The Quaternionic Lattice for $2G_2(4)$ and Its Maximal Subgroups

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In this paper we construct a certain six-dimensional quaternionic linear group $2G$, which is isomorphic to the double cover of the Chevalley group $G_2(4)$ (denoted by $G$). We investigate the associated geometry, and prove

Theorem 1. The group $G$ has eight conjugacy classes of maximal subgroups:

(A) three classes of $p$-local subgroups:

1. $N(2^4) = 2^4 \cdot 2^4 : (A_5 \times A_4)$, the monomial subgroup or coordinate frame stabilizer, of index 1365,
2. $N(2^3) = 2^3 \cdot 2^3 : (3 \times A_3)$, the stabilizer of a certain partition into three orthogonal two-spaces, of index 1365,
3. $N(3) = 3 \cdot L_3(4) \cdot 2$, writable over $\mathbb{U}(\sqrt{-3})$, of index 2080.

(B) five classes of non-local subgroups:

4. $A_5 \times A_5$, of index 69,888,
5. $L_2(13)$, writable over $\mathbb{U}(\sqrt{13})$, of index 230,400,
6. $U_3(3) \cdot 2$, of index 20,800,
7. $U_3(4) \cdot 2$, of index 2016,
8. $J_2$, of index 416,

where $p^a$ denotes an elementary Abelian group of order $p^a$, $AB$ or $A \cdot B$ denotes any extension of $A$ by $B$, and $A : B$ and $A \cdot B$ denote split and nonsplit extensions, respectively. $J_2$ is the Hall–Janko simple group of order 604,800.

Classes (1) and (2) are the maximal parabolic subgroups, containing the Borel subgroup of shape $2^4 \cdot 2^4 : (A_4 \times A_4)$. Classes (3) and (7) are known
from general Lie theory, and classes (6) and (8) from Suzuki's construction of his sporadic simple group [9].

This work was undertaken in preparation for similar work on the Suzuki group, which is now essentially complete and which I hope to publish in due course.

1. THE HURWITZ AND ICOSIAN RINGS OF INTEGRAL QUATERNIONS

The root-lattice of the Lie algebra of type $D_4$ can be embedded in the division algebra of the real quaternions as indicated by Fig. 1, and under this map acquires a ring structure. The units of this ring form a group $2A_4$, consisting up to sign of the elements $1, i, j, k, \omega, \omega^*, \omega^j, \omega^k$, where $\omega = \frac{1}{2}(-1 + i + j + k)$, corresponding, respectively, to the automorphisms $1, (\omega_0)(12), (\omega_1)(01), (\omega_2)(01), (\omega_02), (\omega_01), (\omega_021), (\omega_021)$ of the projective line over $GF(3)$.

Since the unit group is a double cover of the Coxeter group $[3, 3]$ we shall denote this ring $\mathbb{H}_{3,3}$. It is the ring which Tits calls $R_4$ in his paper on $J_2$ [11]. In order to work easily in this ring it is useful to write down the identities $\omega^j = j\omega = \omega k$, together with the inverses and $\omega$-transforms of these. These identities enable us to see at once that the left and right ideals generated by $1 + i$ coincide, i.e., the ideal is two-sided. The quotient ring $\mathbb{H}_{3,3}/(1 + i)$ is isomorphic to $GF(4)$, consisting of the images of $0, 1, \omega$, and $\omega^*$, so we use these same names for elements of $GF(4)$.

The Coxeter group $[3, 5]$ has a double cover realised as the group $2A_5$ of unit quaternions generated by $i$ and $\omega_{ij} = \frac{1}{2}(-1 + \tau i - \sigma j)$, where $\tau = \frac{1}{2}(1 + \sqrt{5}), \sigma = \frac{1}{2}(1 - \sqrt{5}), \tau + \sigma = 1$, and $\tau\sigma = -1$. The integral linear combinations of elements of $2A_5$ form the Icosian ring $\mathbb{H}_{3,5}$, called $R_5$ by Tits [11]. Under the left-linear map $\psi$ defined by $\psi(1) = (1, 1)$, $\psi(\omega_{ij}) = (i, -1)$, $\mathbb{H}_{3,5}$ becomes a two-dimensional lattice over $\mathbb{H}_{3,3}$, consisting of all vectors $(x_1, x_2)$ satisfying $x_1 \equiv x_2 \mod(1 + i)$.

