$

$$\geq 2\cdot\frac{2(n-2)}{2n}\cdot\frac{2(n-1)}{n}\cdot(n-1)!$$

$$\geq \binom{n}{2}(n-3)! = \binom{n}{2}s(n-2,1).$$

The Lemma is now proved by induction on n. The case $n=4$ is easily verified and so we assume that the lemma holds when n is replaced by $n-1$. The case $n=2k-2$ was considered above and hence we may assume that $n \geq 2k-1$. It follows that $n-1 \geq 2(k-1)-2$ and $n-1 \geq 2k-2$, and hence by (1.1) and the induction hypothesis,

$$\binom{n}{2}s(n-2,k-2)=\binom{n}{2}[s(n-3,k-3)+(n-3)s(n-3,k-2)]$$

$$\leq \binom{n}{2}\left[\binom{n-1}{2}^{-1}\frac{2(k-1)^2}{9}s(n-1,k-1)\right.$$

$$\left. +(n-3)\binom{n-1}{2}^{-1}\frac{2k^2}{9}s(n-1,k)\right]$$

$$\leq \frac{2k^2}{9}[s(n-1,k-1)+(n-1)s(n-1,k)]$$

$$=\frac{2k^2}{9}s(n,k). \quad \square$$

Lemma 3.4. $\binom{n}{2}s(n-2,k-1) \leq \frac{1}{4}(k+3)s(n,k)$ *for* $n \geq 4$ *and* $2 \leq k \leq n/2+1$.

Proof. It follows from (1.1) and Lemma 3.2 that

$$\frac{s(n,k)}{\binom{n}{2}s(n-2,k-1)} = \frac{1}{\binom{n}{2}}\left[\frac{s(n-2,k-2)}{s(n-2,k-1)} + (2n-3)\frac{s(n-2,k-1)}{s(n-2,k-1)} + (n-1)(n-2)\frac{s(n-2,k)}{s(n-2,k-1)}\right]$$

$$\geqslant \frac{1}{\binom{n}{2}}\left[2n-4+(n-2)(n-2)\cdot\frac{2(n-k-1)}{(n-2)(k-1)}\right]$$

$$= \frac{4(n-2)(n-2)}{n(n-1)(k-1)} \geqslant \frac{4}{k+3}. \quad \square$$

Lemma 3.5. *Let* $N_n = \sum_{k=1}^{n} ks(n,k)$. *Then*

$$\frac{N_n^{(1)}}{n!} = H_n$$

$$\frac{N_n^{(2)}}{n!} = H_n + H_n^2 - \sum_{k=1}^{n}\frac{1}{k^2}.$$

Proof. It follows from the recursion (1.1) that

$$\frac{N_n^{(1)}}{n!} = \frac{N_{n-1}^{(1)}}{(n-1)!} + \frac{1}{n} = \cdots = H_n.$$

It follows similarly that

$$\frac{N_n^{(2)}}{n!} = \frac{N_{n-1}^{(2)}}{(n-1)!} + \frac{2H_{n-1}}{n} + \frac{1}{n} = \cdots = H_n + 2\sum_{k=1}^{n-1}\frac{H_k}{k+1}$$

$$= H_n + H_n^2 - \sum_{k=1}^{n}\frac{1}{k^2}. \quad \square$$

Lemma 3.6. *Let*

$$a_n^{(j)} = \frac{2}{n!}\sum_{k=1}^{[n/2]}(2k)^j s(n,2k), \qquad b_n^{(j)} = \frac{2}{n!}\sum_{k=1}^{\{n/2\}}(2k-1)^j s(n,2k-1).$$

Then

(i) $\lim_{n\to\infty}[a_n^{(1)} - H_n] = \lim_{n\to\infty}[b_n^{(1)} - H_n] = 0$

(ii) $\lim_{n\to\infty}[a_n^{(2)} - H_n - H_n^2 + \zeta(2)] = \lim_{n\to\infty}[b_n^{(2)} - H_n - H_n^2 + \zeta(2)] = 0.$

Proof. It follows from the recursion (1.1) that

$$a_n^{(1)} = \frac{1}{n} + \frac{n-1}{n} a_{n-1}^{(1)} + \frac{1}{n} b_{n-1}^{(1)},$$

$$b_n^{(1)} = \frac{1}{n} + \frac{1}{n} a_{n-1}^{(1)} + \frac{n-1}{n} b_{n-1}^{(1)}.$$

Consequently, we have

$$a_n^{(1)} - b_n^{(1)} = \left(1 - \frac{2}{n}\right)[a_{n-1}^{(1)} - b_{n-1}^{(1)}]$$

and

$$\lim_{n\to\infty} (a_n^{(1)} - b_n^{(1)}) = 0.$$

Since $a_n^{(1)} + b_n^{(1)} = 2H_n$, the limits of part (i) are established.

