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Constructing the Preprojective Components of an Algebra

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Let k be an algebraically closed field. Let A be a finite-dimensional k-algebra. We may assume that $A = kQ/I$, where Q is a finite connected quiver and I is an admissible ideal of the path algebra kQ; see [5]. The nature of our problem allows us to assume without loss of generality that Q has no oriented cycles.

Consider the category of the finite dimensional left A-modules, mod_A. For each indecomposable non-projective A-module X, the Auslander–Reiten translate $\tau_A X$ is an indecomposable non-injective module. The Auslander–Reiten quiver Γ_A has as vertices representatives of the iso-classes of finite dimensional indecomposable A-modules, and there are as many arrows from X to Y in Γ_A as $\dim_k \text{rad}_A(X,Y)/\text{rad}_A^2(X,Y)$. A connected component $\mathscr{P}$ of Γ_A is said to be *preprojective* if $\mathscr{P}$ has no oriented cycles and each module X in $\mathscr{P}$ has only finitely many predecessors in the path order of $\mathscr{P}$. Several classes of algebras have preprojective components, such as algebras with the separation condition (in particular, tree algebras) [2] and hereditary algebras [9]. A general criterion for the existence of preprojective components was recently established [4].

Given a preprojective component $\mathscr{P}$ of Γ_A, the modules on $\mathscr{P}$ can be easily determined. Starting with the simple projective modules and using the additivity of the dimension function on Auslander–Reiten sequences, the classes $\mathbf{dim}\, X$ in the Grothendieck group $K_0(A)$ of modules $X \in \mathscr{P}$ are obtained; the module X is the unique indecomposable with class $\mathbf{dim}\, X$. This *knitting procedure* has been used since at least 1977 (see [5]).

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The purpose of this work is to give an algorithmic procedure to construct all preprojective components in Γ_A. Indeed, we do the following.

(a) We describe an algorithm which decides whether or not a given simple projective module P_i belongs to a preprojective component; in fact, we show that if by starting with P_i it is possible to use the knitting procedure to construct $N(\dim_k A)$ new modules (where $N(\dim_k A)$ is a certain number depending only on $\dim_k A$), then P_i lies in a preprojective component of Γ_A;

(b) if P_i belongs to a preprojective component $\mathscr{P}$, by applying the procedure (a), we get two functionals $f, g\colon K_0(A) \to \mathbf{R}$ such that an indecomposable module X belongs to $\mathscr{P}$ if and only if one of the following holds: (i) $f(\mathbf{dim}\, X) > 0$ or (ii) $f(\mathbf{dim}\, X) = 0$ and $g(\mathbf{dim}\, X) < 0$.

For the proof of the above statements we show some results on the growth of $(\dim_k \operatorname{Hom}_B(\tau_B^{-t}X, Y))_t$ which are interesting by themselves. Indeed, for a wild connected hereditary algebra $B = k\Delta$ and two indecomposable B-modules X and Y such that X is preprojective and Y is regular or preinjective, we prove that

$$\dim_k \operatorname{Hom}_B\left(\tau_B^{-t}X, Y\right) \geq \left[\frac{t}{d^2}\right]$$

for $t \geq 3d$, where d is the number of vertices of Δ.

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1. DIMENSION OF MODULES IN PREPROJECTIVE COMPONENTS

1.1 Let $A = kQ/I$ be as in the Introduction. We assume that the set of vertices of Q is $Q_0 = \{1, \ldots, n\}$. For each vertex $i \in Q_0$, we have the simple module S_i with $S_i(i) = k$ and $S_i(j) = 0$ for $j \neq i$. The projective cover of S_i is denoted by P_i and its injective envelope by I_i.

Given a module $X \in \operatorname{mod}_A$, its class $\mathbf{dim}\, X \in K_0(A) = \mathbf{Z}^n$ has ith coordinate $\dim_k \operatorname{Hom}_A(P_i, X)$. Since the global dimension gldim A is finite, we get a bilinear form

$$\langle -,- \rangle_A\colon K_0(A) \times K_0(A) \to \mathbf{Z}, \langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle = \sum_{s=0}^{\infty} (-1)^s \dim_k \operatorname{Ext}_A^s(X, Y).$$

Let X be a module in a preprojective component $\mathscr{P}$ of Γ_A. Then $\operatorname{Ext}_A^s(X, X) = 0$ for $s \geq 1$ and $\dim_k \operatorname{End}_A(X) = 1$. Also there is a quotient B of A such that X is a faithful B-module. Then gldim $B \leq 2$ and $p\dim_B X \leq 1$. Hence if Y is an indecomposable A-module with $\mathbf{dim}\, X = \mathbf{dim}\, Y$, then Y is a B-module and $1 = \langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle_B$. Thus $\mathrm{Hom}_A(X, Y) \neq 0$. Similarly, $\mathrm{Hom}_A(Y, X) \neq 0$, which implies that X and Y are isomorphic. See [3, 9].

1.2. Following [6], we say that the module $X \in \mathrm{mod}_A$ is *directing* provided there do not exist indecomposable direct summands X_1, X_2 of X and an indecomposable nonprojective module Y such that $X_1 \leq \tau_A Y$ and $Y \leq X_2$ (we write $Y \leq Z$ for two indecomposable modules Y, Z if there is a chain of non-zero maps $Y = Y_0 \to Y_1 \to Y_2 \to \cdots \to Y_t = Z$). In particular, an indecomposable module X is directing if and only if there is no chain of non-zero, non-isomorphisms $X \to X_1 \to \cdots \to X_t = X$.

For an indecomposable projective module P_i, the radical rad P_i satisfies $P_i/\mathrm{rad}\, P_i \simeq S_i$. In [6] (see also [11]), it is shown that the following assertions are equivalent:

(a) P_i is directing,

(b) rad P_i is directing,

(c) each indecomposable direct summand of rad P_i is directing.

1.3. Given $i \in Q_0$, consider the quotient A^i of A formed as the full subcategory of A with vertices j such that there is no path from j to i in Q (*Note*: There is a trivial path from i to i.).

The following is an inductive criterion for the existence of preprojective components in Γ_A.

