J. Math. Anal. Appl. 374 (2011) 516–528

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Bounds for ratios of modified Bessel functions and associated Turán-type inequalities

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ARTICLE INFO

Article history:
Received 29 January 2010
Available online 17 September 2010
Submitted by Steven G. Krantz

Keywords:
Modified Bessel functions
Riccati equation
Bounds
Turán-type inequalities
Condition numbers

ABSTRACT

New sharp inequalities for the ratios of Bessel functions of consecutive orders are obtained using as main tool the first order difference-differential equations satisfied by these functions; many already known inequalities are also obtainable, and most of them can be either improved or the range of validity extended. It is shown how to generate iteratively upper and lower bounds, which are converging sequences in the case of the I-functions. Few iterations provide simple and effective upper and lower bounds for approximating the ratios $I_\nu(x)/I_{\nu-1}(x)$ and the condition numbers $xI'_\nu(x)/I_\nu(x)$ for any $x,\nu \geqslant 0$; for the ratios $K_\nu(x)/K_{\nu+1}(x)$ the same is possible, but with some restrictions on ν. Using these bounds Turán-type inequalities are established, extending the range of validity of some known inequalities and obtaining new inequalities as well; for instance, it is shown that $K_{\nu+1}(x)K_{\nu-1}(x)/(K_\nu(x))^2 < |\nu|/(|\nu|-1)$, $x>0$, $\nu \notin [-1,1]$ and that the inequality is the best possible; this proves and improves an existing conjecture.

1. Introduction

The ratios of modified Bessel functions $I_\nu(x)/I_{\nu-1}(x)$ and $K_\nu(x)/K_{\nu+1}(x)$ are important functions appearing in a great number of applied scientific areas, particularly in probability and statistics [8,23,9], with applications in chemical kinetics [2,20], optics [24] and signal processing [14], just to cite few examples. In addition, these ratios are key computational tools in the construction of numerical algorithms for computing modified Bessel functions (see for instance Algorithms 12.6 and 12.7 of [11]).

On the other hand, recently there has been a considerable interest in the study of Turán-type inequalities for special functions (see, for example, [3,5–7,12,15–18]), including modified Bessel functions. A good number of bounds for ratios and Turán-type inequalities are available for the ratios of modified Bessel functions (particularly for the ratio $I_\nu(x)/I_{\nu-1}(x)$) which have been proved using various techniques [4,5,10,12,13,18,19,21,22]. Here, we show that simple arguments are enough to prove and generalize most of the known results, to find new inequalities and to prove and extend previous conjectures (some of them implicitly assumed in some applications, as we later discuss).

In this paper, the main tools to be considered are the first order difference-differential equations (DDEs) satisfied by Bessel functions and the associated three-term recurrence relation, together with the analysis of the qualitative behavior of the solutions of the Riccati equations satisfied by the ratios. Recurrence relations and DDEs were also considered in [5] and [18] for obtaining some bounds for the ratios of Bessel functions as well as Turán-type inequalities. These relations will appear later in this paper, and some of them will be improved.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.09.030

2. Bounds from a qualitative analysis of Riccati equations

In this section we prove the following result:

Theorem 1. *For $\nu \geqslant 0$ and $x > 0$ the following holds*:

$$\frac{I_{\nu+1/2}(x)}{I_{\nu-1/2}(x)} < \frac{x}{\nu + \sqrt{\nu^2 + x^2}} \leqslant \frac{K_{\nu-1/2}(x)}{K_{\nu+1/2}(x)}, \tag{1}$$

where the equality takes place only if $\nu = 0$.

The lower bound for K-Bessel functions appears to be new, while the bound for the I-functions already appeared in [4] (but it was said to hold for $\nu \geqslant 1/2$). As we will see, both results can be proved using similar arguments.

2.1. The K-Bessel function

Let us denote

$$h_\nu(x) = \frac{K_{\nu-1/2}(x)}{K_{\nu+1/2}(x)}. \tag{2}$$

In the analysis, we can restrict to positive $\nu > 0$ because

$$K_\nu(x) = K_{-\nu}(x), \quad \nu \in \mathbb{R}. \tag{3}$$

Using the well-known first order difference-differential system [1, 9.6.26]:

$$K'_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x}K_\nu(x), \qquad K'_\nu(x) = -K_{\nu+1}(x) + \frac{\nu}{x}K_\nu(x), \tag{4}$$

we obtain

$$h'_\nu(x) = -1 + \frac{2\nu}{x}h_\nu(x) + h_\nu(x)^2. \tag{5}$$

Next, we analyze the qualitative properties of the solution of (5) with the initial conditions provided by the following lemma.

Lemma 1. *Let $h_\nu(x) = K_{\nu-1/2}(x)/K_{\nu+1/2}(x)$. The following holds for $\nu > 0$.*

$$\lim_{x\to 0+} h_\nu(x) = 0^+, \qquad \lim_{x\to 0+} h'_\nu(x) > 0. \tag{6}$$

Proof. Because for $\nu > 0$

$$K_\nu(x) \sim \frac{1}{2}\Gamma(\nu)(z/2)^{-\nu} \tag{7}$$

as $x \to 0^+$ [1, 9.6.9], we have $h_\nu(x) \sim x/(2\nu - 1)$ if $\nu > 1/2$. On the other hand, for $\nu = 1/2$ we have $h_{1/2}(x) = K_0(x)/K_1(x) \sim -x\log(x/2)$ (see [1, 9.8.5] and [1, 9.8.7]) and for $\nu \in (0, 1/2)$, applying (3) we write $h_\nu(x) = K_{1/2-\nu}(x)/K_{1/2+\nu}(x)$ and using again (7) we get $h_\nu(x) \sim C(\nu)(x/2)^{2\nu}$, $C(\nu) = \Gamma(1/2-\nu)/\Gamma(1/2+\nu)$.

Summarizing we have that $h_\nu(0^+) = 0^+$ for all $\nu > 0$, $h'_\nu(0^+) > 0$ for $\nu > 1/2$ and $h'_\nu(0^+) = +\infty$ for $\nu \in (0, 1/2]$. □

Now, it is straightforward to prove the following

Theorem 2.

$$\frac{K_{\nu-1/2}(x)}{K_{\nu+1/2}(x)} > B_\nu(x) = \frac{x}{\sqrt{x^2+\nu^2}+\nu}, \quad \nu > 0,\ x > 0. \tag{8}$$

For $\nu < 0$ the inequality is reversed and for $\nu = 0$ the equality holds.

Proof. Consider $\nu > 0$ (for $\nu = 0$ the result is trivial). Because of Lemma 1, the ratio $h_\nu(x)$ is positive and increasing in the vicinity of $x = 0$ ($x > 0$). Now, consider (5); the critical line $-1 + \frac{2\nu}{x}z(x) + z(x)^2 = 0$, has the positive solution $z(x) = B_\nu(x)$. Clearly $h'_\nu(x) > 0$ if $h_\nu(x) > B_\nu(x)$; therefore, $h_\nu(x) > B_\nu(x)$ around $x = 0$. But then necessarily $h_\nu(x) > B_\nu(x)$ for all $x > 0$, otherwise we enter in contradiction with the fact that $h_\nu(x) > 0$ for all $x > 0$ ($K_\nu(x) > 0$ for all $\nu \in \mathbb{R}$ and all $x > 0$ (see [1, Eq. 9.6.24])).