![Figure 1](image-url)
2. The Hexacode \( \mathcal{H} \)

The (additive) hexacode \( \mathcal{H} \) is defined as the code over \( GF(4) \) consisting of multiples of the words

\[
(0; 0, 0, 0, 0, 0), \quad (1; 1, 1, 1, 1, 1), \quad (1; 1, \omega, \bar{\omega}, \bar{\omega}, \omega), \\
(0; \bar{\omega}, \omega, \omega, \bar{\omega}), \quad (\omega; \omega, 0, \bar{\omega}, \bar{\omega}, 0), \quad (\bar{\omega}; \bar{\omega}, \omega, 0, 0, \omega)
\]

together with all rotations of the last five co-ordinates. The multiplicative hexacode \( \mathcal{H}^* = i^{\mathcal{H}} \) is then obtained by replacing \( 0, 1, \omega, \bar{\omega} \) by \( 1, i, j, k \), respectively. If we number the co-ordinates \( \infty, 0, 1, 2, 3, 4 \) then all the permutations of \( L_2(5) \) yield automorphisms of the code. The full automorphism group of the hexacode is \( 3 \cdot A_6 \), with the words of weight 6 (i.e., words with six non-zero co-ordinates, in the additive notation) corresponding in groups of three to the six “synthematic totals” of Sylvester.

3. The Lattice \( A = A_6(\frac{1}{3}, 3) \)

Let \( \mathcal{Q} \) be the set of vectors \((1; 1, 1, 1, 1, 1), (i; i, j, k, k, j), (2\omega; 0, 0, 0, 0, 0) \) and \((0; 2\omega, 0, 0, 0, 0, 0)\) together with all rotations of the last five co-ordinates. Then the lattice \( A \) consists of all sums \( \sum a_i v_i \) with \( v_i \) in \( \mathcal{Q} \) and \( a_i \) in \( \mathbb{H}_{1, 3} \) such that \( \sum a_i \) lies in the ideal \((1 + i)\).

Now \( A \subset \mathbb{H}_{1, 3}^6 \) and may be regarded as a left \( \mathbb{H}_{1, 3}^6 \)-module in the usual way; in other words if \( v \subset A \) and \( \lambda \subset \mathbb{H}_{1, 3} \) then \( \lambda v \subset A \). In particular, each vector has 24 unit multiples lying in the same one-space. We do not usually distinguish between different unit multiples of a vector. Now let \( 2G \) be the set of all matrices written over \( \mathbb{Q}(i, \omega) \) which are isometries of \( A \), i.e., invertible linear maps \( g \) for which \( v \in A \Rightarrow vg \in A \). Then \( A \) may similarly be regarded as a right \( 2G \)-module.

If \( 2M \) is the subgroup of \( 2G \) consisting of all the monomial matrices, then by construction we see that \( 2M \) is generated by the following:

1. an elementary Abelian 2-group \( 2E \) consisting of all even sign changes, of order 32,
2. modulo \( 2E \) an elementary Abelian 2-group \( F \) of co-ordinatewise right multiplications by elements of the multiplicative hexacode, of order 64,
3. an \( L_2(5) \) of co-ordinate permutations, of order 60,
4. right multiplication by \( \omega \), of order 3.

and so \( 2M \) has order 368,640. The group \( E \) is clearly normal in \( M \), as is \( E \cdot F \); so \( M \) has the structure \( 2^4 \cdot 2^6 \cdot (3 \times A_5) \). Moreover, the four-group of right multiplications by \( 1, i, j, k \) commutes with the \( A_5 \) and is normalized by
right multiplication by \( \omega \), so we may alternatively write \( M = 2^4 \cdot 2^4 \): \((A_4 \times A_2)\). Anything normalizing \( E \) must preserve the co-ordinate frame, so is contained in \( M \), and hence \( M = N_G(E) \). We write \( \epsilon_{mn} \) for the sign change on the \( m \)th and \( n \)th co-ordinates, and \( i, j, k, \omega \) for right multiplication by \( i, j, k, \omega \), respectively.