THEOREM [4]. *There is a preprojective component in Γ_A if and only if for each vertex $i \in Q_0$ one of the following conditions is satisfied*:

(a) *there is a preprojective component $\mathscr{P}'$ of Γ_{A^i} such that no indecomposable direct summand of* rad P_i *belongs to $\mathscr{P}'$*;

(b) rad P_i *is a directing A^i-module and each indecomposable direct summand of* rad P_i *has only finitely many predecessors in* mod_{A^i}, *all of them directing*.

1.4. Let $\mathscr{C}$ be a component of Γ_A. Let $\mathscr{S}$ be a full subquiver in $\mathscr{C}$. We say that

(i) $\mathscr{S}$ is a *section* if $\mathscr{S}$ is path-closed in $\mathscr{C}$; if $X \in \mathscr{S}$, then $\tau_A X \notin \mathscr{S}$; each $X \in \mathscr{S}$ is directing.

(ii) $\mathscr{S}$ is a *m-complete section* ($m \in \mathbf{N} \cup \{\infty\}$) if:

(a) $\mathscr{S}$ is a section;

(b) $\mathscr{S}$ admits only finitely many precessors in Γ_A, all of them directing;

(c) if $X \to Y$ is an arrow in Γ_A, $X \in \mathscr{S}$, and $Y \notin \mathscr{S}$, then Y is non-projective and $\tau_A Y \in \mathscr{S}$;

(d) if $X \in \mathscr{S}$, $0 \le l \le m$, and Y is a predecessor of $\tau_A^{-l} X$ such that $Y = I_j$ or Y is a direct summand of rad P_j, then Y is proper predecessor of $\mathscr{S}$.

Remarks. (1) Let $\mathscr{S}$ be a m-complete section in $\mathscr{C}$. Then we get a full subquiver $\tau_A^{-l}\mathscr{S}$ of $\mathscr{C}$ formed by the modules $\{\tau_A^{-l}X\colon X \in \mathscr{S}\}$ for any $0 \le l \le m$. The quiver $\tau_A^{-l}\mathscr{S}$ is a $(m - l)$-complete section in $\mathscr{C}$.

Let $k(\mathscr{C})$ be the mesh-category of $\mathscr{C}$; see [5, 8]. Consider any two modules X, Y in $\mathscr{C}$ predecessors of $\tau_A^{-m}\mathscr{S}$, then $\operatorname{Hom}_A(X, Y) = k(\mathscr{C})(X, Y)$.

(2) Let $\mathscr{P}$ be an infinite preprojective component of Γ_A. Consider the modules $\tau_A^{-s_i}P_i$ such that $P_i \in \mathscr{P}$, $\tau_A^{-s}P_i$ is non-injective for all $s \ge 0$, and s_i is minimal such that $\tau_A^{-s_i}P_i$ is no predecessor in $\mathscr{P}$ of an injective or a projective module. Then the full subquiver $\mathscr{S}$ of $\mathscr{P}$ formed by all modules $\tau_A^{-s_i}P_i$ is a maximal ∞-complete section in $\mathscr{P}$. Almost every module in $\mathscr{P}$ belongs to $\bigcup_{t \ge 0}\tau_A^{-t}\mathscr{S}$.

1.5. The main purpose of this section is to show the following

THEOREM. *Let $\mathscr{S}$ be a connected component of a m-complete section in a component $\mathscr{C}$ of Γ_A. Assume that $\mathscr{S}$ is not of Dynkin type. Then for every $3n \le t \le m + 1$ and $X \in \mathscr{S}$, we have*

$$\left[\frac{t}{n^2}\right] \le \dim_k \tau_A^{-t} X.$$

Let us show the useful consequence.

COROLLARY. *Let $\mathscr{S}$ be a connected component of a m-complete section in a component $\mathscr{C}$ of Γ_A and assume that $\mathscr{S}$ is not of Dynkin type. Let M be the maximal of all $\dim_k Y$, where $Y = I_j$ or Y is a direct summand of a* rad P_j, *for some $j \in Q_0$. Suppose $m + 1 > Mn^2$. Then $\mathscr{S}$ is ∞-complete section.*

Proof. Let $X \in \mathscr{S}$. Since $\mathscr{S}$ is a m-complete section, $\tau_A^{-m}X$ is non-injective. Therefore $\tau_A^{-(m+1)}X$ is a well-defined and $\dim_k \tau^{-(m+1)}X \ge [(m + 1)/n^2] > M$. Hence $\tau_A^{-(m+1)}X$ is not isomorphic to I_j or to a direct summand of rad P_j for some $j \in Q_0$. Thus $\mathscr{S}$ is a $(m + 1)$-complete section.

Since $[(m + 2)/n^2] > M$, we may continue inductively to get that $\mathscr{S}$ is a ∞-complete section. ∎

1.6. We shall reduce the proof of (1.5) to the case of preprojective components of tilted algebras. We recall that for a given module $X \in \operatorname{mod}_A$, the *support* of X is $\operatorname{supp} X = \{i \in Q_0 \colon X(i) \neq 0\}$.

LEMMA. *Let $\mathscr{C}$ be a component of Γ_A and $\mathscr{S}$ be a connected component of a m-complete section in $\mathscr{C}$ which is not of Dynkin type. Let $X_1, \dots, X_d$ be the modules in $\mathscr{S}$ and B be the full subcategory of A in the vertices of $\bigcup_{i=1}^{d} \operatorname{supp} X_i$. Then*

(a) *There is a preprojective component $\mathscr{C}'$ of Γ_B containing $\mathscr{S}$. Moreover $\mathscr{S}$ is a m-complete section in $\mathscr{C}'$.*

(b) *The module $\oplus_{i=1}^{d} X_i$ is a B-tilting module.*

(c) *The modules $\tau_A^{-l} X$, for $X \in \mathscr{S}$ and $0 \leq l \leq m+1$ are B-modules and $\tau_B^{-l} X = \tau_A^{-l} X$.*

(d) *Let ϕ_A (resp. ϕ_B) be the Coxeter matrix of A (resp. B), then for any $X \in \mathscr{S}$, $0 \leq l \leq m+1$, we have*

$$\mathbf{dim}\, \tau_A^{-l} X = (\mathbf{dim}\, X)\, \phi_B^{-l}$$

Proof. Clearly, $\mathscr{S}$ is formed by B-modules. Since $\mathscr{S}$ is a m-complete section, the modules $\tau_A^{-l} X$ with $X \in \mathscr{S}$ and $0 \leq l \leq m+1$ are also B-modules and $\tau_B^{-l} X = \tau_A^{-l} X$. This shows (c).

