Discrete Mathematics 109 (1992) 45–57
North-Holland

Endomorphism spectra of graphs

Michael Böttcher and Ulrich Knauer
Fachbereich Mathematik, Universität Oldenburg, W-2900 Oldenburg, Germany

Received May 1990
Revised 23 April 1991

Dedicated to Professor Gert Sabidussi on the occasion of his 60th birthday.

Abstract

Böttcher, M. and U. Knauer, Endomorphism spectra of graphs, Discrete Mathematics 109 (1992) 45–57.

In this paper we give an account of the different ways to define homomorphisms of graphs. This leads to six classes of endomorphisms for each graph, which as sets always form a chain by inclusion. The endomorphism spectrum is defined as a six-tuple containing the cardinalities of these six sets, and the endomorphism type is a number between 0 and 31 indicating which classes coincide. The well-known constructions by Hedrlin and Pultr (1965) and by Hell (1979) of graphs with a prescribed endomorphism monoid always give graphs of endomorphism type 0 mod 2.

After the basic definitions in Section 1, we discuss some properties of the endomorphism classes in Section 2. Section 3 contains what is known about existence of certain endomorphism types, Section 4 gives a list of graphs with given endomorphism type, except for some cases where none have been found so far. Finally we formulate some problems connected with concepts presented here.

The graphs considered are finite and undirected without multiple edges and loops although all these restrictions are not essential as far as the definitions of endomorphism classes go. Only the colorgraphs mentioned in the proof of 2.3 may have loops and have directed edges which are called arcs. The vertex set $V(X)$ of a graph X is also denoted just by X, and the edge set is denoted by $E(X)$. If $x, x' \in X$ are adjacent denote the edge connecting x and x' by $\{x, x'\}$ and write $\{x, x'\} \in E(X)$.

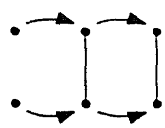
This paper mainly contains definitions and many examples. Most of these are found and computed by a computer program. This, as usual, raises the problem of verification. So we are dealing with 'Mathematischer Zoologie' as E. Hecke phrased it in 1937 (cited after C.A. Kaloujnine, R. Pöschel, EIK 22 (1980) 5–24). As justification for writing this paper we take the rich algebraic structure which is put on a graph by its endomorphism classes and the numerous questions connected with these concepts.

1. Definitions and first examples

Following each definition, we give an example of a homomorphism belonging to the class defined but not to the class defined next. The images of vertices are indicated by arrows. Vertices without starting arrow are fixed. Let X and Y be graphs, $f: X \to Y$ a mapping. In the examples one always has $X = Y$.

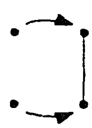
1.1. The mapping f is called an (*ordinary*) *homomorphism* if $\{x, x'\} \in E(X)$ implies $\{f(x), f(x')\} \in E(Y)$.

Symbol: $f \in \mathrm{Hom}(X, Y)$. Example:



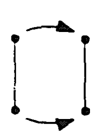
1.2. The homomorphism f is called a *halfstrong homomorphism* if $\{f(x), f(x')\} \in E(Y)$ implies the existence of preimages $\bar{x}$, $\bar{x}'$, i.e., $f(\bar{x}) = f(x)$, $f(\bar{x}') = f(x')$, such that $\{\bar{x}, \bar{x}'\} \in E(X)$.

Symbol: $f \in \mathrm{HHom}(X, Y)$. Example:



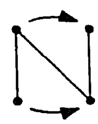
1.3. The homomorphism f is called a *locally strong homomorphism* if $\{f(x), f(x')\} \in E(Y)$ implies that for every preimage $\bar{x} \in X$ of $f(x)$ there exists a preimage $\bar{x}' \in X$ of $f(x')$, such that $\{\bar{x}, \bar{x}'\} \in E(X)$ and analogously for every preimage of $f(x')$.

Symbol: $f \in \mathrm{LHom}(X, Y)$. Example:



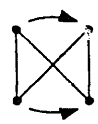
1.4. The homomorphism f is called a *quasi-strong homomorphism* if $\{f(x), f(x')\} \in E(Y)$ implies that there exists a preimage $\bar{x} \in X$ of $f(x)$ which is adjacent to every preimage of $f(x')$, and analogously for preimages of $f(x')$.