We can now find the following representatives for the four orbits of \( 2M \) on the minimal vectors of the lattice:

- \((i+j+k, 1, 1, 1, 1, 1)\) orbit size = \( 4 \times 6 \times 32 \times 64 \times 3 = 6144 \times 24 \)
- \((0, 0, 1+k, 1+j, 1+j, 1+k)\) orbit size = \( 15 \times 16 \times 64 \times 3 = 1920 \times 24 \)
- \((2, 0, 0, 0, 0, 0)\) orbit size = \( 15 \times 4 \times 16 \times 3 = 120 \times 24 \)
- \((2+2i, 0, 0, 0, 0, 0)\) orbit size = \( 6 \times 2 \times 4 \times 3 = 6 \times 24 \)

Total = \( 196,560 = 8190 \times 24 \)

(Note that these 196,560 vectors correspond to the 196,560 minimal vectors of the Leech lattice (see Section 6 below for details)).

If we call two vectors equivalent when one is congruent to a unit multiple of the other modulo \((1+i)A\) (i.e., \( v_1 \lambda v_2 = (1+i)w \) for some unit \( \lambda \) in \( \mathbb{H}/3,3 \) and some vector \( w \) in the lattice \( A \)), then they group further into 1365 "co-ordinate frames," each consisting of vectors in six orthogonal one-spaces. These likewise fall into four orbits under \( 2M \), with the following representatives:

- \((i+j+k, 1, 1, 1, 1, 1)\)
- \((1, i+j+k, 1, -1, -1, 1)\)
- \((1, 1, i+j+k, 1, -1, -1)\)
- \((1, -1, 1, i+j+k, 1)\)
- \((1, 1, -1, 1, i+j+k)\)
- \((0, 0, 1+k, 1+j, 1+j, 1+k)\)
- \((0, 0, 1+k, -1-j, -1-j, 1+k)\)
- \((1+j, 1+j, 1+i, 0, 0, -1-i)\)
- \((1+k, -1-k, 0, 1+i, -1-i, 0)\)
- \((1+k, -1-k, 0, -1-i, 1+i, 0)\)
- \((1+j, 1+j, -1-i, 0, 0, 1+i)\)
- \((2, 2, 0, 0, 0, 0)\)
- \((2, -2, 0, 0, 0, 0)\)
- \((0, 0, 2, 0, 0, 2)\)
- \((0, 0, 2, 2, 0)\)
- \((0, 0, 2, -2, 0)\)
- \((0, 0, 2, 0, -2)\)

orbit length = 1024

orbit length = 320

orbit length = 20
MAXIMAL SUBGROUPS OF $G_2(4)$

(2 + 2i, 0, 0, 0, 0, 0)
(0, 2 + 2i, 0, 0, 0, 0)
(0, 0, 2 + 2i, 0, 0, 0)
(0, 0, 0, 2 + 2i, 0, 0)
(0, 0, 0, 0, 2 + 2i, 0)
(0, 0, 0, 0, 0, 2 + 2i)

orbit length = 1

Total number of co-ordinate frames = 1365

4. The Automorphism Group of the Lattice

Calculation shows that the following six vectors form an integral basis for the lattice over the ring $\mathbb{Z}_{3,3}$:

$v_1 = (2 + 2i, 0, 0, 0, 0, 0)$
$v_2 = (2, 0, 0, 0, 0, 0)$
$v_3 = (0, 2, 0, 0, 0, 0)$
$v_4 = (i + j + k, 1, 1, 1, 1, 1)$
$v_5 = (0, 0, 1 + k, 1 + j, 1 + j, 1 + k)$
$v_6 = (0, 1 + j, 1 + j, 1 + k, 0, 1 + k)$.