Indeed, if $h_\nu(x_0) = B_\nu(x_0)$ for some $x_0 > 0$ then, because $B'_\nu(x) > 0$ and $h'_\nu(x_0) = 0$, the curve $h_\nu(x)$ would cross the critical line and enter the region $0 < h_\nu(x) < B_\nu(x)$, where $h'_\nu(x) < 0$. But taking the derivative of (5), we have $h''_\nu(x) = -2\nu x^{-2}h_\nu(x) + h'_\nu(x)(2\nu/x + h_\nu(x)) < 0$, and the graph of $h_\nu(x)$ would cross the x-axis, becoming negative. □

2.2. The I-Bessel function

For the I-Bessel functions similar arguments follow for $\nu \geqslant 0$.
We consider the function

$$g_\nu(x) = \frac{I_{\nu+1/2}(x)}{I_{\nu-1/2}(x)}. \tag{9}$$

Now, considering the relations [1, 9.6.26]:

$$I'_\nu(x) = I_{\nu+1}(x) + \frac{\nu}{x} I_\nu(x), \qquad I'_\nu(x) = I_{\nu-1}(x) - \frac{\nu}{x} I_\nu(x). \tag{10}$$

We obtain the following Riccati equation:

$$g'_\nu(x) = 1 - \frac{2\nu}{x} g_\nu(x) - g_\nu(x)^2. \tag{11}$$

We are next analyzing the qualitative behavior of the solutions of the Riccati equation satisfying the conditions of the next lemma.

Lemma 2. *Let $g_\nu(x) = I_{\nu+1/2}(x)/I_{\nu-1/2}(x)$, then*

1. *If $\nu \geqslant -1/2$ then $g_\nu(0^+) = 0^+$ and $g'_\nu(0^+) > 0$.*
2. *If $\nu < -1/2$, $\nu \neq -(k+1/2)$, $k \in \mathbb{N}$, then $g_\nu(0^+) = 0^-$, $g'_\nu(0^+) < 0$.*
3. *If $\nu = k + 1/2$, $k \in \mathbb{Z}$, $g_\nu(x) = (g_{-\nu}(x))^{-1}$.*

Additionally, $g_\nu(x) = 1 - \frac{\nu}{x} + \nu\frac{\nu-1}{2x^2} + \mathcal{O}(x^{-3})$ as $x \to +\infty$ and, therefore $B_\nu(x) - g_\nu(x) = \frac{\nu}{2x^2} + \mathcal{O}(x^{-3})$.

Proof. Because $I_\nu(x) \sim (x/2)^\nu/\Gamma(\nu+1)$ if $\nu \neq -1, -2, \ldots$ [1, 9.6.7], then $g_\nu \sim x/(2\nu+1)$, $\nu \neq -1/2, -3/2, \ldots$ and the behavior close to $x = 0$ is proved. For $\nu = k + 1/2$, $k \in \mathbb{Z}$ the result follows from the symmetry relation $I_n(x) = I_{-n}(x)$, $n \in \mathbb{N}$. Finally, considering the asymptotic expansion [1, 9.7.1] the asymptotic approximation for $g_\nu(x)$ follows easily. □

With the aid of the previous lemma, we can prove the next theorem.

Theorem 3.

$$0 < \frac{I_{\nu+1/2}(x)}{I_{\nu-1/2}(x)} < B_\nu(x) = x/\bigl(\nu + \sqrt{\nu^2 + x^2}\bigr), \quad \nu \geqslant 0,\ x > 0. \tag{12}$$

Proof. Let $g_\nu(x) = I_{\nu+1/2}(x)/I_{\nu-1/2}(x)$. Because of Lemma 2, the ratio $g_\nu(x)$ is positive and increasing in the vicinity of $x = 0$ ($x > 0$). Now, consider (11); the critical equation $1 - \frac{2\nu}{x} z(x) - z(x)^2 = 0$ has the positive solution $z_+(x) = B_\nu(x)$. Because $g'_\nu(x) > 0$ if $0 \leqslant g_\nu(x) < B_\nu(x)$ and $g'_\nu(x) < 0$ if $g_\nu(x) > B_\nu(x)$, it is clear than $g_\nu(x) < B_\nu(x)$ close to $x = 0$. But then necessarily $g_\nu(x) < B_\nu(x)$ for all $x > 0$. Indeed, a value x_0 such that $g_\nu(x_0) = B_\nu(x_0)$ (and therefore $g'_\nu(x_0) = 0$) can not be reached, because $g_\nu(x)$ approaches the graph of the increasing function $B_\nu(x)$ from below and for this it is necessary that $g'_\nu(x_0) > B'_\nu(x_0)$, in contradiction with the fact that $g'_\nu(x_0) = 0$. □

For $\nu < 0$, it is also possible to follow a qualitative analysis of the solution of the Riccati equation. However, the results are far less simple and interesting, particularly when $\nu < -1/2$, due to the appearance of zeros of $I_{\nu+1/2}(x)$ and $I_{\nu-1/2}(x)$. Also, they are far less interesting in applications. We only mention that for large enough x, the inequality $I_{\nu+1/2}(x)/I_{\nu-1/2}(x) > B_\nu(x)$ must hold for negative ν (see last result of Lemma 2), but that only if $\nu = -1/2, -3/2, \ldots$ it holds for all $x > 0$. We omit the details.

3. Iteratively refined inequalities

For the I-function, the upper and lower bounds can be iteratively refined; bounds can be obtained which converge to the ratio of functions for any $\nu \geqslant 0$. For the K-function, successive bounds can be also obtained using similar ideas, although the range of applicability as a function of ν decreases in each iteration. For the K-function, new and tighter bounds and Turán-type inequalities can be proved using these refined bounds. Also, we will extend the range of validity of some bounds and inequalities for the I-function.

3.1. The I-Bessel function

We denote

$$r_\nu(x)=\frac{1}{x}\frac{I_\nu(x)}{I_{\nu-1}(x)}. \tag{13}$$

The following holds:

Theorem 4. *Let $u_\nu^{(0)}(x)$ and $l_\nu^{(0)}(x)$ be such that*

$$0<l_\nu^{(0)}(x)<r_\nu(x)<u_\nu^{(0)}(x),\quad \nu\geqslant\nu_0\geqslant 0 \tag{14}$$

and consider the sequences

$$\begin{aligned} l_\nu^{(k+1)}(x)&=\big(2\nu+x^2u_{\nu+1}^{(k)}(x)\big)^{-1},\\ u_\nu^{(k+1)}(x)&=\big(2\nu+x^2l_{\nu+1}^{(k)}(x)\big)^{-1}, \end{aligned} \tag{15}$$

then

$$0<l_\nu^{(k)}(x)<r_\nu(x)<u_\nu^{(k)}(x),\quad \nu\geqslant\max\{\nu_0-k,0\}. \tag{16}$$

Proof. The result follows by induction. First, we assume that

$$0<l_\nu^{(k-1)}(x)<r_\nu(x)<u_\nu^{(k-1)}(x),\quad \nu\geqslant\max\{\nu_0-k+1,0\} \tag{17}$$

(which holds for $k=1$).