It also follows from recursion (1.1) that

$$a_n^{(2)} = \frac{1}{n}[1 + 2b_{n-1}^{(1)} + (n-1)a_{n-1}^{(2)} + b_{n-1}^{(2)}]$$

$$b_n^{(2)} = \frac{1}{n}[1 + 2a_{n-1}^{(1)} + a_{n-1}^{(2)} + (n-1)b_{n-1}^{(2)}],$$

and so, here too,

$$\lim_{n\to\infty} [a_n^{(2)} - b_n^{(2)}] = 0$$

and consequently, since $a_n^{(2)} + b_n^{(2)} = 2[H_n + H_n^2 + \zeta_n(2)]$,

$$\lim_{n\to\infty} [a_n^{(2)} - H_n - H_n^2 + \zeta(2)] = \lim_{n\to\infty} [b_n^{(2)} - H_n - H_n^2 + \zeta(2)] = 0. \quad \square$$

Lemma 3.7. *For each integer $i \geq 0$ and each real number $\varepsilon > 0$ there is an integer $n_i(\varepsilon)$ such that*

$$\frac{1}{n!}\sum_{k=1}^{n} k^i s(n, k) \leq (1+\varepsilon)\ln^i n \quad \textit{for } n \geq n_i(\varepsilon).$$

Proof. Set

$$f_i(n) = \frac{1}{n!}\sum_{k=1}^{n} k^i s(n, k).$$

It follows from (1.1) that

$$f_i(n) = f_i(n-1) + \frac{1}{n}\sum_{j=0}^{i-1}\binom{i}{j} f_j(n-1),$$

and consequently, by the telescoping sum technique,

$$f_i(n) - f_i(1) = \sum_{m=2}^{n} \frac{1}{m} \sum_{j=0}^{i-1} \binom{i}{j} f_j(m-1).$$

The lemma is clearly valid for $i = 0$ and so we proceed by induction on i. If we choose $\frac{3}{4} > \varepsilon > 0$, then we have for $n \geqslant n_j(\varepsilon/3)$, $j = 0, 1, \ldots, i-1$

$$f_i(n) \leqslant 1 + \sum_{m=2}^{n} \frac{1}{m} \sum_{j=0}^{i-1} \binom{i}{j} \left(1 + \frac{\varepsilon}{3}\right) \ln^j(m-1)$$
$$< 1 + \left(1 + \frac{\varepsilon}{3}\right) \sum_{j=0}^{i-1} \binom{i}{j} \sum_{m=2}^{n} \frac{\ln^j m}{m}.$$

However, the function $\ln^j x / x$ is concave downwards and attains a maximum value of $(j/e)^j$ at $x = e^j$, and so

$$1 + \sum_{m=2}^{n} \frac{\ln^j m}{m} \leqslant 1 + \int_2^n \frac{\ln^j x}{x} dx + \left(\frac{j}{e}\right)^j$$
$$< \left(1 + \frac{\varepsilon}{3}\right) \frac{\ln^{j+1} n}{j+1} \quad \text{for } n \geqslant n_j', \quad j = 0, 1, \ldots, i-1.$$

Thus,

$$f_i(n) < \left(1 + \frac{\varepsilon}{3}\right)^2 \sum_{j=0}^{i-1} \binom{i}{j} \frac{\ln^{j+1} n}{j+1}$$
$$< \left(1 + \frac{3\varepsilon}{4}\right) \ln^i n + (i-1) 2^i \ln^{i-1} n$$
$$< (1 + \varepsilon) \ln^i n \quad \text{for } n \geqslant n_i(\varepsilon). \quad \square$$

Lemma 3.8. *For each integer $j \geqslant 0$,*

$$\lim_{n \to \infty} \frac{n^j}{n!} s(n, \{n^{1/5}\}) = 0.$$

Proof. Rather than derive this limit from the estimates of $s(n, k)$ given in [12], it is actually easier to modify their proof so as to obtain an upper bound that yields this limit.