By the definition of $B, \mathscr{C}'$ contains all the indecomposable projective B-modules P'_j, $1 \leq j \leq m$. To show that $\mathscr{S}$ is a *slice* in B (in the sense of [9, 4.2]), it is enough to observe that $\oplus_{i=1}^{d} X_i$ is a sincere B-module. This proves (a) and (b).

Let $X \in \mathscr{S}$ and $0 \leq l \leq m$. Then $\operatorname{Hom}_B(\tau_B^{-l-1} X, B) = 0$ implies that $i \dim_B \tau_B^{-l} X \leq 1$. Moreover, since $\tau_B^{-l} X$ has no injective predecessors in Γ_B, then $\mathbf{dim}\, \tau_B^{-l-1} X = (\mathbf{dim}\, \tau_B^{-l} X) \phi_B^{-1}$; see [9, 2.4]. Therefore (d) follows by induction. ∎

1.7. For the proof of (1.5) we need some results on the growth of $\dim_k \operatorname{Hom}_B(\tau_B^{-t} X, Y)$ in the case B is a hereditary algebra.

We recall that a connected algebra $B = k\Delta$ is of *tame* representation-infinite type if Δ is a quiver of Euclidean type; B is of *wild type* if either Δ contains a quiver of the form  with at least 3 arrows or Δ contains properly a convex subquiver of Euclidean type.

PROPOSITION. *Let $B = k\Delta$ be a representation-infinite, connected hereditary algebra and $\Delta_0 = \{1, \dots, d\}$. Let X and Y be indecomposable B-modules.*

Assume that X is preprojective and Y is preinjective (resp., preinjective or regular) if B is tame (resp., B is wild). Then

$$\dim_k \operatorname{Hom}_B(\tau_B^{-t} X, Y) > \left[\frac{t}{d^2}\right], \qquad \textit{for } t \geq 3d.$$

Proof. We shall divide the proof in several steps:

(1) For any two vertices $i, j \in \Delta_0$, we show that $\dim_k \tau_B^{-t} P_i(j) \geq 4[t/d^2]$, for $t \geq d$.

(a) We consider first the case where Δ is of Euclidean type. Let $\mathscr{T}_1, \ldots, \mathscr{T}_s$ be the non-homogeneous tubular components of Γ_B; let $X_1^{(i)}, \ldots, X_{n_i}^{(i)}$ be the modules in the mouth of the tube $\mathscr{T}_i$. Then $\sum_{j=1}^{s}(n_i - 1) = d - 2$ and $\sum_{j=1}^{n_i} \mathbf{dim}\, X_j^{(i)} = z$ $(1 \leq i \leq s)$, where z is a sincere vector generating the space $\{v \in \mathbf{Q}^n \colon \langle \mathbf{dim}\, X_j^{(i)}, v\rangle_B = 0,\ 1 \leq i \leq s,\ i \leq j \leq n_i\}$. Let m be the least common multiple of $n_1, \ldots, n_s$. Then

$$\begin{aligned}\langle \mathbf{dim}\, X_j^{(i)}, \mathbf{dim}\, \tau_B^{-m} P_i\rangle_B &= \langle \mathbf{dim}\, \tau_B^{-m} X_j^{(i)}, \mathbf{dim}\, P_i\rangle_B \\ &= \langle \mathbf{dim}\, X_j^{(i)}, \mathbf{dim}\, P_i\rangle_B.\end{aligned}$$

Hence $\mathbf{dim}\, \tau_B^{-m} P_{\rm i} = \mathbf{dim}\, P_i + az$ for some $a \in \mathbf{Z}$. Therefore for any $t \in \mathbf{N}$, we write $t = mc + e$ with $c \geq 0$ and $0 \leq e \leq m$. We get

$$\mathbf{dim}\, \tau_B^{-t} P_i = (\mathbf{dim}\, P_i)\, \phi_B^{-t} = acz + \mathbf{dim}\, \tau_B^{-e} P_i \geq acz.$$

Hence $a > 0$; moreover, we have the following bounds for m:

$$\Delta \text{ of type } \tilde{\mathbf{A}}_{d-1}\colon m \leq \left[\tfrac{1}{2}(d-1)\right]^2 < \tfrac{1}{4} d^2;$$

$$\Delta \text{ of type } \tilde{\mathbf{D}}_{d-1}\colon m \leq 2(d-3);$$

$$\Delta \text{ of type } \tilde{\mathbf{E}}_6\colon m = 3;$$

$$\Delta \text{ of type } \tilde{\mathbf{E}}_7\colon m = 4;$$

$$\Delta \text{ of type } \tilde{\mathbf{E}}_8\colon m = 6$$

Therefore $\dim_k \tau_B^{-t} P_i(j) \geq [t/m] \geq 4[t/d^2]$.

(b) We consider the case where Δ is of the form $a \overset{\cdot}{\underset{\cdot}{\cdot}} b$ with $s \geq 3$ arrows. The inverse of the Coxeter matrix of B is

$$\phi_B^{-1} = \begin{bmatrix} -1 & -s \\ s & s^2 - 1 \end{bmatrix}.$$

Then for $(a_0, b_0) = \mathbf{dim}\, P_b$, we write $(a_t, b_t) = \mathbf{dim}\, \tau_B^{-t} P_b$. We get inductively $b_t \geq a_t$; $b_t \geq s^{t+1}$ and $a_t \geq (s-1)s^t$ for $t \geq 1$. Similarly, for $(c_0, d_0) = \mathbf{dim}\, P_a$, we write $(c_t, d_t) = \mathbf{dim}\, \tau_B^{-t} P_a$, and we get $d_t \geq c_t$; $d_t \geq s^t$ and $c_t \geq (s-1)s^{t-1}$ for $t \geq 1$. Certainly, $s^{t-2} \geq [t/2]$, for $t \geq 2$.