Now, because $I_{\nu-1}(x)=\frac{2\nu}{x}I_\nu(x)+I_{\nu+1}$ we have

$$r_\nu(x)=\frac{x^{-2}}{2\nu x^{-2}+r_{\nu+1}(x)}, \tag{18}$$

and considering Eq. (17) we have $0<l_{\nu+1}^{(k-1)}(x)<r_{\nu+1}(x)<l_{\nu+1}^{(k-1)}(x)$, $\nu\geqslant\max\{\nu_0-k,0\}$, and provided that $\nu\geqslant 0$ (in which case $r_\nu(x)>0$ and $r_{\nu+1}(x)>0$), using (18) we have

$$0<\frac{x^{-2}}{2\nu x^{-2}+u_{\nu+1}^{(k-1)}(x)}<r_\nu(x)<\frac{x^{-2}}{2\nu x^{-2}+l_{\nu+1}^{(k-1)}(x)},\quad \nu\geqslant\max\{\nu_0,0\} \tag{19}$$

and the theorem is proved. □

In the previous theorem we can choose (Theorem 3)

$$u_\nu^{(0)}(x)=\big(\nu-1/2+\sqrt{(\nu-1/2)^2+x^2}\big)^{-1} \tag{20}$$

and the result applies with $\nu_0=1/2$.

For a lower bound, we could consider, for instance (see [18])

$$l_\nu^{(0)}(x)=\big(\nu+\sqrt{\nu^2+x^2}\big)^{-1}, \tag{21}$$

and the theorem applies for $\nu_0=0$. We note that we also have the bound

$$r_\nu(x)>l_\nu^{(1)}(x)=\big(\nu-1/2+\sqrt{(\nu+1/2)^2+x^2}\big)^{-1},\quad \nu\geqslant 0, \tag{22}$$

and that $l_\nu^{(1)}(x)>l_\nu^{(0)}(x)$ for all $\nu\geqslant 0$, $x>0$, therefore improving (21). The bound (22) appears in [4] (but a more restrictive range for ν is given).

With the initial values chosen as (20) and (21) some relations between bounds are as follows. First, because $l_\nu^{(1)}(x)>l_\nu^{(0)}(x)$ we have $u_\nu^{(2i+2)}(x)<u_\nu^{(2i+1)}(x)$ and $l_\nu^{(2i+1)}(x)>l_\nu^{(2i)}(x)$, $i=0,1,\ldots.$ On the other hand, it is also easy to check that

$$l_\nu^{(0)}(x)<l_\nu^{(2)}(x)<\cdots<r_\nu(x)<\cdots<u_\nu^{(3)}(x)<u_\nu^{(1)}(x). \tag{23}$$

Therefore we have monotonic and bounded sequences, and thus convergent sequences. That the limit is precisely $r_\nu(x)$ follows from the fact that, the continued fraction approximants

$$H_{\nu,0}=\frac{1}{2\nu},\qquad H_{\nu,k}=\frac{1}{2\nu+x^2H_{\nu+1,k-1}}=\frac{1}{2\nu+}\,\frac{x^2}{2(\nu+1)+}\cdots\frac{x^2}{2(\nu+k)} \tag{24}$$

converge to $r_\nu(x)$ as $k \to +\infty$ as a consequence of Pincherle's theorem.[1] And because it is clear that $0 < r_\nu(x) < u_\nu^{(i)}(x) < H_{\nu,0} = 1/(2\nu)$, $i = 1, 2$, if $\nu \geqslant 0$ then, using arguments similar to those in the proof of 4 we have $r_\nu(x) < u_\nu^{(2k+i)}(x) < H_{\nu,2k}$ and $H_{\nu,2k+1} < l_\nu^{(2k+1+i)}(x) < r_\nu(x)$, $i = 1, 2$, and convergence to $r_\nu(x)$ follows immediately.

3.2. The K-Bessel function

We denote

$$R_\nu(x) = \frac{1}{x}\frac{K_\nu(x)}{K_{\nu+1}(x)}. \tag{25}$$

Proceeding similarly as for the *I*-functions, one obtains the following theorem.

Theorem 5. *Let $U_\nu^{(0)}(x)$ and $L_\nu^{(0)}(x)$ be such that*

$$0 < L_\nu^{(0)}(x) < R_\nu(x) < U_\nu^{(0)}(x), \quad \nu \geqslant \nu_0 \tag{26}$$

and consider the sequences

$$L_\nu^{(i+1)}(x) = \left(2\nu + x^2 U_{\nu-1}^{(i)}(x)\right)^{-1}, \qquad U_\nu^{(i+1)}(x) = \left(2\nu + x^2 L_{\nu-1}^{(i)}(x)\right)^{-1} \tag{27}$$

then

$$xL_\nu^{(i)}(x) \leqslant \frac{K_\nu(x)}{K_{\nu+1}(x)} < xU_\nu^{(i)}(x), \quad \nu \geqslant \max\{\nu_0 + i, 0\}. \tag{28}$$

Proof. The proof is very similar to that in Theorem 4 and follows from the use of the relation

$$R_\nu(x) = \frac{x^{-2}}{2\nu x^{-2} + R_{\nu-1}(x)}. \qquad \square \tag{29}$$

In Theorem 5 we can take

$$L_\nu^{(0)}(x) = \left(\nu + 1/2 + \sqrt{(\nu + 1/2)^2 + x^2}\right)^{-1} \tag{30}$$

(see Theorem 2); for this bound $\nu_0 = -1/2$ and the inequalities generated (for $U_\nu^{(i)}(x)$, $i = 1, 3, 5, \ldots$ and $L_\nu^{(i)}(x)$, $i = 0, 2, 4, \ldots$) turn to equalities when $\nu = -1/2 + i$.

Later we prove that the following upper bound can be considered,

$$U_\nu^{(0)}(x) = \left(\nu + \sqrt{\nu^2 + x^2}\right)^{-1}, \tag{31}$$

which holds for any $\nu \in \mathbb{R}$ (and then $\nu_0 = -\infty$), but is only sharp when $\nu > 0$. This upper bound is easily obtainable from the bound [18, Thm. 2] using the symmetry relation (3).

From (31) and using Theorem 5, we obtain the lower bound $L_\nu^{(1)}(x) = (2\nu + x^2 U_{\nu-1}^{(0)}(x))^{-1}$. This lower bound, as the upper bound, also holds for any $\nu \in \mathbb{R}$: it is clear that it must hold for $\nu \geqslant 0$, and for $\nu < 0$ it is also easy to check that it holds (because $2\nu + x^2 U_{\nu-1}^{(0)}(x) > 0$, $R_{\nu-1}(x) + 2\nu x^{-2} > 0$ and using (29)). After this, $U_\nu^{(2)}(x)$ holds for $\nu \geqslant 0$, $L_\nu^{(3)}(x)$ for $\nu \geqslant 1$, $U_\nu^{(4)}(x)$ for $\nu \geqslant 2$ and so on.