We remind the reader that $s(n, k)$ is in fact the coefficient of z^k in the polynomial $(z + n - 1)_{(n)} = z(z+1) \cdots (z + n - 1)$. Consequently, by the Residue Theorem,

$$s(n, k) = \frac{1}{2\pi i} \int_C \frac{(z + n - 1)_{(n)}}{z^{k+1}} \mathrm{d}z,$$

where C is a circle of radius R centered at the origin. Setting $z = \mathrm{Re}^{i\theta}$, we have

$$s(n, k) = \frac{1}{2\pi i} \int_C \frac{\Gamma(z+n)}{z^{k+1} \Gamma(z)} \mathrm{d}z = \frac{\Gamma(R+n)}{2\pi \Gamma(R) R^k} \int_{-\pi}^{\pi} F(\theta) \, \mathrm{d}\theta,$$

where

$$F(\theta)=\frac{\Gamma(\mathrm{Re}^{i\theta}+n)\Gamma(R)}{\Gamma(R+n)\Gamma(\mathrm{Re}^{i\theta})\mathrm{e}^{ik\theta}}=\mathrm{e}^{-ik\theta}\prod_{h=1}^{n-1}\frac{\mathrm{Re}^{i\theta}+h}{R+h}.$$

Since

$$\left|\frac{\mathrm{Re}^{i\theta}+h}{R+h}\right|\leqslant 1$$

it follows that $|F(\theta)|\leqslant 1$ and so

$$\left|\int_{-\pi}^{\pi}F(\theta)\,\mathrm{d}\theta\right|\leqslant 2\pi.$$

Consequently,

$$s(n,k)\leqslant\frac{\Gamma(R+n)}{R^k\Gamma(R)}.$$

If we now choose $k=\{n^{1/5}\}$ and $R=[k/\ln k]$, then

$$\frac{n^j}{n!}s(n,k)\leqslant\frac{n^j}{n!}\frac{\Gamma(R+n)}{R^k}\leqslant\frac{n^j(R+n-1)_{(R-1)}}{R^k}$$
$$\leqslant\frac{(2n)^{R+j}}{R^k}.$$

However, it is easily verified that the logarithm of the last term diverges to $-\infty$ as n becomes large. Hence

$$\lim_{n\to\infty}\frac{n^j}{n!}s(n,\{n^{1/5}\})=0.\quad\square$$

4. Afterthoughts and conjectures

It would be interesting to know where the mode and the median of the random variable X_q describing the embeddings of the bouquet B_q occur. It is known [3] that the mode of the random variable Y_n associated with the Stirling numbers $s(n,k)$ occurs within one unit of $k=H_{2q}\approx\ln 2q$. It would seem that the same holds for the median of Y_n. Consequently we conjecture the following for large q.

Conjecture 4.1. The mode of the random variable X_q associated with the bouquet B_q occurs within one unit of H_{2q}.

Conjecture 4.2. The median of X_q differs from H_{2q} by at most one unit.

The genus distributions of the ladders and other narrow graphs [4] notwithstanding, the author believes that the behavior of the bouquet is typical. For each graph G let X_G be the random variable describing its embeddings. Then the following is conjectured.

Conjecture 4.3. For almost all graphs G, $\mu(X_G)$ and $\sigma^2(X_G)$ are close to H_{2q} and $H_{2q} - \zeta(2)$ respectively, and the mode and the median of X_G are close to H_{2q}.

Robert Rieper [13] has shown that if G is the dipole on n edges, i.e., the multigraph consisting of q edges joining two vertices, then $X_G = Y_q$ where Y_q is the random variable describing the region distribution of all graphs on q edges.

The following lemma and its proof are generalizations of Theorem 4.1 of [5] and its proof. They are included here for the sake of completeness and because of some implications which will be pointed out below.