Symbol: f $\in$ QHom(X, Y). Example:



1.5. The homomorphism f is called a *strong homomorphism* if $\{f(x), f(x')\} \in E(Y)$, implies that any preimage of $f(x)$ is adjacent to any preimage of $f(x')$.

Symbol: $f \in$ SHom(X, Y). Example:



1.6. The homomorphism f is called an *isomorphism* if f is bijective and f^{-1} is a homomorphism.

Symbol: $f \in$ Iso(X, Y).

If $X = Y$ we speak of endomorphisms with the respective epitheta, or of automorphisms, and write End $X \supset$ HEnd $X \supset$ LEnd $x \supset$ QEnd $X \supset$ SEnd $X \supset$ Aut X, where the inclusions of the sets are indicated.

Comments 1.7. Ordinary homomorphisms are used everywhere and are mostly called homomorphisms.

Halfstrong homomorphisms were called full homomorphisms by Hell [4] and by Sabidussi [15] and partially adjacent homomorphism by Antohe and Olaru [2].

Surjective locally strong homomorphisms appear in Pultr and Trnková [14].

Quasi-strong homomorphisms have, as far as we know, not yet appeared in literature.

Strong homomorphisms were probably first considered by K. Čulík [3] under the name of homomorphisms and later on used by many authors (cf. for example [6, 11]).

The names selected here are mildly suggestive, and the notation gives an alphabetic order (except for Aut).

Before we go into details, we repeat some standard terminology and notation.

The *complete graph on n vertices* is denoted by K_n for any natural number $n \geq 1$.

The *totally disconnected graph on n vertices* has no edges, it is denoted by $\bar{K}_n$

The *circuit on n vertices* is denoted by C_n.

The *path on n vertices* is denoted by P_n.

Let X, Y be graphs with $V(X) \cap V(Y) = \emptyset$.

The *union* $X \cup Y$ is just the set theoretic union of X and Y.
The *join* $X + Y$ is the graph with vertex set $V(X) \cup V(Y)$ and edge set

$$E(X + Y) = E(X) \cup E(Y) \cup \{\{x, y\} \mid x \in X, y \in Y\}.$$

From [7] we recall the definition of the *generalized lexicographic product*, sometimes called X-join (cf. [16]), $X[(Y_x)_{x\in X}]$ of the graph X with the graphs $(Y_x)_{x\in X}$ which is defined by

$$V(X[(Y_x)_{x\in X}]) = \{(x, y_x \mid x \in X_x, y_x \in Y_x\}$$

and

$$\{(x, y_x), (x', y'_{x'})\} \in E(X[(Y_x)_{x\in X}])$$

if and only if $\{x, x'\} \in E(X)$ or $x = x'$ and $\{y_x, y'_x\} \in E(Y_x)$.

So far End X, SEnd X and Aut X have been studied separately and in their relations to each other, their properties w.r.t. to graph operations specially the join and the lexicographic product, and in the special case where Aut $X = 1$. In this case X is called *asymmetric.*

The following definitions are used. According to [6] the graph X is called S-*unretractive* (or, more precisely, S – A-*unretractive*) if SEnd X = Aut X, and *unretractive* (or, more precisely, E – A-*retractive*) if End X = Aut X. The graph X is called E – S-*unretractive* if End X = SEnd X. Graphs with End $X = 1$ are called *rigid*, [17]. This concept can easily be extended to the other classes of endomorphisms mentioned. Perminov [12] calls C – B-rigid what would be called here C – B-unretractive, where B, C $\subset$ End X.

Nowakowski and Rival [10] call a graph X *retract rigid* if it has no nontrivial idempotent endomorphism. The same notion is called a *retract-free* graph by Bang–Jensen, Hell, and MacGillivray (this Vol.).

Remark 1.8. A finite graph is retract rigid if and only if it is unretractive.