We don't give explicit expressions except for the first few bounds generated, which are collected in the following theorem, together with some relations between consecutive bounds.

Theorem 6. *Let*

$$B_{\lambda,\gamma}(x) = \left(\lambda + \sqrt{\gamma^2 + x^2}\right)^{-1}, \tag{32}$$

and $L_\nu^{(0)}$, $U_\nu^{(0)}$ as in (30) *and* (31).

Then

1. $$B_{\nu+\frac{1}{2},\nu+\frac{1}{2}}(x) = L_\nu^{(0)} \leqslant R_\nu(x) \leqslant 1, \quad \nu \geqslant -1/2 \tag{33}$$

   and the equalities only hold for $\nu = -1/2$.

[1] This is so because the function $i_\nu(x) = x^{-\nu} I_\nu(x)$ is the minimal solution of the recurrence $x^2 i_{\nu+1}(x) + 2\nu i_\nu(x) - i_{\nu-1}(x) = 0$ (a dominant solution being $e^{i\pi\nu} x^{-\nu} K_\nu(x)$) and Pincherle's theorem [11, Thm. 4.7] guarantees the convergence of the associated continued fraction, with approximants given by (24), to the ratio $i_\nu(x)/i_{\nu-1}(x) = r_\nu(x)$.

2. $$B_{\nu+1,\nu-1}(x) = L_\nu^{(1)} < R_\nu(x) < U_\nu^{(0)} = B_{\nu,\nu}(x), \quad \nu \in \mathbb{R}. \tag{34}$$

3. $$R_\nu(x) \leqslant U_\nu^{(1)} = B_{\nu+\frac{1}{2},\nu-\frac{1}{2}}(x), \quad \nu \geqslant 1/2 \tag{35}$$

and the equality only holds for $\nu = 1/2$.

Some comparisons between bounds are as follows:

1. $U_\nu^{(2i+1)}(x) < U_\nu^{(2i)}(x)$ *in the range of validity of* $U_\nu^{(2i+1)}(x)$.
2. $L_\nu^{(2i+2)}(x) > L_\nu^{(2i+1)}(x)$ *in the range of validity of* $L_\nu^{(2i+2)}(x)$.
3. $U_\nu^{(0)}(x) < 1 \iff \nu > 0$.
4. $U_\nu^{(2i+2)}(x) < U_\nu^{(2i)}(x) \iff \nu > 2i+1$.
5. $L_\nu^{(2i+3)}(x) < U_\nu^{(2i+1)}(x) \iff \nu > 2(i+1)$.

Proof. The first three results are proved from the previous discussion except the upper bound in (33). Because $K_\nu(x)$ is increasing as a function of ν when $\nu \geqslant 0$ (as becomes evident considering [1, 9.6.24]) the result is obviously true if $\nu \geqslant 0$. For $\nu \in [-1/2, 0)$, because $K_\nu(x) = K_{-\nu}(x)$ we have $K_{\nu+1}(x)/K_\nu(x) = K_{\nu+1}(x)/K_{-\nu}(x) \geqslant 1$ and the equality is only reached when $\nu = -1/2$.

The comparison between bounds follows from simple algebraic manipulations and by induction. □

Observe that the comparison between bounds shows (1 and 2) that the bounds generated from $L_\nu^{(0)}$ are generally preferable. On the other hand (4 and 5) the bounds tend to be sharper in each iteration for large enough ν.

4. Consequences

4.1. Bounds obtained from Turán-type inequalities and Amos' bounds

Let us observe that from Theorems 4 and 5 the following follows straightforwardly:

Corollary 1. *Let $U_\nu^{(i)}(x)$, $L_\nu^{(i)}(x)$, $u_\nu^{(i)}(x)$, $l_\nu^{(i)}(x)$ as in Theorems 4 and 5, then*

$$\begin{aligned} &\frac{l_{\nu+1}^{(i)}(x)}{u_\nu^{(k)}(x)} < \frac{I_{\nu-1}(x)}{I_\nu(x)}\frac{I_{\nu+1}(x)}{I_\nu(x)} < \frac{u_{\nu+1}^{(i)}(x)}{l_\nu^{(k)}(x)}, \\ &\frac{L_{\nu-1}^{(i)}(x)}{U_\nu^{(k)}(x)} \leqslant \frac{K_{\nu-1}(x)}{K_\nu(x)}\frac{K_{\nu+1}(x)}{K_\nu(x)} < \frac{U_{\nu-1}^{(i)}(x)}{L_\nu^{(k)}(x)} \end{aligned} \tag{36}$$

in the range of validity of the upper and lower bounds.

From this double-sided inequalities, one can obtain additional bounds using the three-term recurrence relation and solving the resulting inequations.

Indeed, if

$$0 < \mathcal{L}_\nu^K(x) < \frac{K_{\nu+1}(x)}{K_\nu(x)}\frac{K_{\nu-1}(x)}{K_\nu(x)} < \mathcal{U}_\nu^K(x) \tag{37}$$

using $K_{\nu-1}(x) = K_{\nu+1}(x) - 2\frac{\nu}{x}K_\nu(x)$ and solving the resulting inequality for the ratio $R_\nu(x) = x^{-1}K_\nu/K_{\nu+1}(x)$, assuming it is positive (which is true for all $x > 0$ and $\nu \in \mathbb{R}$), we get

$$\left(\nu + \sqrt{\nu^2 + x^2\mathcal{U}_\nu^K(x)}\right)^{-1} < R_\nu(x) < \left(\nu + \sqrt{\nu^2 + x^2\mathcal{L}_\nu^K(x)}\right)^{-1}. \tag{38}$$

Similarly, if

$$0 < \mathcal{L}_\nu^I(x) < \frac{I_{\nu+1}(x)}{I_\nu(x)}\frac{I_{\nu-1}(x)}{I_\nu(x)} < \mathcal{U}_\nu^I(x) \tag{39}$$

using $I_{\nu+1}(x) = I_{\nu-1}(x) - 2\frac{\nu}{x}I_\nu(x)$ and solving the resulting inequality for the ratio $r_\nu(x) = x^{-1}I_\nu(x)/I_{\nu-1}(x)$, assuming it is positive (which is true for all $x > 0$ if $\nu \geqslant 0$), we get

$$\left(\nu + \sqrt{\nu^2 + x^2\mathcal{U}_\nu^I(x)}\right)^{-1} < r_\nu(x) < \left(\nu + \sqrt{\nu^2 + x^2\mathcal{L}_\nu^I(x)}\right)^{-1}. \tag{40}$$