To check whether any particular invertible matrix belongs to $2G$ it is sufficient to show that it takes $v_1, \ldots, v_6$ to vectors in the lattice. The vector $(q_0, q_1, q_2, q_3, q_4)$ belongs to $\mathcal{A}$ iff the following conditions are satisfied:

(i) $q_2 \equiv q_3 \equiv q_4 \mod (1 + i)$,
(ii) $(q_1 + q_4)\bar{\omega} + (q_2 + q_3)\omega \equiv (q_0 + q_1)\omega + (q_2 + q_4)\bar{\omega} \equiv 0 \mod 2$,
(iii) $-q_0(i + j + k) + q_0 + q_1 + q_2 + q_3 + q_4 \equiv 0 \mod (2 + 2i)$.

Now any matrix in $2G$ must take the standard co-ordinate frame to one of the co-ordinate frames given above, so it is a simple if tedious matter to check that the following four matrices form a complete set of double coset representatives of $2M$ in $2G$:

$$\frac{1}{4}(i - j) \begin{bmatrix}
  i + j + k & 1 & 1 & 1 & 1 & 1 \\
  1 & i + j + k & 1 & -1 & -1 & 1 \\
  1 & 1 & i + j + k & 1 & -1 & -1 \\
  1 & -1 & 1 & i + j + k & 1 & -1 \\
  1 & -1 & -1 & 1 & i + j + k & 1 \\
  1 & 1 & -1 & -1 & 1 & i + j + k
\end{bmatrix}.$$
Furthermore, we can easily check that the third matrix given above takes the vectors \((2 + 2i, 0, 0, 0, 0, 0), (0, 2, 2, 0, 0, 0),\) and \((i + j + k, 1, 1, 1, 1, 1)\) to \((-2j, -2j, 0, 0, 0, 0), (k - j, j - k, j - i, 0, 0, j - i),\) and \((i - j, i - k, j - i, i - k, 0, 0),\) respectively, and thus the automorphism group \(2G\) is transitive on the 1365 co-ordinate frames, and so has order \(368,640 \times 1365 = 503,193,600.\)

### 5. The Centralizer of a Central Involution in \(G\)

We shall discuss the centralizer of the involution \(e_{\infty 0},\) which is of class 2A (see Table I), and in order to do this we find it convenient to change base from \(\{e_\infty, e_0, e_1, e_2, e_3, e_4\}\) to \(\{f_1, f_2, f_3, f_4, f_5, f_6\} = \{e_\infty, e_0, \omega e_1, \omega e_4, \omega e_3, \omega e_2\}.\) The involution determines a two-space \(\langle f_1, f_2 \rangle\) as being the eigenspace corresponding to the eigenvalue \(-1,\) and this two-space contains ten vectors of the lattice, which fall naturally into five pairs of mutually orthogonal vectors. Each of these pairs extends to a "co-ordinate frame" in the sense of Section 3, and it can then be seen that there is a unique splitting of the four-space \(\langle f_1, f_4, f_5, f_6 \rangle\) into two orthogonal two-spaces with the property that all the vectors in the five co-ordinate frames belong to one of
TABLE I