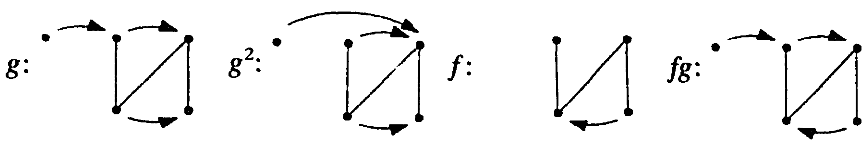
Proof. Assume that X has no nontrivial idempotent endomorphisms. Take $f \in$ End X and let f^i be its idempotent power, which exists since X and hence also End X is finite. By assumption $f^i = \mathrm{id}_X$ and thus $f \in$ Aut X. The converse is obvious. □

2. Some basic properties

It is well known that SEnd X forms a monoid with respect to composition of mappings, and, as always, End X is a monoid and Aut X a group.

Proposition 2.1. HEnd X, LEnd X, QEnd X *do not form monoids.*

Proof. Select X and $g \in \mathrm{QEnd}\,X$ as indicated. Then $g^2 \in \mathrm{HEnd}\,X \setminus \mathrm{LEnd}\,X$. Take f as indicated. Then $f \in \mathrm{HEnd}\,X \setminus \mathrm{LEnd}\,X$ but $fg \notin \mathrm{HEnd}\,X$:



□

Proposition 2.2. *Idempotent endomorphisms are elements of* HEnd.

Proof. Obvious. □

Now we recall the standard construction of the Cayley color graph for a given monoid. Let M be a monoid. The Cayley *colorgraph* $F(M)$ of M has $V(F(M)) = M$ and edges are constructed as follows. Take a set $C \subset M \setminus 1$ of generating elements of M and represent M by say left translations by elements of C acting on M. That means we draw an edge of color c leading from a to b, $a, b \in M$ if $ca = b$. This way $F(M)$ becomes a directed graph with colored edges. Any endomorphism of $F(M)$ which preserves directions and colors will be called a *color endomorphism.*

Note that also the other classes of endomorphisms can be defined analogously for color endomorphisms, paying special attention to the symmetry in the definitions of local and quasistrong endomorphisms.

Lemma 2.3. *Let M be a monoid. All color preserving endomorphisms of $F(M)$ are halfstrong.*

Proof. Let f be a color endomorphism of $F(M)$. Assume that the edge $(f(a), f(b))$ has color c for $a, b \in M$, $c \in C$. Set $f^{-1}(a) = \{a_1, \ldots, a_n\}$, $a = a_1$, and $f^{-1}(b) = \{b_1, \ldots, b_m\}$, $b = b_1$. From every a_i, $i = 1, \ldots, n$ there starts exactly one edge of color c and they all end in some vertex of $\{b_1, \ldots, b_m\}$. For, if (a_i, d) would be an edge of color c with $d \notin \{b_1, \ldots, b_m\}$, then $(f(a_i), f(d)) = (a, f(d)) \neq (a, b)$ would be an edge of color c which is impossible since in a, as in every vertex, there starts exactly one edge of each color. □

Lemma 2.4. *Let M be a monoid, which has a right zero x, i.e., $Mx = x$, and is not right solvable, i.e., there exist $e, c \in M$ such that $e \notin cM$. Then there exists a color endomorphism of $F(M)$ which is not locally strong.*

Proof. As x is a right zero, the constant mapping $c_x : F(M) \to \{x\} \subset F(M)$ is a color endomorphism. By hypothesis, there does not end an edge of color c at e. Therefore, for $e \in c_x^{-1}(x)$ there does not exist a vertex $m \in F(M)$ with color c. Thus, c_x is not locally strong. □

The following result is remarkable as every monoid is the halfstrong endmorphism monoid of a suitable graph, although in general halfstrong endomorphisms of a graph do not form a monoid.