Amos built sequences of upper and lower bounds for $I_{\nu+1}(x)/I_\nu(x)$ following this type of procedure. The upper and lower bounds considered in his paper correspond to the case $k = i+1$ in Corollary 1, and starting with $u_\nu^{(0)}(x)$ (20). But this is equivalent to the bounds generated with Theorem 4, starting with $u_\nu^{(0)}(x)$. Indeed, taking $\mathcal{L}_\nu(x) = u_{\nu+1}^{(i)}/l_\nu^{(i+1)}(x)$ it is immediate to check that

$$r_\nu(x) < \left(\nu + \sqrt{\nu^2 + x^2 u_{\nu+1}^{(i)}/l_\nu^{(i+1)}(x)}\right)^{-1} = l_\nu^{(i+1)}(x). \tag{41}$$

4.2. Best possible Turán-type inequalities (old and new) and associated bounds

Let us note that the lower bound in Corollary 1 has larger range of applicability than the corollary seems to indicate. Indeed, because $0 < I_{\nu+1}/I_\nu(x) < x u_{\nu+1}^{(i)}(x)$ if $\nu > -1$ then, using (10),

$$I_{\nu-1}(x)I_{\nu+1}(x) = \left(\frac{I_\nu(x)}{x}\right)^2 x\frac{I_{\nu+1}(x)}{I_\nu(x)}\left(x\frac{I_{\nu+1}(x)}{I_\nu(x)} + 2\nu\right) < \left(I_\nu(x)\right)^2 u_{\nu+1}^{(i)}(x)\left(x^2 u_{\nu+1}^{(i)} + 2\nu\right) \tag{42}$$

therefore

$$\frac{I_{\nu+1}(x)}{I_\nu(x)}\frac{I_{\nu-1}(x)}{I_\nu(x)} < u_{\nu+1}^{(i)}(x)/l_\nu^{(i+1)}(x), \quad \nu \geqslant -1. \tag{43}$$

Considering this, we can prove in a simple way the known Turán inequality for I-Bessel functions, extending its range of validity (it was known to hold for $\nu \geqslant -1/2$).

Theorem 7. *For $x > 0$ and $\nu \geqslant -1$ the following holds*

$$I_{\nu-1}(x)I_{\nu+1}(x) < I_\nu(x)^2. \tag{44}$$

Proof. Considering the previous discussion we have

$$\frac{I_{\nu-1}(x)}{I_\nu(x)}\frac{I_{\nu+1}(x)}{I_\nu(x)} < \frac{u_{\nu+1}^{(0)}(x)}{l_\nu^{(1)}(x)}, \quad \nu \geqslant -1 \tag{45}$$

and it is easy to check that

$$\frac{u_{\nu+1}^{(0)}(x)}{l_\nu^{(1)}(x)} = 1 - u_{\nu+1}^{(0)}(x) < 1. \tag{46}$$

On the other hand, because $\lim_{x\to+\infty}\frac{I_\nu(x)}{I_\mu(x)} = 1$ for any ν, μ, the inequality is the best possible one not depending on x. □

And because,

$$I_{\nu-1}(x)I_{\nu+1}(x) = \left(\frac{I_\nu(x)}{x}\right)^2\left[\left(x\frac{I'_\nu(x)}{I_\nu(x)}\right)^2 - \nu^2\right] \tag{47}$$

this implies that

Corollary 2.

$$\left|x\frac{I'_\nu(x)}{I_\nu(x)}\right| < \sqrt{\nu^2 + x^2}, \quad \nu \geqslant -1. \tag{48}$$

Of course Corollary 2 also holds without absolute value. For the case $\nu \geqslant 0$ this is named Amos' inequality, that we extend.

Now, we prove the known Turán-type inequality for K-Bessel functions in a simple way.

Theorem 8.

$$\frac{K_{\nu+1}(x)K_{\nu-1}(x)}{K_\nu(x)^2} > 1, \quad \nu \in \mathbb{R}. \tag{49}$$

Proof. For $\nu \geqslant 1/2$,

$$\frac{K_{\nu-1}(x)}{K_\nu(x)}\frac{K_{\nu+1}(x)}{K_\nu(x)} > \frac{L_{\nu-1}^{(0)}(x)}{U_\nu^{(1)}(x)} = 1 + L_{\nu-1}^{(0)}(x) > 1. \tag{50}$$

With this, the result is proved for $\nu \geqslant 1/2$. In order to complete the proof for all ν we need to consider two additional properties, namely the symmetry relation $K_\nu(x) = K_{-\nu}(x)$ for all ν and the fact that $K_\nu(x)$ is an increasing function of ν when $\nu \geqslant 0$ (which is easily seen considering [1, 9.6.24]). Now, if $\nu \in [0, 1/2)$ the Turán inequality also holds because

$$\frac{K_{\nu-1}(x)}{K_\nu(x)} = \frac{K_{1-\nu}(x)}{K_\nu(x)},$$

$1 - \nu > \nu > 0$ and $K_\nu(x)$ increases as a function of ν. Then it is obvious that the Turán inequality holds for any $\nu \geqslant 0$. But then it holds for any real ν because $K_\nu(x) = K_{-\nu}(x)$.

On the other hand, because $\lim_{x\to+\infty} \frac{K_\nu(x)}{K_\mu(x)} = 1$ for any ν, μ, the inequality is the best possible one not depending on x. □

Incidentally, let us observe that $L_\nu^{(0)}(x) = u_{\nu+1}^{(0)}(x)$ and comparing (46) with (50) a natural question arises: does the expression

$$\frac{K_\nu(x)}{K_{\nu+1}(x)} \frac{K_{\nu+2}(x)}{K_{\nu+1}(x)} + \frac{I_{\nu-1}(x)}{I_\nu(x)} \frac{I_{\nu+1}(x)}{I_\nu(x)} - 2$$

have defined sign? The answer is no.

The Turán inequality (7) was proved in [19], where the more general result $I_\nu(x)^2 - I_{\nu-\epsilon}(x) I_{\nu+\epsilon}(x) > 0$ was obtained using an integral representation for the I function, but only for $\nu \geqslant -1/2$. We see that the less general version (7) can be proved with simple arguments in a larger range.

Let us observe that the Turán inequalities are obtained as a consequence of Theorem 1. Which is the only bound used in the proofs (together with some iterative refinements).

From the known Turán inequalities proved in this section (and extended for I-functions), additional bounds can be obtained by considering Eqs. (37)–(40), namely:

Corollary 3. *For $x > 0$ the following holds for the ratios* (13) *and* (25)

$$r_\nu(x) > \left(\nu + \sqrt{\nu^2 + x^2}\right)^{-1}, \quad \nu \geqslant 0 \tag{51}$$

and

$$R_\nu(x) < \left(\nu + \sqrt{\nu^2 + x^2}\right)^{-1}, \quad \nu \in \mathbb{R}. \tag{52}$$

As previously discussed, the first bound is weaker than the bound $l_\nu^{(1)}(x)$. The bound for K-Bessel functions will be useful for obtaining a new Turán-type inequality which proves and extends a conjecture [5] and has interesting applied consequences, as we next discuss.[2]

Considering Theorem 4 and Corollary 1 we can prove the following result.