(c) In the general case, we may assume that $B = B'[R]$ is a one-point extension of the representation-infinite hereditary algebra B'. By induction hypothesis $\dim_k \tau_{B'}^{-t} P_i(j) \geq 4[t/(d-1)^2]$ for any two vertices of the quiver Δ' of B' and for $t \geq d-1$. Let ω the extension vertex of B, that is, $\operatorname{rad} P_\omega = R$. Consider the preprojective component $\mathscr{P}$ (resp. $\mathscr{P}'$) of B (resp., B'). Then $\mathscr{P}$ (resp., $\mathscr{P}'$) is a standard component of the form $\mathbf{N}\Delta^{\mathrm{op}}$ (resp., $\mathbf{N}\Delta'^{\mathrm{op}}$). For any set S of vertices of $\mathscr{P}$, we denote by $k(\mathscr{P})/S$ the quotient category of the mesh category $k(\mathscr{P})$ by the ideal generated by all paths factorizing through some vertex in S.

Let i, j be any two vertices in Δ'_0. Let S_1 be the τ-orbit $\{\tau_B^{-t} P_\omega : t \geq 0\}$, then $k(\mathscr{P})/S_1 = k(\mathscr{P}')$. Hence

$$\dim_k \tau_B^{-t} P_i(j) = \dim_k \operatorname{Hom}_B\left(P_j, \tau_B^{-t} P_i\right) \geq \dim_k \operatorname{Hom}_{B'}\left(P_j, \tau_{B'}^{-t} P_i\right) \geq 4\left[\frac{t}{(d-1)^2}\right] \geq 4\left[\frac{t}{d^2}\right],$$

for $t \geq d$. Let i be in Δ'_0. Let S_2 be the set of vertices $\{\tau_B^{-t} P_\omega : t \geq 1\}$, then $k(\mathscr{P})/S_2(X, Y) = k(\mathscr{P}')(X, Y)$ for $X, Y \in \mathscr{P}'$ and $k(\mathscr{P})/S_2(P_\omega, Y) = k(\mathscr{P}')(\tau_{B'}^{-1} R, Y)$, for $Y \in \mathscr{P}'$. Hence

$$\begin{aligned}\dim_k \tau_B^{-t} P_i(\omega) &\geq \dim_k \operatorname{Hom}_{B'}\left(\tau_{B'}^{-1} R, \tau_{B'}^{-1} P_i\right) \\ &= \dim_k \operatorname{Hom}_{B'}\left(R, \tau_{B'}^{-t+1} P_i\right) \geq 4\left[\frac{t-1}{(d-1)^2}\right] \geq 4\left[\frac{t}{d^2}\right],\end{aligned}$$

for $t \geq d$.

Similarly, we get $\dim_k \tau_B^{-t} P_\omega(i) \geq 4[t/d^2]$, for any $i \in \Delta_0$. This finishes the proof of our first claim.

(2) Let X be a preprojective B-module and Y be a preinjective B-module. Let $t \geq d$ and assume $X = \tau_B^{-m} P_i$ and $Y = \tau_B^{q} I_j$. Then

$$\begin{aligned}\dim_k \operatorname{Hom}_B\left(\tau_B^{-t} X, Y\right) &= \dim_k \operatorname{Hom}_B\left(\tau_B^{-(t+m+q)} P_i, I_j\right) \geq 4\left[\frac{t+m+q}{d^2}\right] \\ &\geq 4\left[\frac{t}{d^2}\right],\end{aligned}$$

applying (1).

(3) For the rest of the proof we may assume that B is wild.

Let Y be an indecomposable regular B-module. We show that there is an integer $1 \leq s \leq 2d$ and an exact sequence

$$0 \to Y' \to \tau_B^s Y \to C \to 0$$

of B-modules where Y' (resp., C) is a direct sum of regular (resp., preinjective) B-modules. Indeed, assume first that Y is a simple regular module. As in [1, (1.1)] we obtain an exact sequence

$$0 \to Y \to \tau_B^s Y \to C \to 0$$

where $1 \leq s \leq 2d$ and C is a direct sum of preinjective modules. Assume now that $0 \to Y' \to Y \to Y'' \to 0$ is an exact sequence with Y' and Y'' regular B-modules. By induction hypothesis, there is an exact sequence

$$0 \to R \to \tau_B^s Y'' \to C \to 0$$

for some $1 \leq s \leq 2d$, where R (resp., C) is a direct sum of regular (resp., preinjective) modules. We complete the exact and commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \tau_B^s Y' & \to & E & \to & R & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \tau_B^s Y' & \to & \tau_B^s Y & \to & \tau_B^s Y'' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where E is a regular module. This shows claim (3).

(4) Let X be a preprojective and Y be a regular B-module. By (3), we may construct a sequence

$$0 \longrightarrow Y' \longrightarrow \tau_B^s Y \longrightarrow C \longrightarrow 0$$

where $1 \leq s \leq 2d$ and Y' (resp., C) is a direct sum of regular (resp., preinjective) modules. Let $t \geq 3d$. For $t' = t - s \geq d$, we get an exact sequence

$$\operatorname{Hom}_B\left(\tau_B^{-t'} X, \tau_B^s Y\right) \to \operatorname{Hom}_B\left(\tau_B^{-t'} X, C\right) \to \operatorname{Ext}_B^1\left(\tau_B^{-t'} X, Y'\right) = 0.$$

From (2), we get that

$$\dim_k \operatorname{Hom}_B\left(\tau_B^{-t} X, Y\right) = \dim_k \operatorname{Hom}_B\left(\tau_B^{-t+s} X, \tau_B^s Y\right) \geq 4\left[\frac{t-s}{d^2}\right] > \left[\frac{t}{d^2}\right].$$

This completes the proof of the proposition. ∎

1.8. *Proof of the Theorem* 1.5. Clearly, by (1.6) we may assume that A is a tilted algebra and $\mathscr{S}$ is a m-complete section not of Dynkin type in the preprojective component $\mathscr{C}$ of Γ_A. Moreover, if $X_1, \dots, X_n$ are the modules in $\mathscr{S}$, then $\oplus_{i=1}^n X_i$ is a tilting module.

Let $B = k\Delta$ be a hereditary algebra and ${}_BT$ be a tilting module with $A = \mathrm{End}_B(T)$. The functor $\Sigma = \mathrm{Hom}_B(T, -)$ (resp., $\Sigma' = \mathrm{Ext}^1_B(T, -)$) induces an equivalence between the full subcategories $\mathscr{G}(T) = \{X\colon \mathrm{Ext}^1_B(T, X) = 0\}$ of mod_B and $\mathscr{Y}(T) = \{M\colon \mathrm{Tor}_1^A(T, M) = 0\}$ of mod_A (resp., $\mathscr{F}(T) = \{X\colon \mathrm{Hom}_B(T, X) = 0\}$ and $\chi(T) = \{M\colon T \otimes_A M = 0\}$). We may choose ${}_BT$ such that $\mathscr{S}$ is formed by the modules ΣQ_x, $x \in \Delta_0$, where Q_x (resp., P_x) is the indecomposable injective (resp., projective) B-module associated with x. Observe that $\tau_A^{-1}\Sigma Q_x = \Sigma' P_x$ and $\tau_A^{-t}\Sigma Q_x = \Sigma'\tau_B^{-t+1}P_x$, for $1 \le t \le m$. See [9, 4.1].