Proposition 2.5. *For every monoid M there exists a graph X such that $M \cong \operatorname{End} X$.*

Proof (Sketch). Hedrlin and Pultr [13] (cf. also Hell [5]) proved that for every monoid M there exists a graph X with $\operatorname{End} X \cong M$. Both proofs first consider the color graph $F(M)$. It is quite straight forward to see that M is isomorphic to the monoid of color endomorphisms of $F(M)$ which in turn is half-strong by Lemma 2.3. The proofs in [13] and [5] consist in getting rid of colors and directions, but thereby preserving the monoid which will then be the endomorphism monoid of the resulting graph (which is going to be much bigger than the colorgraph). These constructions replace directed colored edges by graphs which reflect different colors by their inner structure and by the property to be mutually rigid. In this way the constructed graphs admit only such endomorphisms which formerly preserved colors and directions. The replacements do not touch the property of the endomorphisms to be halfstrong. The details for the construction by Hell [5] are worked out in [1]. □

Remark 2.6. *The graphs with given monoid constructed by Hedrlin and Pultr* [13] *or Hell* [5] *do not have the property that all endomorphisms are locally strong.*

Proof. Use Lemma 2.4 and argue as in 2.5. □

Proposition 2.7. *If* $\operatorname{SEnd} X$ *is not a group, then* $\operatorname{SEnd} X$ *contains two idempotent elements different from the identity which are right identities to each other.*

Proof. Using Lemma 1.1 of [9], we know that $N(x_1) = N(x_2)$ for $x_1, x_2 \in X$ and $f \in \operatorname{SEnd} X$ such that $f(x_1) = f(x_2) \neq x_2$. Here $N(y) = \{x \mid \{x, y\} \in E(X)\}$, $y \in X$. Define

$$\begin{aligned} g: X \to X \quad & \text{by } g(x_1) = g(x_2) = f(x_1) \quad \text{and} \\ & g(x) = x \quad \text{for } x \neq x_1, x_2; \\ h: X \to X \quad & \text{by } h(x_1) = h(x_2) = x_2, \\ & h(x) = x \quad \text{for } x \neq x_1, x_2. \end{aligned}$$

Then obviously, $g, h \in \operatorname{SEnd} X$ and $g^2 = g$, $h^2 = h$, $gh = g$, $hg = h$. □

Remark 2.8. Proposition 2.7 shows that not every monoid can be the strong endomorphism monoid of a graph. It was shown in [9] that monoids of strong endomorphisms are always von Neumann regular monoids. Moreover, in 3.8 of [9] it was shown that $\operatorname{SEnd} X$ cannot have 2 or an odd number of elements $\leqslant 29$, except for 27, unless $\operatorname{SEnd} X = \operatorname{Aut} X$.

3. The endomorphism type of a graph

For a more systematic treatment of different endomorphisms we define the endomorphism spectrum and the endomorphism type of a graph.

For the graph X consider the sequence

$$\operatorname{End} X \supset \operatorname{HEnd} X \supset \operatorname{LEnd} X \supset \operatorname{QEnd} X \supset \operatorname{SEnd} X \supset \operatorname{Aut} X.$$

With this sequence associate the sequence of respective cardinalities

$$\operatorname{Endospec} X = (|\operatorname{End} X|, |\operatorname{HEnd} X|, |\operatorname{LEnd} X|, |\operatorname{QEnd} X|, |\operatorname{SEnd} X|, |\operatorname{Aut} X|)$$

and call this 6-tuple the *endospectrum* or *endomorphism spectrum of X*. Secondly, associate with the above sequence a 5-tuple $(s_1, s_2, s_3, s_4, s_5)$ with

$$s_i \in \{0, 1\}, \quad i = 1, \ldots, 5, \quad \text{where 1 stands } \neq \text{ and 0 stands for } =$$

at the respective position in the above sequence, i.e., $s_1 = 1$ indicates that $|\operatorname{End} X| \neq |\operatorname{HEnd} X|$ etc.

The integer $\sum_{i=1}^{5} s_i 2^{i-1}$ is called the *endotype* or *endomorphism type* of X and is denoted by Endotype X.

In principle there are 32 possibilities, i.e., endotype 0 up to endotype 31. Endotype 0 describes unretractive graphs. Enotype 0 up to 15 describe S-unretractive graphs. Endotype 16 describes E − S-unretractive graphs which are not unretractive (cf. [6, 7, 9, 11]). Endotype 31 describes graphs for which all 6 sets are different.