<table>
<thead>
<tr>
<th>Conjugacy class</th>
<th>Centralizer order</th>
<th>Classes in the double cover</th>
<th>Character</th>
<th>Permutation character</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1A)</td>
<td>251,596,800</td>
<td>1, 2</td>
<td>12</td>
<td>416</td>
</tr>
<tr>
<td>(2A = 4A^2 = 4B^2 = 4C^2)</td>
<td>61,440</td>
<td>2. 2</td>
<td>4</td>
<td>32</td>
</tr>
<tr>
<td>(2R)</td>
<td>3,840</td>
<td>4</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>(3A)</td>
<td>60,480</td>
<td>3. 6</td>
<td>-6</td>
<td>56</td>
</tr>
<tr>
<td>(3B)</td>
<td>180</td>
<td>3. 6</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>(4A = 8A^2)</td>
<td>1,536</td>
<td>4. 4</td>
<td>-4</td>
<td>16</td>
</tr>
<tr>
<td>(4B)</td>
<td>768</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4C = 8B^2)</td>
<td>512</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(5A/B)</td>
<td>300</td>
<td>5/5, 10/10</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(5C/D)</td>
<td>300</td>
<td>5/5, 10/10</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>(6A = 2A \cdot 3A)</td>
<td>192</td>
<td>6. 6</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>(6B = 2B \cdot 3B)</td>
<td>12</td>
<td>12</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(7A)</td>
<td>21</td>
<td>7. 14</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(8A)</td>
<td>32</td>
<td>8</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(8B)</td>
<td>32</td>
<td>8. 8</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>(10A/B = 2A \cdot 5C/D)</td>
<td>20</td>
<td>10/10, 10/10</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>(10C/D = 2B \cdot 5A/B)</td>
<td>20</td>
<td>20/20</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(12A = 3A \cdot 4A)</td>
<td>48</td>
<td>12, 12</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>(12B/C = 3B \cdot 4A)</td>
<td>48</td>
<td>12/12</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(13A/B)</td>
<td>13</td>
<td>13/13, 26/26</td>
<td>-i</td>
<td>0</td>
</tr>
<tr>
<td>(15A/B = 3A \cdot 5A/B)</td>
<td>15</td>
<td>15/15, 30/30</td>
<td>-i</td>
<td>1</td>
</tr>
<tr>
<td>(15C/D = 3B \cdot 5C/D)</td>
<td>15</td>
<td>15/15, 30/30</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(21A/B = 3A \cdot 7A)</td>
<td>21</td>
<td>21/21, 42/42</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

these two-spaces, viz. \(\langle f_3, f_4 \rangle \oplus \langle f_5, f_6 \rangle\). Thus \(\varepsilon_{\infty,0}\) determines the two other involutions, \(\varepsilon_{14}\) and \(\varepsilon_{23}\), which have eigenvalue \(-1\) on the other two two-spaces, and so \(C(\varepsilon_{\infty,0}) = C(\varepsilon_{14}) = C(\varepsilon_{23}) = C(P_1)\), where \(P_1 = \langle \varepsilon_{\infty,0}, \varepsilon_{14} \rangle\).

On each two-space separately and for each of the five pairs of orthogonal vectors, consider (modulo \(-1\)) the automorphisms which fix one member of the pair and negate the other. These are the matrices

\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}
\]

which we denote \(G, H, I, J, K\), respectively. Modulo \(-1\), these five matrices generate an abstract elementary Abelian group \(X\) of order 16, satisfying \(GHIJK = 1\) and fixing all five of the pairs of orthogonal vectors. Now let \(2L\) be the group of matrices, written with respect to the new base, of the form \(g_1 \oplus g_2 \oplus g_3\), where the \(g_i\) belong to \(2X\) and have product \(\pm 1\). Then modulo \(P_1\), \(L\) forms an elementary Abelian group of order 256. It is also the
stabilizer in \( C_G(\epsilon_{x,0}) \) of all the five special co-ordinate frames determined by \( \epsilon_{x,0} \), and so is normal in that group.