Since $\mathscr{S}$ has only finitely many predecessors in Γ_A, there is at least one preinjective summand T_j of T. Then $N = \tau_B T_j \in \mathscr{F}(T)$ and $I_j = \Sigma' N$ is an indecomposable injective A-module.

Taking $X = \Sigma Q_x$ in $\mathscr{S}$ and $3n \le t \le m + 1$, we get using (1.7) $\dim_k \tau_A^{-t}X \ge \dim_k \mathrm{Hom}_A(\tau_A^{-t}X, I_j) = \dim_k \mathrm{Hom}_B(\tau_B^{-t+1}P_x, N) > [(t-1)/n^2]$. The result follows. ∎

1.9. *Remark.* In some cases the lower bounds given in (1.5) may be improved. Namely, let $B = k\Delta$ be a wild hereditary connected algebra with $d = |\Delta_0|$. Then there are constants $0 < a$, $1 < \mu$ such that for any projective P and $t \ge 0$, we have

$$\dim_k \tau_B^{-t}P \ge a\mu^m.$$

Moreover, the constant μ may be chosen independent of Δ.

Proof. Consider $\phi = \phi_B$, the Coxeter matrix of B. Since B is wild, the spectral radius $\rho = \rho(\phi) > 1$ [10]. There is a vector y^+ with positive coordinates such that $y^+\phi = \rho y^+$ [7, 12]. Hence

$$\left(\dim_k \tau_B^{-t}P_i\right)|y^+| > \left\langle (\mathbf{dim}\, P_i)\phi^{-t}, y^+ \right\rangle_B = \rho^t\left\langle \mathbf{dim}\, P_i, y^+ \right\rangle_B = \rho^t y^+(i) > 0.$$

Therefore, $\dim_k \tau_B^{-t}P_i \ge \rho^t y^+(i)/|y^+|$, where $|y^+| = \sum_{j\in\Delta_0} y^+(j)$.

As in [7, 4.3], we have that for any $j \in \Delta_0$,

$$\rho^{-d} \le y^+(i)/y^+(j) \qquad \text{and} \qquad \frac{1}{d\rho^d} \le y^+(i)/|y^+|.$$

Finally, we recall that since B is wild, $\rho \ge \mu$, where μ is the largest root of the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$; approximately $\mu \approx 1.1762$; see [13]. ∎

2. WHEN DOES A MODULE BELONG TO A GIVEN PREPROJECTIVE COMPONENT?

2.1. Let $\mathscr{P}$ be an infinite preprojective component of the Auslander–Reiten quiver Γ_A of the algebra $A = kQ/I$. The purpose of this section is to prove the following:

PROPOSITION. *There are functionals $f, g: K_0(A) \to \mathbf{R}$ such that an indecomposable module X belongs to $\mathscr{P}$ if and only if one of the following two conditions holds*

(i) $f(\mathbf{dim}\, X) > 0$

(ii) $f(\mathbf{dim}\, X) = 0$ *and* $g(\mathbf{dim}\, X) < 0$.

In fact, we will explicitly construct f and g.

2.2. For the proof of the proposition, we consider a section $\mathscr{S}$ of $\mathscr{P}$ as in (1.4, (2)), that is, $\mathscr{S}$ a *maximal* ∞-*complete section* in $\mathscr{P}$. Let $\mathscr{S}_1, \dots, \mathscr{S}_r$ be the connected components of $\mathscr{S}$. As in Lemma 1.6, we consider the full subcategory B of A in the vertices of $\bigcup_{x \in \mathscr{S}} \operatorname{supp} X$.

LEMMA. *The algebra $B = \coprod_{i=1}^r B_i$ is a coproduct of tilded algebras $B_1, \dots, B_r$, such that for $1 \le i \le r$, $\mathscr{S}_i$ is a slice in a preprojective component $\mathscr{P}_i$ of Γ_{B_i}. Moreover, $B_i = k\Delta_i/J_i$ for a path-closed subquiver Δ_i of Q, $1 \le i \le r$.*

Proof. For each $1 \le i \le r$, let $s(i) = \operatorname{supp} \mathscr{S}_i = \bigcup_{X \in \mathscr{S}_i} \operatorname{supp} X$. Observe that $B_i = \operatorname{End}_A(\oplus_{j \in s(i)} P_j)^{\mathrm{op}}$ is a tilted algebra having slice $\mathscr{S}_i$ in a preprojective component. For $1 \le i, j \le r$, $i \ne j$, the algebra $B_{ij} = \operatorname{End}_A(\oplus_{t \in s(i) \cup s(j)} P_t)^{\mathrm{op}}$ is also a tilted algebra with slice $\mathscr{S}_i \amalg \mathscr{S}_j$. Let $H = k\Delta$ be a hereditary algebra with a tilting module ${}_H T$ such that $B_{ij} = \operatorname{End}_A(T)$. Then $\Delta^{\mathrm{op}} = \mathscr{S}_i \amalg \mathscr{S}_j$ and $H = H_1 \amalg H_2$ with $H_i = k\Delta_i$ a connected hereditary algebra such that $\Delta_i^{\mathrm{op}} = \mathscr{S}_i$, $i = 1, 2$. Hence $B_{ij} = B_i \amalg B_j$ and $B = \coprod_{i=1}^r B_i$.