Theorem 9. *If $\nu \geqslant 0$ and $x > 0$ then*

$$T_\nu^{(I)}(x) = \frac{I_{\nu-1}(x)}{I_\nu(x)} \frac{I_{\nu+1}(x)}{I_\nu(x)} > \frac{\nu}{\nu+1}. \tag{53}$$

The inequality is the best possible when $\nu \geqslant 0$.

Proof. This is easily obtained for Corollary 1 and simple algebraic manipulations:

$$T_\nu^{(I)}(x) > \frac{l_{\nu+1}^{(0)}}{u_\nu^{(1)}} = 1 - \frac{2}{\nu + 1 + \sqrt{(\nu+1)^2 + x^2}} > 1 - \frac{1}{\nu+1} \tag{54}$$

which is valid for $\nu \geqslant 0$. Because $\lim_{x\to+\infty} T_\nu^{(I)}(x) = \nu/(\nu+1)$, $\nu \geqslant 0$, the inequality is the best possible. □

The previous Turán inequality is already known (see for instance [5]).

Considering Theorem 5 and Corollary 1 we can prove the following new result.

[2] It is interesting to note that also (51) is a useful relation in applications. In [20], the mean number of molecules of a given class dissolved in a water droplet is compared using the so-called classical and stochastic approaches. Let N_{clas} and N_{stoc} be the respective numbers, then, with some redefinition of the variables it turns out that $N_{clas} = \frac{x^2}{4} l_{\nu+1}^{(0)}(x)$ and $N_{sto} = \frac{x}{4} I_{\nu+1}(x)/I_\nu(x)$. Therefore the bound (51) implies that $N_{clas} < N_{sto}$, $\nu \geqslant -1$, which was not known in general, but only for x close to zero and for large x.

Theorem 10. *For $x > 0$ and $\nu \notin [-1, 1]$ the following holds*

$$T_\nu^{(K)}(x) = \frac{K_{\nu-1}(x)}{K_\nu(x)}\frac{K_{\nu+1}(x)}{K_\nu(x)} < \frac{|\nu|}{|\nu|-1} \tag{55}$$

and the inequality is the best possible.

Proof. We only need to prove the result for $\nu > 1$. For $\nu < -1$ it follows by considering (3).

The result is again a consequence of Corollary 1

$$T_\nu^{(K)}(x) < \frac{U_{\nu-1}^{(0)}}{L_\nu^{(1)}} = 1 + \frac{2}{\nu - 1 + \sqrt{(\nu-1)^2 + x^2}} \tag{56}$$

which is valid for $\nu \geqslant 1/2$. From this, we have $T_\nu^{(K)}(x) < 1 + \frac{1}{\nu-1}$ for $\nu > 1$.

On the other hand, because $\lim_{x\to+\infty} T_\nu^{(K)}(x) = |\nu|/(|\nu|-1)$ for $|\nu| > 1$, the inequality is the best possible not depending on x. □

Theorem 10 appears to be new, but it is used for bounding the variance of a distribution in [2], although a sketch of proof for the Turán-type inequality is not given.[3] The same combination of K-functions also appears in [8, Eq. (37)], related with the variance of a different distribution.

Theorem 10 proves and improves an inequality conjectured by Baricz, which we now state as a theorem:

Theorem 11. *The modified Bessel functions $K_\nu(x)$ satisfy the following Turán-type inequalities*:

$$\begin{aligned}
&(\nu-1)K_{\nu-1}(x)K_{\nu+1}(x) - (2\nu-1)\big(K_\nu(x)\big)^2 < 0, \quad \nu \geqslant 0,\\
&(\nu+1)K_{\nu-1}(x)K_{\nu+1}(x) - (2\nu+1)\big(K_\nu(x)\big)^2 > 0, \quad \nu \leqslant 0.
\end{aligned} \tag{59}$$

Proof. Only the first inequality needs to be proved; the second follows from the first and taking into account that $K_\nu(x) = K_{-\nu}(x)$.

The first inequality is already proved for $\nu > 1$: indeed, Theorem 10 gives a sharper inequality because $(2\nu-1)/(\nu-1) > \nu/(\nu-1)$ if $\nu > 1$. For $\nu \in [1/2, 1]$, the result is obvious because the K-functions are positive. Finally, for $\nu \in [0, 1/2)$ the inequality reads

$$\frac{K_\nu(x)}{K_{\nu-1}(x)}\frac{K_\nu(x)}{K_{\nu+1}(x)} < 1 + \frac{\nu}{1-2\nu} \tag{60}$$

but this holds on account of Theorem 8. □

Considering Theorems 9 and 10 and the discussion of Section 4.1 we can obtain additional bounds, which we show in the next theorem, together with bounds on the condition numbers $xI_\nu'(x)/I_\nu(x)$ and $xK_\nu'(x)/K_\nu(x)$ (using the last equations in (10) and (4)).

Corollary 4. *For $x > 0$ the following holds*:

$$r_\nu(x) < \big(\nu + \sqrt{\nu^2 + x^2\nu/(\nu+1)}\big)^{-1}, \quad \nu \geqslant 0 \tag{61}$$

and

$$R_\nu(x) > \big(\nu + \sqrt{\nu^2 + x^2\nu/(\nu-1)}\big)^{-1}, \quad \nu \geqslant 1. \tag{62}$$

---

[3] In [2] the effective variance of the generalized Gaussian distribution is written as

$$v_{eff} = 1 - \frac{K_{\lambda-1}(x)K_{\lambda+1}(x)}{K_\lambda(x)^2}, \quad \lambda = 4 + \nu \tag{57}$$

where ν is called order of the distribution. From the Turán-type inequalities, we see that

$$0 < v_{eff} < \frac{1}{|\lambda|-1}, \quad \lambda \notin [-1, 1]. \tag{58}$$

This result is mentioned in [2] (as a function of ν), but a proof is not given. The result is now proved to be true. In the same paper, the effective radius of the distribution is proportional to $K_\lambda(x)/K_{\lambda-1}(x)$, for which we have also given upper and lower bounds.

As a consequence,

$$x\frac{I'_\nu(x)}{I_\nu(x)} > \sqrt{\nu^2 + x^2\nu/(\nu+1)}, \quad \nu \geqslant 0 \tag{63}$$

and

$$x\frac{K'_\nu(x)}{K_\nu(x)} > -\sqrt{\nu^2 + x^2\nu/(\nu-1)}, \quad \nu \geqslant 1. \tag{64}$$

The bound (63) was already known to hold [5], but (64) was only known to hold for positive integer ν, although it was conjectured in [5] that it should hold for any $\nu > 1$. This is now proved.

It is easy to check that the bound $u_\nu^{(1)}(x)$ in (23) and the bound $L_\nu^{(1)}(x)$ in (34) improve the bounds (61) and (62) respectively. Consequently, we can expect that the known bounds on the condition numbers can also be easily improved, as we next show.