Before analyzing the endotypes of graphs in some more detail, we consider all endotypes with respect to whether or not $\operatorname{Aut} X = 1$.

Remark 3.1. (4.13 of [6]). *Let X be a graph. If* $\operatorname{Aut} X = 1$, *then* $\operatorname{SEnd} X = 1$.

The preceding remark shows that for endotypes 16 up to 31 we always have $\operatorname{Aut} X \neq 1$, since $\operatorname{SEnd} X \neq \operatorname{Aut} X$ in these cases. We also know that there exist rigid graphs and unretractive graphs which are not rigid (see for example [7] and many more places). So we add for endotypes 0 to 15 an additional *a* denoting asymmetry, i.e., $\operatorname{Aut} X = 1$. It can be proved that endotypes 1 and 17 do not exist.

Proposition 3.2. *Let X be a graph such that* $\operatorname{End} X \neq \operatorname{HEnd} X$. *Then* $\operatorname{HEnd} X \neq \operatorname{SEnd} X$.

Proof. Take $f \in \operatorname{End} X \setminus \operatorname{HEnd} X$. Then there exists $\{f(x), f(x')\} \in E(X)$ but for all $\bar{x}$, $\bar{x}'$ with $f(\bar{x}) = f(x)$ and $f(\bar{x}') = f(x')$ one has $\{\bar{x}, \bar{x}'\} \notin E(X)$. From finiteness of End X we get an idempotent power f^i of f, i.e., $(f^i)^2 \in f^i$, and from 2.2 we know that $f^i \in \operatorname{HEnd} X$. In particular, since $\{f^i(x), f^i(x')\} \in E(x)$ we have that $f^i(x)$ and $f^i(x')$ are fixed under f^i, and thus they are adjacent preimages. Moreover, $f^i \notin \mathrm{SEnd}\, X$, since not all preimages are adjacent: $\{x, x'\} \notin E(X)$. □

Remark 3.3. If it could be proved that $\mathrm{End}\, X \neq \mathrm{HEnd}\, X$ also implies that $\mathrm{HEnd}\, X \neq \mathrm{QEnd}\, X$, then endotypes 9 and 25 would not be possible.

4. Graphs with given endotypes

For two monoids $M(X)$ and $M(Y)$ acting on X and Y respectively—as for example $\mathrm{End}\, X$ acts on X—define the *sum of monoids*

$$M(X) + M(Y) = \{g + h \mid g \in M(X),\, h \in M(Y)\}$$

acting on $X \cup Y$ by $(g + h)(x) = g(x)$ and $(g + h)(y) = h(y)$ for $x \in X$, $y \in Y$.

It is almost trivial to see that for two graphs X and Y we have

$$\mathrm{End}(X + Y) = \mathrm{End}\, X + \mathrm{End}\, Y$$

if and only if

$$f(X) \subset X \quad \text{and} \quad f(Y) \subset Y \quad \text{for all } f \in \mathrm{End}(X + Y).$$

This condition may be hard to verify for given graphs. But if we know that $\mathrm{End}(X + Y) = \mathrm{End}\, X + \mathrm{End}\, Y$ for graphs X and Y, such that, for example, $\mathrm{End}\, X = \mathrm{HEnd}\, X$ and $\mathrm{End}\, Y \neq \mathrm{HEnd}\, Y$, then $\mathrm{End}(X + Y) \neq \mathrm{HEnd}(X + Y)$. This observation may be helpful in constructing new graphs with a certain endotype, using joins and boolean addition of endotypes.