Let $B_i = k\Delta_i/J_i$ and $a = a_0 \to a_1 \to \cdots \to a_t \to a_{t+1} = b$ and a, b vertices in Δ_i (that is, $a, b \in s(i)$). Assume that $c = a_t \notin s(i)$. Then P_c is a predecessor of some $X_1 \in \mathscr{S}_i$ with $X_1(a) \ne 0$ and I_c is a successor of some $X_2 \in \mathscr{S}_i$ with $X_2(b) \ne 0$. Since $\operatorname{Hom}_A(P_c, I_c) \ne 0$ and $\mathscr{S}$ is a maximal ∞-complete section, there is some $Z \in \mathscr{S}$ such that $Z(c) \ne 0$. Assume $Z \in \mathscr{S}_j$. Since $c \notin s(i)$, then $j \ne i$. But then $c \in \operatorname{supp} I_b \cap \operatorname{supp} Z \subset s(i) \cap s(j)$, which contradicts that $B_{ij} = B_i \amalg B_j$. ∎

2.3. Keep the notation as in (2.1) and (2.2) above. Let $T_1^{(i)}, \dots, T_{m_i}^{(i)}$ be the vertices of $\mathscr{S}_i$ and $T^{(i)} = \oplus_{j=1}^{m_i} T_j^{(i)}$. Then $H_i = \operatorname{End}_A(T^{(i)})$ is a representation-infinite connected hereditary algebra; write $H_i = k\Delta_i$ where $\Delta_i = \mathscr{S}_i^{\mathrm{op}}$ is a quiver not of Dynkin type. Consider the isometry

$$\sigma_i\colon K_0(B_i) \to K_0(H_i), \dim X \mapsto \left(\dim_k \operatorname{Hom}_{B_i}\left(T_j^{(i)}, X\right) - \dim_k \operatorname{Ext}^1_{B_i}\left(T_j^{(i)}, X\right)\right)_j.$$

Moreover, let $1 \leq \rho_i$ be the spectral radius of the Coxeter matrix ϕ_{B_i}. Let y_i^- be a vector with positive coordinates such that $y_i^- \phi_{B_i} = \rho_i^{-1} y_i^-$. The following is a slight modification of [7, (2.2)].

LEMMA. *Let X be an indecomposable B_i-module. Then X belongs to $\mathscr{P}_i$ if and only if $\langle y_i^-, \sigma_i(\mathbf{dim}\, X)\rangle_{H_i} < 0$.*

Proof. The tilting module $T^{(i)}$ defines a torsion theory $(\mathscr{F}, \mathscr{T})$ in mod_{B_i}. Observe that the indecomposable torsion-free modules (thus in $\mathscr{F}$) belong to the preprojective component $\mathscr{P}_i$ of Γ_{B_i}. Moreover, the modules of the form $\operatorname{Hom}_{B_i}(T^{(i)}, X)$ with $X \in \mathscr{T} \cap \mathscr{P}_i$ are the vertices of the preprojective component $\mathscr{C}_i$ of Γ_{H_i}.

If $X \in \mathscr{P}_i$, we distinguish two possibilities. If $X \in \mathscr{T}$, then $\sigma_i(\mathbf{dim}\, X) = \mathbf{dim}\operatorname{Hom}_{B_i}(T^{(i)}, X)$ and $\operatorname{Hom}_{B_i}(T^{(i)}, X) \in \mathscr{C}_i$. Hence [7, (2.2)] implies that $\langle y_i^-, \sigma_i(\mathbf{dim}\, X)\rangle_{H_i} < 0$. If $X \in \mathscr{F}$, then $\sigma_i(\mathbf{dim}\, X) = -\mathbf{dim}\operatorname{Ext}^1_{B_i}(T^{(i)}, X)$ and $\operatorname{Ext}^1_{B_i}(T^{(i)}, X) \notin \mathscr{C}_i$. In this case, $\langle y_i^-, \sigma_i(\mathbf{dim}\, X)\rangle_{H_i} = -\langle y_i^-, \mathbf{dim}\operatorname{Ext}^1_{B_i}(T^{(i)}, X)\rangle_{H_i} < 0$. For the converse assume that $\langle y_i^-, \sigma_i(\mathbf{dim}\, X)\rangle_{H_i} < 0$. If $X \in \mathscr{F}$, trivially $X \in \mathscr{P}_i$. Otherwise $X \in \mathscr{T}$ and $\operatorname{Hom}_{B_i}(T^{(i)}, X) \in \mathscr{C}_i$, applying [7, (2.2)]. Therefore $X \in \mathscr{P}_i$. ▮

2.4. *Proof of* (2.1). Let J be a direct sum of all injective modules $I_i \in \mathscr{P}$. We define $f\colon K_0(A) \to \mathbf{Z}$, $\mathbf{dim}\, X \mapsto \langle \mathbf{dim}\, X, \mathbf{dim}\, J\rangle_A$.

Consider $B_1, \ldots, B_r$ as in (2.2) and let $\varepsilon_i\colon K_0(B_i) \to K_0(A)$ be the canonical inclusion, $i = 1, \ldots, r$. We define

$$g\colon K_0(A) \to \mathbf{R}, \mathbf{dim}\, X \mapsto \sum_{i=1}^{r} \left\langle \varepsilon_i \sigma_i^{-1}(y_i^-), \mathbf{dim}\, X\right\rangle_A.$$

First assume that X is an indecomposable in $\mathscr{P}$. If $X(j) \neq 0$ for some $I_j \in \mathscr{P}$, then

$$f(\mathbf{dim}\, X) = \sum_{I_i \in \mathscr{P}} \dim_k \operatorname{Hom}_A(X, I_i) \geq \dim_k \operatorname{Hom}_A\left(X, I_j\right) > 0.$$

Otherwise, $f(\mathbf{dim}\, X) = 0$ and X is a B-module. Assume that X is a B_i-module. Then X lies in the preprojective component $\mathscr{P}_i$ of Γ_{B_i}. By (2.2) and (2.3), $\langle \varepsilon_i \sigma_i^{-1}(y_i^-), \mathbf{dim}\, X\rangle_A = \langle \sigma_i^{-1}(y_i^-), \mathbf{dim}\, X\rangle_{B_i} < 0$. Thus $g(\mathbf{dim}\, X) < 0$.