Lemma 4.4. *Let $f_{n,k} \geq 0$, $n, k \in \mathbb{Z}$ satisfy the recursion*

$$f_{n,k} = c_{n,r} f_{n-r,k-1} + d_{n,r} f_{n-1,k} \quad \text{for } n > n_0,$$

where $c_{n,r}$ and $d_{n,r}$ are nonnegative reals, r is a positive integer and n_0 is some fixed integer. If relations A–D *below hold for all $n \leq n_0$ then these relations hold for all n:*

$$\left.\begin{aligned} &A(n,k,j)\colon f_{n,k} f_{n-j,k} \geq f_{n,k+1} f_{n-j,k-1} \\ &B(n,k,j)\colon f_{n,k} f_{n-j,k} \geq f_{n,k-1} f_{n-j,k+1} \\ &C(n,k,j)\colon f_{n,k} f_{n-j,k-1} \geq f_{n,k-1} f_{n-j,k} \\ &D(n,k,j)\colon f_{n,k} f_{n-j,k-1} \geq f_{n,k+1} f_{n-j,k-2} \end{aligned}\right\} j = 0, 1, 2, \ldots, r.$$

Proof. By induction on n, the values $n \leq n_0$ providing the anchor. We thus assume that A–D hold whenever n is replaced by any smaller value. As this will not give rise to any ambiguities, we abbreviate $c_{n,r}$ and $d_{n,r}$ to c and d respectively. We first dispose of $A(n, k, 0)$. By $A(n-r, k-1, 0)$, $A(n-1, k, 0)$, $C(n-1, k, r-1)$, and $D(n-1, k, r-1)$:

$$\begin{aligned} f_{n,k} f_{n-0,k} &= c^2 f^2_{n-r,k-1} + d^2 f^2_{n-1,k} + 2cd f_{n-1,k} f_{n-r,k-1} \\ &\geq c^2 f_{n-r,k} f_{n-r,k-2} + d^2 f_{n-1,k+1} f_{n-1,k-1} \\ &\quad + cd f_{n-1,k-1} f_{n-r,k} + cd f_{n-r,k-2} f_{n-1,k+1} \\ &= (c f_{n-r,k} + d f_{n-1,k+1})(c f_{n-r,k-2} + d f_{n-1,k-1}) \\ &= f_{n,k+1} f_{n,k-1}. \end{aligned}$$

Having verified $A(n, k, 0)$ we note that it is logically equivalent to $B(n, k, 0)$. Relation $C(n, k, 0)$ is trivial, and $D(n, k, 0)$ follows from $A(n, k-1, 0)$ and

$A(n, k, 0)$ as follows:

$$\frac{f_{n,k-1}}{f_{n,k-2}} \geqslant \frac{f_{n,k}}{f_{n,k-1}} \geqslant \frac{f_{n,k+1}}{f_{n,k}}.$$

It may therefore be assumed that $j > 0$. We now verify $A(n, k, j)$, where $j > 0$. By $C(n-j, k, r-j)$ and $A(n-1, k, j-1)$

$$\begin{aligned} f_{n,k}f_{n-j,k} &= cf_{n-j,k}f_{n-r,k-1} + df_{n-1,k}f_{n-j,k} \\ &\geqslant cf_{n-j,k-1}f_{n-r,k} + df_{n-1,k+1}f_{n-j,k-1} \\ &= f_{n,k+1}f_{n-j,k-1}. \end{aligned}$$

Similarly, $B(n, k, j)$ follows from $D(n-j, k, r-j)$ and $B(n-1, k, j-1)$, $C(n, k, j)$ follows from $A(n-j, k-1, r-j)$ and $C(n-1, k, j-1)$, and $D(n, k, j)$ follows from $B(n-j, k-1, r-j)$ and $D(n-1, k, j-1)$. □

Let $s(n, k, d)$ denote the number of permutations of n symbols with k orbits each of which has length at least d. Then $s(n, k, 1) = s(n, k)$ and $s(n, k, 2)$ is the number of derangements with k orbits. It is known that [1, p. 257] that

$$s(n, k, d) = (n-1)_{(d-1)}s(n-d, k-1, d) + (n-1)s(n-1, k, d).$$

Consequently we have

Corollary 4.5. *For fixed n and d the sequence $s(n, k, d)$ is strongly unimodal.*

Proof. This follows easily from the previous lemma with $n_0 = d$. Note that relation $A(n, k, 0)$ is tantamount to strong unimodality. □

The bound obtained in Theorem 2.2 is an improvement on the one that appears in Lemma 2.3 of [16]. The author believes that it can be further improved as follows.

Conjecture 4.6. If $\lambda = (\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_r)$ and $r \geqslant 2$, then

$$|\chi_\lambda^{(1^i, n-i)}| \leqslant \binom{r-1}{i},$$

with quality holding only when $\lambda = (1^n)$.

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