The full centralizer in \( G \) of \( \epsilon_{x,0} \) is then a split extension of \( L \) by an \( A_5 \), which may be taken to be generated (modulo \(-1\)) by the three matrices \( h_i \oplus h_i \oplus h_i \), where \( h_1, h_2 \) and \( h_3 \) are the matrices

\[
\begin{pmatrix}
  j & 0 \\
  0 & k
\end{pmatrix}, \quad \begin{pmatrix}
  \omega & 0 \\
  0 & \omega
\end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix}
  j + k & j - k \\
  j - k & j + k
\end{pmatrix},
\]

respectively, and may be identified, respectively, with the Icosian units \( i, \omega \), and \( i_H = \frac{1}{2}(i + aj + tk) \) of Section 1. The \( A_5 \) then acts by permuting the five special co-ordinate frames, and so also the matrices \( G, H, I, J, K \), in the natural manner.

(Note that if we change base again to \((\omega f_1, \omega f_2, \omega f_3, \omega f_4, \omega f_5, \omega f_6)\) then \( h_1, h_2, h_3 \) become

\[
\begin{pmatrix}
  i & 0 \\
  0 & i
\end{pmatrix}, \quad \begin{pmatrix}
  \omega & 0 \\
  0 & \omega
\end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix}
  i + j & j - k \\
  j - k & j + k
\end{pmatrix},
\]

respectively, and hence the element \( \sigma \) of the Icosian ring corresponds to the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \). The matrices \( G, H, I, J, K \) then become

\[
\begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
  0 & \omega \\
  \omega & 0
\end{pmatrix}, \quad \begin{pmatrix}
  0 & \omega^i \\
  \omega^i & 0
\end{pmatrix}, \quad \begin{pmatrix}
  0 & \omega^j \\
  \omega^j & 0
\end{pmatrix}, \quad \begin{pmatrix}
  0 & \omega^k \\
  \omega^k & 0
\end{pmatrix},
\]

respectively.)

Finally we note that \( N(P_1)/C(P_1) \) has order 3 and is generated by the image of the co-ordinate permutation \((f_1, f_3, f_5)(f_2, f_4, f_6)\), whose centralizer in \( G \) is exactly the \( A_5 \) just described. Hence we see that \( N_G(P_1) \) has the structure \( 2^2 \cdot 2^8 : (3 \times A_5) \). At this point we could apply Thomas’s characterization theorem [10] to establish the isomorphism of \( G \) with \( G_2(4) \).

Note. The two subgroups \( M \) and \( N_P(P_1) \) of index 1365 which we have constructed are representatives of the two conjugacy classes of maximal parabolic subgroups, and their intersection, of shape \( 2^4 \cdot 2^4 : (A_4 \times A_4) \), is a Borel subgroup, and has index 5 in each of them. There is a generalized hexagon of order \((4, 4)\) consisting of the co-ordinate frames for points, and the four-groups whose involutions have two-dimensional \((-1)\)-eigenspaces spanning the whole six-space for lines. In this generalized hexagon, \( M \) is the stabilizer of a point, \( N_G(P_1) \) is the stabilizer of a line, and the Borel subgroup is the stabilizer of a flag.
MAXIMAL SUBGROUPS OF $G_2(4)$

6. CONNECTIONS WITH THE LEECH LATTICE AND THE HALL–JANKO GROUP

By doubling the vectors of $A$ and writing them in the array

\[
\begin{array}{ccccccc}
q_\infty & q_0 & q_1 & q_2 & q_4 & q_3 \\
1 & 1 & 1 & 1 & 1 & 1 \\
k & j & k & j & k & j \\
i & k & i & k & i & k \\
j & i & j & i & j & i \\
\end{array}
\]

so that for instance the vector $(-1 + j - k, i, j, k, k, j)$ becomes

\[
\begin{array}{ccccccc}
-2 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 2 \\
2 & 2 & 2 & 0 & 2 & 0 \\
\end{array}
\]

(note the reversal of the last two co-ordinates) we obtain the vectors of the Leech lattice in a version of Curtis’s M.O.G. notation (see [4]). This process yields an injection $2A_4 \circ 2G_2(4) \to 2Co_1$, where $2A_4$ is the group of left unit-multiplications of $A$, $2G_2(4)$ is the automorphism group of $A$, and $\circ$ denotes the central product in which the left and right multiplications by $-1$ are identified.