4.3. *Improved bounds on the condition numbers*

In this section we denote

$$C\big(I_\nu(x)\big) = x\frac{I'_\nu(x)}{I_\nu(x)}, \qquad C\big(K_\nu(x)\big) = -x\frac{K'_\nu(x)}{K_\nu(x)} \tag{65}$$

which are positive quantities if $\nu \geqslant 0$ ($C(K_\nu(x))$ is positive for all ν).

Given a function $f(x)$, the condition number $C(f(x)) = |xf'(x)/f(x)|$ measures the ratio of the relative error in the evaluation of the functions $f(x)$ over the relative error in x. This is a crucial piece of information when computing functions, because it informs us about how the error is amplified or diminished with respect to the error in the variable. Condition numbers can be easily evaluated for elementary functions, but it is not common that sharp upper and lower bounds can be given for non-elementary functions, particularly when it depends on several variables.

Additionally, from this bounds one can extract bounds for ratios of functions of different arguments. For instance, if $A_\nu(x) < C(I_\nu(x)) < B_\nu(x)$ then, integrating, we have

$$\exp\left(\int_y^z \frac{A_\nu(x)}{x}\right) < \frac{I_\nu(z)}{I_\nu(y)} < \exp\left(\int_y^z \frac{B_\nu(x)}{x}\right). \tag{66}$$

Therefore, and improvement on the bounds of the condition numbers also implies and improvement over the bounds for these type of ratios. We will not give explicit formulas for this, and only concentrate on condition numbers.

Considering Theorem 4 and the relations (10) the following corollary follows straightforwardly.

Corollary 5. *For any* $i = 0, 1, \ldots$

$$x^2 l_{\nu+1}^{(i)}(x) + \nu < C\big(I_\nu(x)\big) < x^2 u_{\nu+1}^{(i)}(x) + \nu, \tag{67}$$

and

$$\big(u_\nu^{(i)}\big)^{-1} - \nu < C\big(I_\nu(x)\big) < \big(l_\nu^{(i)}\big)^{-1} - \nu, \tag{68}$$

with the range of validity corresponding to the bounds given by Theorem 4.

Using (15) we observe that (68) is this the same as (67) with i replaced by $i-1$. Therefore, both equations give the same bounds except if $i = 0$; if $i = 0$ (68) must be considered separately.

Similarly, we have for the K-functions the following

Corollary 6.

$$x^2 L_{\nu-1}^{(i)}(x) + \nu < C\big(K_\nu(x)\big) < x^2 U_{\nu-1}^{(i)}(x) + \nu, \tag{69}$$

and

$$\big(U_\nu^{(i)}(x)\big)^{-1} - \nu < C\big(K_\nu(x)\big) < \big(L_\nu^{(i)}(x)\big)^{-1} - \nu. \tag{70}$$

Similarly as for the I-functions except when $i = 0$ (69) and (70) are related. We only need to consider (70) if $i = 0$.

Interestingly, it appears that the known upper (lower) bounds for $C(I_\nu(x))$ ($C(K_\nu(x))$, including (63) and (64), appear from the use of (68) and (70) for $i = 0$. Using (67) and (69) iterative refinement is possible.

For instance, considering (68) with $u_\nu^{(0)}(x)$ and $l_\nu^{(0)}(x)$ as in (20) and (21), we obtain

$$\sqrt{(\nu-1/2)^2+x^2}-1/2<C\big(I_\nu(x)\big)<\sqrt{\nu^2+x^2}. \tag{71}$$

The upper bound was already proved in Corollary 22, and it was shown that it holds for $\nu\geqslant -1$. The lower bound holds for $\nu\geqslant 1/2$.

And using (67) with $i=0$:

$$\sqrt{(\nu+1)^2+x^2}-1<C\big(I_\nu(x)\big)<\sqrt{(\nu+1/2)^2+x^2}-1/2, \tag{72}$$

the lower bound holds for $\nu\geqslant -1$ and improves (63) when $\nu\geqslant 0$. The upper bound holds for $\nu\geqslant -1/2$, and is sharper that Amos inequality (Corollary 22).

The lower bound can be easily improved. Because, as we discussed, $l_\nu^{(1)}(x)>l_\nu^{(0)}(x)$ if $\nu\geqslant 0$ considering $i=1$ in (67) we have

$$x\frac{I'_\nu(x)}{I_\nu(x)}>\nu+\frac{x^2}{\nu+1/2+\sqrt{(\nu+3/2)^2+x^2}},\quad \nu\geqslant -1 \tag{73}$$

which indeed improves the lower bound when $\nu\geqslant 0$. Further iterative refinement is always possible.

Proceeding similarly for the K-Bessel function, with $L_\nu^{(0)}(x)$ and $U_\nu^{(0)}(x)$ as in (30) and (31) and using (70) fir $i=0$ we get

$$\sqrt{\nu^2+x^2}<C\big(K_\nu(x)\big)<\frac{1}{2}+\sqrt{(\nu+1/2)^2+x^2},\quad \nu\geqslant 0 \tag{74}$$

($K'_\nu(x)/K_\nu(x)<0$ for all ν).

Considering (69) for $i=0$ instead, we have the bounds:

$$\sqrt{(\nu-1/2)^2+x^2}+1/2<C\big(K_\nu(x)\big)<\sqrt{(\nu-1)^2+x^2}+1,\quad \nu\geqslant 1 \tag{75}$$

where the upper bound also holds for $\nu\geqslant 0$. Both bounds are sharper than the bounds in (74), although the lower bound is valid in a more restricted range. The upper bound also improves (64).

Additional bounds can be built, but either the bounds are more restricted in the ν variable, or they not an improvement for any x and ν or an explicit formula becomes too involved. We don't give further details.

4.4. Accuracy of the bounds and comparison with rational bounds

In Theorem 4, we can consider starting the process with the lower bound $l_\nu^{(0)}=0$. In this case, the successive iterations give the continued fraction approximants of the ratio $r_\nu(x)$ (24) which, as a consequence of this theorem are themselves rational bounds for the ratios of I-functions, as is well known. The approximants $H_{\nu,2k}$ are monotonically decreasing upper bounds converging to $r_\nu(x)$ and the approximants $H_{\nu,2k+1}$ are monotonically increasing lower bounds.

Similarly, one can build sequences of rational lower bounds for the K-functions (although they don't converge for any fixed $\nu>0$).

Next, we compare the new bounds (non-rational) with the continued fraction approximants and with other rational bounds appearing in the literature [21,10]. First, we start with the continued fraction approximants.

For the I-function, the next result follows by comparing the first bounds and by induction:

Theorem 12. *The following holds for any $x,\nu>0$:*

$$H_{\nu,j}>u_\nu^{(j+k)},\quad k\geqslant 1,\ j=0,2,\ldots,$$
$$H_{\nu,j}<l_\nu^{(j+k)},\quad k\geqslant 1,\ j=1,3,\ldots.$$

Some continued fraction approximants may be better for small x; for instance, one can check that $H_{\nu,j}<u_\nu^{(j)}$, $j=1,3,\ldots,$ only if $x<\sqrt{2}\sqrt{\nu+j}$ and a similar relation holds for lower bounds. However, it is guaranteed that only one additional iteration of the non-rational bounds provides improvement over the rational bounds.