Boolean addition of endotypes means, in the dual representation,

$$(s_1, \ldots, s_5) \dot{+} (s'_1, \ldots, s'_5) = (s_1 \dot{+} s'_1, \ldots, s_5 \dot{+} s'_5)$$

with $0 \dot{+} 0 = 0$ and $1 \dot{+} 0 = 0 \dot{+} 1 = 1 \dot{+} 1 = 1$ in every component. We shall say that the join is *additive* with respect to the endotype if

$$\mathrm{Endotype}(X + Y) = \mathrm{Endotype}\, X + \mathrm{Endotype}\, Y$$

and *superadditive*, if

$$\mathrm{Endotype}(X + Y) > \mathrm{Endotype}\, X + \mathrm{Endotype}\, Y.$$

An analysis of the different endomorphism classes shows that join and union of graphs are always additive or superadditive and, moreover, one has

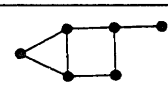
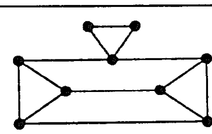
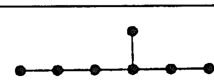
$$\mathrm{QEnd}(X + Y) \neq \mathrm{SEnd}(X + Y) \quad \text{or} \quad \mathrm{SEnd}(X + Y) \neq \mathrm{Aut}(X + Y)$$

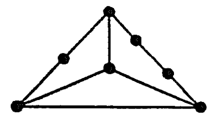
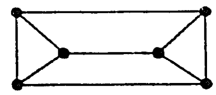
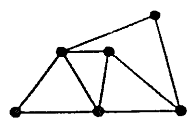
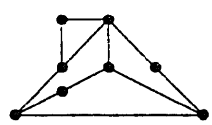
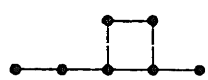
if and only if the corresponding is true for X or Y; analogously for the union of graphs. The corresponding result is obviously not true for the other three inequalities.

Next we give a table of graphs without loops ordered according to endotypes. The graphs with endospectra up to 9 vertices were found using a computer program. The endomorphism spectrum of the graphs constructed by joins were also computed by a computer program. Since by 3.2 graphs with endotypes 1 and 17 do not exist, there is no way to construct graphs of endotypes 2, 3, 4, 5, 8, 9, and 16 using boolean addition. So far graphs of endotype 9 and 25 have not been found, nor graphs of endotypes 5a, 8a, 9a and 13a.

Table 4.1. In the following table we abbreviate the endomorphism classes by their first letters. Graphs are given by constructions, the usual pictures or, if there are too many edges, we give the upper triangle of the adjacency matrix, lines written consecutively, always starting with the element right next to the diagonal element.

At places where we give a join, say 8 + 6, this indicates that the join is between the smallest graphs of endotype 8 and endotype 6 given before in the list unless stated differently. If the join is superadditive, we write *8 + 6* in italics. The cardinalities of the respective endomorphism sets always refer to the first graph given in the last column.

Endo Type	Explicit form and cardinality of the classes											Graph
0a	E	=	H	=	L	=	Q	=	S	=	A	There exist 10 rigid Graphs with 8 vertices (cf. e.g. [7])
0	2		2		2		2		2		2	K_2, K_n, C_{2n+1}, $n \leq 1$
	E	=	H	≠	L	=	Q	=	S	=	A	
2a	101		101		1		1		1		1	
2	6		6		2		2		2		2	$K_2 \cup K_1$
	E	≠	H	≠	L	=	Q	=	S	=	A	
3a	75		71		1		1		1		1	
3	44		36		2		2		2		2	$P_4[K_1, K_2, K_1, K_1]$
	E	=	H	=	L	≠	Q	=	S	=	A	
4a	37		37		37		1		1		1	10100101, 1001001, 100010, 10000, 1100, 100, 10, 1
4	16		16		16		8		8		8	$K_2 \cup K_2$
	E	≠	H	=	L	≠	Q	=	S	=	A	
5a												?
5	184		168		168		8		8		8	
	E	=	H	≠	L	≠	Q	=	S	=	A	
6a	387		387		13		1		1		1	
6	80		80		32		8		8		8	$K_2 \cup K_2 \cup K_1$, 4 + 2