For the converse, if $f(\mathbf{dim}\, X) > 0$, clearly X is a predecessor of some $I_i \in \mathcal{P}$ and $X \in \mathcal{P}$. If $f(\mathbf{dim}\, X) = 0$ and $g(\mathbf{dim}\, X) < 0$, as above, X is a B-module and we apply (2.3). ∎

3. THE ALGORITHM

3.1. Let $A = kQ/I$ be a finite dimensional k-algebra and $Q_0 = \{1, \dots, n\}$ be the set of vertices of Q.

For each writex $i \in Q_0$, we consider the indecomposable decomposition $\operatorname{rad} P_i = \oplus_{j=1}^{t_i} R_j^{(i)}$. Let $\sim$ be the minimal equivalence relation on $\{1, \dots, t_i\}$ such that $j \sim j'$ if $\operatorname{succ}(\operatorname{supp} R_j^{(i)}) \cap \operatorname{succ}(\operatorname{supp} R_{j'}^{(i)}) \neq \emptyset$, where $\operatorname{succ}(L)$ denotes the set of vertices $x \in Q_0$ such that there is an oriented path from some $l \in L$ to x. We may assume that $1, \dots, s_i$ $(\leq t_i)$ are representatives of the equivalence classes $\{1, \dots, t_i\}/\sim$.

We fix $M = \max\{\dim_k I_i, \dim_k R_j^{(i)}: 1 \leq i \leq n,\ 1 \leq j \leq t_i\}$.

3.2. Let $x \in Q_0$ be a sink. We describe an *inductive procedure* to decide whether or not the simple projective module P_x belongs to a preprojective component. Namely, starting with $\mathcal{P}_0 = \{\mathbf{dim}\, P_x\}$, we will define inductively a procedure for constructing a new set $\mathcal{P}_{s+1} \subset K_0(A)$ from $\mathcal{P}_s \subset K_0(A)$. The procedure may fail; in that case, $\mathcal{P}_{s+1}$ is not defined and the procedure stops indicating that P_x does not belong to a preprojective component. Otherwise, the procedure continues.

More precisely, assume $\mathcal{P}_s \subset K_0(A)$ is a well-defined finite set satisfying

(a) for each $y \in \mathcal{P}_s$ there is a unique indecomposable $\hat{y}$ with $\mathbf{dim}\, \hat{y} = y$;

(b) the set $\{\hat{y}: y \in \mathcal{P}_s\}$ is closed under predecessors in Γ_A and $\mathcal{P}_{s-1} \subset \mathcal{P}_s$;

(c) each module $\hat{y}$ (for $y \in \mathcal{P}_s$) is directing.

Let $\mathcal{S}^{(s)}$ be the full subquiver of Γ_A formed by those $\hat{y}$ with $y \in \mathcal{P}_s$ such that $\hat{y}$ is not injective and $\mathbf{dim}\, \tau_A^{-1}\hat{y} \notin \mathcal{P}_s$. Then $\mathcal{S}^{(s)}$ is a section; see (1.4). Consider the full subquiver $\mathcal{S}_1^{(s)}$ of Γ_A formed by $\tau_A^{-1}\hat{y}$ with $\hat{y} \in \mathcal{S}^{(s)}$. We distinguish several situations:

(1) if none of the modules $Y \in \mathcal{S}_1^{(s)}$ has $\mathbf{dim}\, Y = \mathbf{dim}\, X$ for X a direct summand of $\operatorname{rad} P_i$, $i \in Q_0$, then we define

$$\mathcal{P}_{s+1} := \mathcal{P}_s \cup \left\{\mathbf{dim}\, Y: Y \in \mathcal{S}_1^{(s)}\right\};$$

(2) assume $Y \in \mathcal{S}_1^{(s)}$ has $\mathbf{dim}\, Y = \mathbf{dim}\, R_j^{(i)}$ for some $i \in Q_0$, $1 \leq j \leq t_i$. Then consider the algebra A^i as defined in (1.3). All $R_l^{(i)}$ are A^i-modules.

Let $S^{(i)}$ be the set of all vertices $y \in \bigcup_{l=1}^{s_i} \mathrm{succ}(\mathrm{supp}\, R_l^{(i)})$ such that P_y is simple projective, that is, a sink of the quiver Q^i of A^i. Since Q^i has less than n vertices, then our algorithm decides whether or not P_x, $x \in S^{(i)}$, lies in a preprojective component of Γ_{A^i}. We may encounter the following situations:

(2.i) there is a P_y, $y \in S^{(i)}$, which does not lie in a preprojective component of Γ_{A^i}. Then we say that the *procedure fails* and $\mathscr{P}_{s+1}$ is not defined.

Otherwise, all P_y, $y \in S^{(i)}$, lie in preprojective components $\mathscr{C}_1, \dots, \mathscr{C}_s$ of Γ_{A^i}. Using the functionals defined in Section 2, we may decide whether or not $R_l^{(i)}$, $1 \le l \le t_i$, lies in some $\mathscr{C}_t$.

(2.ii) There is some $R_l^{(i)}$ not lying in $\bigcup_{t=1}^{s} \mathscr{C}_t$, then the *procedure fails*. Otherwise, all $R_l^{(i)}$, $1 \le l \le t_i$, lie in $\bigcup_{t=1}^{s} \mathscr{C}_t$. Then,

(2.iii) If $\oplus_{l=1}^{t_i} R_l^{(i)} = \mathrm{rad}\, P_i$ is not directing, then the *procedure fails*.

(2.iv) Assume that $\mathrm{rad}\, P_i$ is directing in mod_{A^i}. Then we may construct a set $\mathscr{P}^{(i)} \subset K_0(A^i)$ satisfying conditions (a)–(c) above such that $\mathbf{dim}\, R_l^{(i)} \in \mathscr{P}^{(i)}$, $1 \le l \le t_i$. Let

$$R^{(s)} = \left\{ i \in Q_0 \colon \text{there is some } Y \in \mathscr{S}_1^{(s)}, \mathbf{dim}\, Y = \mathbf{dim}\, R_j^{(i)} \text{ for some } j \right\}$$

and assume we have constructed $\mathscr{P}^{(i)}$ for all $i \in R^{(s)}$, then

$$\mathscr{P}_{s+1} = \mathscr{P}_s \cup \left(\bigcup_{i \in R^{(s)}} \mathscr{P}^{(i)} \right) \cup \left\{ \mathbf{dim}\, P_i \colon i \in R^{(s)} \right\}.$$

If $\mathscr{P}_{s+1}$ is defined we say that the *procedure is successful* at the step s. Using (1.1) and (1.2), it is clear that $\mathscr{P}_{s+1}$ satisfies the conditions (a)–(c).