The contrary is not true and for large x/ν the non-rational bounds are very superior. For instance, for $\nu=5$, $x=100$ we have $u_\nu^{(4)}(x)<H_{\nu,24}$. The reason behind this behavior is the fact that the non-rational bounds provide sharp approximations both around $x=0$ and $x=\infty$ with fixed $\nu>0$ and also for $\nu\to+\infty$ with $x>0$ fixed. Indeed, it is easy to check that

$$\lim_{x\to 0,+\infty} r_\nu(x)/u_\nu^{(i)}(x)=\lim_{x\to 0,+\infty} r_\nu(x)/l_\nu^{(i)}(x)=1,$$
$$\lim_{\nu\to+\infty} r_\nu(x)/u_\nu^{(i)}(x)=\lim_{\nu\to+\infty} r_\nu(x)/l_\nu^{(i)}(x)=1 \tag{76}$$

with the only exception of $u_\nu^{(0)}(x)$ as $x\to 0$. Contrary, the rational bounds $H_{\nu,k}$ are not sharp as $x\to+\infty$.

Table 1
The minimal number of accurate digits D_M given by the bound $u_\nu^{(i)}(x)$ for approximating the function $r_\nu(x)$. The abscissa x_M where this minimal value is reached is also given.

	$\nu=0$		$\nu=100$	
	D_M	x_M	D_M	x_M
$i=2$	1.4	5.75	3.45	534
$i=4$	1.86	18	3.7	933
$i=8$	2.55	66	3.98	1775
$i=16$	3.14	258	4.28	3560
$i=32$	3.75	1025	4.61	7530

Then, we can expect that the non-rational bounds are sharp in a much wider range than the continued fraction approximants. As a numerical illustration we give in Table 1 the minimal number of correct digits given by the bounds. We define the number of correct digits of a bound B as $D(B) = -\log_{10}(\epsilon_r)$, with ϵ_r the relative error. Numerical experiments show that the lowest number of accurate digits occurs for the smallest ν ($\nu = 0$), and that $D(B)$ reaches a relative minimum for some $x > 0$.

The advantage over the continued fraction approximants is very clear. For instance, the lower bound $H_{\nu,9}$ gives for $x = 66$ a number of correct digits $D = 0.25$, to be compared with the third row of the previous table. Of course, as x becomes larger the accuracy of the continued fraction approximants worsens.

Rational bounds which are sharp for large x can be obtained by approximating the initial values for iteration with Theorem 4. For instance, among other possibilities, we may consider starting with (20):

$$\frac{1}{x}\frac{I_\nu(x)}{I_{\nu-1}(x)} < u_\nu^{(0)}(x) < \frac{1}{x} \equiv \hat{u}_\nu^{(0)}(x), \quad \nu > 1/2 \tag{77}$$

and we can generate rational bounds $l_\nu^{(2i+1)}(x)$, $u_\nu^{(2i)}(x)$, $i = 0, 1, \ldots.$

More sophisticated rational bounds were obtained by Nåsell [21] which are increasingly sharper both as $x \to 0, +\infty$. A systematic comparison of the non-rational bounds with Nåsell's bounds seems difficult, but numerical experiments show that the non-rational bounds are sharper for not so large ν (larger as x becomes larger). For instance, using the sharpest lower bound explicitly given in [21] and with the notation of this paper, we have that $L_{\nu-1,3,0}(x)/x < l_\nu^{(1)}$ ($l_\nu^{(1)}$ given by (22)) if $x = 10$ and $\nu > 5$, if $x = 100$ and $\nu > 15$ and if $x = 1000$ and $\nu > 45$ (and the non-rational bounds are superior for this values).

For the K-function, as discussed in Section 3.2, the ν-range becomes more restricted in each iteration. The bounds generated from $U_\nu^{(0)}(x)$ are less restricted than those from $L_\nu^{(0)}(x)$, however they are generally less sharp for small ν; for instance, Theorem 6 shows that $U_\nu^{(0)}(x)$ is the best bound only when $\nu \in (0, 1/2)$.

For checking numerical accuracy, we concentrate on the bounds generated from $L_\nu^{(0)}(x)$ which, apart from being sharp bounds as $x \to 0, +\infty$ (with fixed ν) and $\nu \to +\infty$, hold with the equal sign for the lower value of ν for which they are valid (as a consequence of Theorem 1).

Denoting by $D(B)$ the number of exact digits given by the bound B, numerical experiments show that the bounds $U_\nu^k(x)$, $k = 1, 3, \ldots$ and $L_\nu^k(x)$, $k = 0, 2, \ldots$ give more accuracy in each iteration; these are strict bounds for $\nu > -1/2 + k$ and they hold with the equality sign for $\nu = -1/2 + k$. For the first bound we have no accuracy and $D(L_\nu^{(0)}) < 0$ ($\nu \geqslant -1/2$) and the worst case is for small x; if we limit the x range we have, for instance, $D(L_\nu^{(0)}) > 1.3$ if $x > 5$. The rest of bounds give always some accuracy ($D(B) > 0$) in all their rage in validity. For instance $D(U_\nu^{(1)}) > 1.5$ ($\nu \geqslant 1/2$), $D(L_\nu^{(2)}) > 2.2$ ($\nu \geqslant 3/2$), $D(U_\nu^{(3)}) > 3$ ($\nu \geqslant 5/2$), $D(L_\nu^{(4)}) > 3.5$ ($\nu \geqslant 7/2$).

Rational bounds for the K-functions can also be built. For instance, considering that $U_\nu^{(0)}(x) < 1/(2\nu)$, we can build bounds starting from the upper bound $1/(2\nu)$; however, the resulting continued fraction-like bounds are always weaker that the bounds generated from $U_\nu^{(0)}(x) < 1/(2\nu)$. A slightly more interesting possibility consists in starting from the upper bound in (33), which is the sharpest upper bound if $\nu \in (-1/2, 0)$ (for $\nu > 0$, $U_\nu^{(0)}(x)$ is better). This fact gives the chance to improve some bounds in restricted intervals. Let us call $\tilde{H}_{\nu,k}$ the bounds generated in this way, that is, the bounds

$$\tilde{H}_{\nu,k} = \frac{1}{2\nu+}\frac{x^2}{2(\nu-1)+}\cdots\frac{x^2}{2(\nu-k+2)+}\frac{x^2}{2(\nu-k+1)+x^2}. \tag{78}$$

Then, comparing the first bounds and using induction, it is not difficult to prove the following:

Theorem 13. *The bound $\tilde{H}_{\nu,k}$ is sharper than any non-rational bound generated from* (30) *or* (31) *only if $x < (\nu-k+1)/(\nu-k+1/2)$ with $\nu \in [-1/2+k, k)$.*

The rational bounds obtained in this way are exactly the same bounds obtained in [10], although they consider negative orders. We see that the non-rational bounds are superior except in a restricted region.

Acknowledgments

The author acknowledges financial support of Ministerio de Ciencia e Innovación, project MTM2009-11686. The author thanks the reviewer for useful suggestions.

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