Endo Type	Explicit form and cardinality of the classes											Graph
	E	≠	H	≠	L	≠	Q	=	S	=	A	
7a	123		121		7		1		1		1	 , 4a + 3a
7	252		200		32		2		2		2	$P_5 \cup K_1$, 4 + 3
	E	=	H	=	L	=	Q	≠	S	=	A	
8a												? 
8	36		36		36		36		12		12	
	E	≠	H	=	L	=	Q	≠	S	=	A	
9a											1	?
9											≠1	?
	E	=	H	≠	L	=	Q	≠	S	=	A	
10a	153		153		25		25		1		1	000110, 01011, 1110 000, 11, 1
10	16		16		8		8		2		2	P_4, 8 + 2
	E	≠	H	≠	L	=	Q	≠	S	=	A	
11a	117		97		11		11		1		1	
11	80		64		12		12		2		2	$P_4 \cup K_1$, 8 + 3
	E	=	H	=	L	≠	Q	≠	S	=	A	
12a	73		73		73		25		1		1	0111010, 011101, 01111, 0110, 011, 11, 0
12	194		194		194		50		2		2	0111010, 011101, 01111, 0110, 011, 11, 0, 8 + 4
	E	≠	H	=	L	≠	Q	≠	S	=	A	
13a												?
13	73152		71712		71712		288		96		96	8 + 5
	E	=	H	≠	L	≠	Q	≠	S	=	A	
14a	163		163		13		7		1		1	
14	540		540		96		36		12		12	$K_3 \cup P_5$, 8 + 6
	E	≠	H	≠	L	≠	Q	≠	S	=	A	
15a	759		615		71		7		1		1	
15	144		136		44		28		4		4	$P_4 \cup K_2$, 8 + 7
16	E	=	H	=	L	=	Q	=	S	≠	A	
	4		4		4		4		4		2	$\bar{K}_2$, $\bar{K}_n$, C_4, $K_n \backslash k$, $n \geq 3$, $k \in E(K_n)$

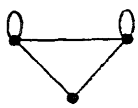
Endo Type	Explicit form and cardinality of the classes	Graph
18	E = H ≠ L = Q = S ≠ A 32 32 8 8 8 4	$K_2 \cup \bar{K}_2$
19	E ≠ H ≠ L = Q = S ≠ A 24 20 6 6 6 2	$P_3 \cup K_1$, 16 + 3, *16 + 2*
20	E = H = L ≠ Q = S ≠ A 240 240 240 48 48 16	11100011, 1010000, 110000, 10000, 1100, 111, 11, 1, 16($= C_5[\bar{K}_2, K_1, K_1, K_1, K_1]$) + 4($=(K_1 + (K_2 \cup K_2)$)
21	E ≠ H = L ≠ Q = S ≠ A 5888 5376 5376 256 256 32	16($= C_7[\bar{K}_2, K_1, K_1, K_1, K_1]$) + 5
22	E = H ≠ L ≠ Q = S ≠ A 48 48 32 20 20 4	$P_3 \cup K_2$, 16 + 6
23	E ≠ H ≠ L ≠ Q = S ≠ A 234 194 32 6 6 4	$P_5[K_1, \bar{K}_2, K_1, K_1, K_1]$, 16 + 7, 16 + 4 + 3, *16 + 4 + 2*
24	E = H = L = Q ≠ S ≠ A 68 68 68 68 20 4	11010111, 1100010, 110101, 11001, 1100, 110, 10, 1
25	E ≠ H = L = Q ≠ S ≠ A	?
26	E = H ≠ L = Q ≠ S ≠ A 60 60 14 14 6 2	$P_4[\bar{K}_2, K_1, K_1, K_1]$, *16 + 8*
27	E ≠ H ≠ L = Q ≠ S ≠ A 94 86 40 40 6 2	$P_4[K_1, \bar{K}_2, K_1, K_1]$ 16 + 11, *16 + 10*
28	E = H = L ≠ Q ≠ S ≠ A 76 76 76 28 4 2	11001111, 1100010, 110101, 11001, 1100, 110, 10, 1
29	E ≠ H = L ≠ Q ≠ S ≠ A 254816 253728 253728 544 160 32	24 + 5
30	E = H ≠ L ≠ Q ≠ S ≠ A 276 276 66 36 4 2	0111000, 011100, 01110, 0111, 001, 11, 0, *16 + 12*
31	E ≠ H ≠ L ≠ Q ≠ S ≠ A 316 292 112 40 8 4	$P_5[K_1, K_1, \bar{K}_2, K_1, K_1]$, 16 + 15, *16 + 14*

Comments 4.2. All graphs in the table which are not joins are, as far as we know, minimal w.r.t. the number of vertices (first criterion), the number of edges (second criterion), an |End| (third criterion).