3.3. THEOREM. *The simple projective P_x belongs to a preprojective component of Γ_A if and only if the procedure is successful for every step $s \ge 0$. Moreover,*

(a) *if the procedure fails, then $\mathscr{P}_s$ is not defined for some $s \le s_0 := 2n \cdot \max\{Mn^2, 16\}$.*

(b) *If the procedure has been successful for all $s \le s_0$, then for any $s \ge s_0$ we have $\mathscr{P}_s = \mathscr{P}_{s_0} \cup \{\mathbf{dim}\, \tau_A^{-l} X \colon X \in \mathscr{S}^{(s_0)}, 0 \le l \le s - s_0\}$ where $\mathscr{S}^{(s_0)}$ is the section defined in* (3.2).

Proof. If P_x belongs to a preprojective component $\mathscr{P}$, then clearly $\mathscr{P}_s \subset \{\mathbf{dim}\, X \colon X \in \mathscr{P}\}$ is well-defined for all $s \ge 0$.

Conversely, if all $\mathscr{P}_s$ are well-defined, then the set of modules $\mathscr{P} = \{\hat{y}\colon y \in \bigcup_{s \ge 0} \mathscr{P}_s\}$ yields a connected component of Γ_A. This component is directing and each module has only finitely many predecessors, thus it is preprojective.

(a) Assume that $\mathscr{P}_s$ is defined and considered the section $\mathscr{S}^{(s)}$. We claim the following:

If $\mathscr{P}_{s+t} = \mathscr{P}_s \cup \{\mathbf{dim}\, \tau_A^{-l}X\colon\ X \in \mathscr{S}^{(s)},\ 0 \le l \le t\}$ for $0 \le t \le \max\{Mn^2, 16\}$, then $\mathscr{P}_{s+t} = \mathscr{P}_s \cup \{\mathbf{dim}\, \tau_A^{-l}X\colon X \in \mathscr{S}^{(s)},\ 0 \le l \le t\}$ for all $t \ge 0$ and P_x belongs to a preprojective component.

Indeed, under this hypothesis the section $\mathscr{S}^{(s)}$ is a m-complete section with $m = \max\{Mn, 16\}$. Let $\mathscr{S}_1, \dots, \mathscr{S}_t$ be the connected components of $\mathscr{S}^{(s)}$. First we show that no $\mathscr{S}_i$ is of Dynkin type. Assume that $\mathscr{S}_i$ is of Dynkin type, then the full subquiver determined by $\{\tau_A^{-l}X\colon X \in \mathscr{S}_i,\ 0 \le l \le m\}$ is contained in $\mathbf{N}\mathscr{S}_i$ (the translation quiver with vertices (x, r) for $x \in \mathscr{S}_r$, $r \in \mathbf{N}$, arrows $(x, r) \to (y, r) \to (x, r+1)$ for each arrow $x \to y$ in $\mathscr{S}_i$, and translation $\tau(x, r) = (x, r-1)$). For $X_1, X_2 \in \mathscr{S}_i$, $Y = \tau_A^{-l}X_2$, $0 \le l \le m$, we have

$$\dim_k \operatorname{Hom}_A(X_1, Y) = \dim_k k(\mathbf{N}\mathscr{S}_i)((X_1, 0), (X_2, l)).$$

By [5], this dimension is zero for

$$l \ge \begin{cases} n-1, & \text{if } \mathscr{S}_i \text{ is of type } \mathbf{A}_s \text{ or } \mathbf{D}_s, \text{ with } s \le n \\ 15, & \text{if } \mathscr{S}_i \text{ is of type } \mathbf{E}_p, 6 \le p \le 8. \end{cases}$$

Therefore, for any $X \in \mathscr{S}_i$, $\tau_A^{-l}X$ is injective for some $l < m$. A contradiction showing $\mathscr{S}_i$ is not of Dynkin type.

Now, since $m + 1 > Mn^2$, then each of $\mathscr{S}_i$ is a ∞-complete section by Corollary 1.5. Thus $\mathscr{P}_{s+t} = \mathscr{P}_s \cup \{\mathbf{dim}\, \tau_A^{-l}X\colon 0 \le l \le t\}$ for all $t \ge 0$. Hence the set of modules $\{\hat{y}\colon y \in \bigcup_{t \ge 0} \mathscr{P}_t\}$ yields a preprojective component of Γ_A. This shows the claim.

Now, assume that $\mathscr{P}_s$ is defined for $0 \le s \le b$ and there is no $0 \le a \le b$ such that $\mathscr{P}_{a+t} = \mathscr{P}_a \cup \{\mathbf{dim}\, \tau_A^{-l}X\colon\ X \in \mathscr{S}^{(a)},\ 0 \le l \le t\}$ for $0 \le t \le \max\{Mn^2, 16\}$. Then there are numbers $0 = a_0 < a_1 < a_2 < \cdots < a_r < a_{r+1} = b$ such that $a_{i+1} - a_i < \max\{Mn^2, 16\}$ and $\mathscr{S}^{(a_i)}$ and $\mathscr{S}^{(a_{i+1})}$ are not isomorphic, $i = 0, \dots, r$. We may assume that $\mathscr{S}^{(a_i+1)}$ already coincides with $\mathscr{S}^{(a_{i+1})}$ $(0 \le i \le r)$. Therefore, there is a module Y in $\mathscr{S}^{(a_i)}$ which is either injective or a direct summand of the radical of a projective module; that is, either $I_j \in \mathscr{S}^{(a_i)}$ or $P_j \in \mathscr{S}^{(a_i+1)}$ for some $j \in Q_0$. Since this may only happen $2n$ times, we get that $r \le 2n$. Therefore $b < 2n \cdot \max\{Mn^2, 16\}$. This proves (a).

(b) follows also from the proof of (a). ∎

3.4. *Remarks.* (1) Suppose we apply the procedure (3.2) starting with the simple projective P_x and we get $\mathscr{P}_s$ for $0 \le s \le s_0 = 2n \cdot \max\{Mn^2, 16\}$.

Then P_x belongs to a preprojective component $\mathscr{P}$ of Γ_A. The section $\mathscr{S}^{(s_0)}$ is a maximal ∞-complete section (3.3, b). At that moment we have gotten all information needed to construct the functionals f, g: $K_0(A) \to \mathbf{R}$ deciding which modules belong to $\mathscr{P}$ (2.1).

(2) The remark above also shows that the step (2.ii) in the algorithm (3.2) can be carried out by means of the same procedure (inductively).

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