The endoptypes 13, 21 and 29 so far exist only as the given joins.

If we admit graphs with loops then we have

 with endotype 8a, Endospec = (2, 2, 2, 2, 1, 1)

 with endotype 25, Endospec = (12, 10, 10, 10, 4, 2).

Note, that the values in the endospectrum of the join under endotype 21 are the products of the respective values of the two summands. The same is true for the join given under endotype 20. This is not the case for the first three values in the endospectrum of the joins of endotypes 13 and 29.

5. Some open problems

5.1. Do there exist graphs of endotypes 9 or 25? Do there exist asymmetric graphs of endoptyes 5, 8 or 13?

5.2. Under which conditions do the sets HEnd X, LEnd X, QEnd X form monoids?

5.3. Under which conditions coincide idempotent endomorphisms (retractions) with the classes LEnd\Aut, QEnd\Aut, SEnd\Aut? Note that idempotent endomorphisms always belong to HEnd (cf. [8]).

5.4. Find conditions on X for various unretractivities of X.

5.5. Which monoids are isomorphic to LEnd X, QEnd X or SEnd X for a suitable graph X?

References

[1] H. Ahrends, Endomorphismenmonoide von Graphen und Distributivgesetze für Kompositionen von Graphen, Diplomarbeit, Oldenburg 1987.
[2] St. Antohe and E. Olaru, Über Homomorphismen und Kongruenzen von Graphen, Bul. Univ. Galati 11 (1978) 15–23.
[3] K. Čulík, Zur Theorie der Graphen, Časopis Pěst. Mat 83 (1958) 133–155.
[4] P. Hell, Subdirect products of bipartite graphs, Coll. Math. Janos Bolyai 10. Infinite and Finite Sets 1973, Vol. II (North-Holland, Amsterdam, 1975) 857–866.
[5] P. Hell, On some strongly rigid families of graphs and the full embeddings they induce, Alg. Univ. 4 (1979) 108–126.

[6] U. Knauer, Unretractive and S-unretractive joins and lexicographic products of graphs, J. Graph Theory 11 (1987) 3, 429–440.

[7] U. Knauer, Endomorphisms of graphs. II. Various unretractive graphs, Arch. Math. 55 (1990) 199–203.

[8] U. Knauer, Endomorphism types of trees, Proc. Conf. Words, Languages and Combinatorics, Kyoto 1990, World Scientific, Singapore et al. (1992) 273–287.

[9] U. Knauer and M. Nieporte, Endomorphisms of graphs I. The monoid of strong endomorphisms, Arch. Math. 52 (1989) 607–614.

[10] R. Nowakowski and I. Rival, Retract rigid cartesian products of graphs, Discr. Math. 70 (1988) 169–184.

[11] U. Nummert, On the monoid of strong endomorphisms of wreath products of graphs, Matem. Zam. 41 (1987) 844–853 (Russian, translated into English).

[12] E.A. Perminov, Rigid graphs and lattices, Dissertation, Sverdlovsk 1985 (in Russian).

[13] A. Pultr and Z. Hedrlín, Symmetric relations (undirected graphs) with given semigroups, Monatsh. Math. 69 (1965) 318–322.

[14] A. Pultr and Trnková, Combinational, Algebraic and Topological Representations of Groups, Semigroups and Categories, Amsterdam 1980.

[15] G. Sabidussi, Subdirect representations of graphs, Coll. Math. Janos Bolyai 10. Infinite and Finite Sets 1973, Vol. III (North-Holland, Amsterdam, 1975) 1199–1226.

[16] G. Sabidussi, Graph derivatives, Math. Z. 76 (1961) 385–401.

[17] Z. Hedrlín and A. Pultr, On rigid undirected graphs, Canad. J. Math. 18 (1966) 1237–1242.