If also (again reversing the last two co-ordinates of $A$) we identify pairs of elements of $\Gamma_{3,3}$ with elements of $\Gamma_{3,5}$ by the inverse of the map $\psi$ defined in Section 1, then we obtain a three-dimensional lattice over $\Gamma_{3,5}$, which is discussed briefly in [2] and at greater length in [11]. The automorphism group of this lattice is the double cover of the Hall–Janko group $J_3$.

7. GENERAL REMARKS ON MAXIMAL SUBGROUPS

If $K$ is a maximal subgroup of a simple group $G$, and $N$ is a minimal normal subgroup of $K$, then $N$ is clearly a characteristically simple group whose normalizer is $K$. Thus in looking for maximal subgroups of a simple group it is sufficient to look for the normalizers of characteristically simple subgroups. We treat the Abelian case in Sections 9 and 10, and the non-Abelian case in Section 8. We assume throughout the character table of $G_2(4)$, which can be found in [12], a portion of which is appended as Table 1, and we make substantial use of the well-known technique used by Finkelstein in his determination of the maximal subgroups of $McL$ and $Co_3$. 
TABLE II
Structure Constants

| $H$ | $X$ | $Y$ | $Z$ | $\sum(1/|C(H)|)$ | $H$ | $X$ | $Y$ | $Z$ | $\sum(1/|C(H)|)$ |
|-----|-----|-----|-----|------------------|-----|-----|-----|-----|------------------|
| $2^3$ | 2A 2A 2A | 62/61 440 | $S_4$ | 2A 3A 4A | 0 |
| 2A 2A 2B | 0 | 2A 3A 4B | 0 |
| 2A 2B 2B | 75/3 840 | 2A 3A 4C | 0 |
| 2B 2B 2B | 152/3 840 | 2A 3B 4A | 128/1 536 |
| 2A 3B 4B | 128/768 |
| $A_5$ | 2A 3A 5A | 0 | 2A 3B 4C | 0 |
| 2A 3A 5C | 0 | 2B 3A 4A | 0 |
| 2A 3B 5A | 4/15 | 2B 3A 4B | 0 |
| 2A 3B 5C | 0 | 2B 3A 4C | 0 |
| 2B 3A 5A | 0 | 2B 3B 4A | 0 |
| 2B 3A 5C | 1/60 | 2B 3B 4B | 192/768 |
| 2B 3B 5A | 1 | 2B 3B 4C | 256/512 |
| 2B 3B 5C | 1 | $A_5$ | 2B 3B 3B | 14/3 |

Note. Let $x, y, z$ be elements of classes $X, Y, Z$ with $xyz = 1$ and $H = \langle x, y \rangle$.

(see [6]), which uses the character table to find the number of subgroups of certain types. In particular we shall use the formula

$$\sum \frac{1}{|C(x, y)|} = \frac{|G|}{|C(x)| |C(y)| |C(z)|} \sum \frac{\chi(x) \chi(y) \chi(z)}{\deg(\chi)},$$

where $xyz = 1$, the left-hand sum is taken over all conjugacy classes of ordered pairs $(x, y)$ with $x, y, z$ in given conjugacy classes, and the right-hand sum is taken over all irreducible characters. Table II contains values of this function, calculated by computer.

8. Non-local Subgroups

Any subgroup of $G_2(4)$ must either have index less than 252 or order less than 1,000,000. Since the only faithful irreducible characters of $G_2(4)$ of degree less than 252 have degrees 65 and 78, it is trivial to check that there is no permutation character of degree less than 252, and thus there is no subgroup of index less than 252. Thus all characteristically simple subgroups
of $G_2(4)$ are derivable from Hall's list of simple groups of order less than 1,000,000 (see, for example, [8]), and so are contained in the following list:

<table>
<thead>
<tr>
<th>Group</th>
<th>Order</th>
<th>Group</th>
<th>Order</th>
<th>Group</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_5$</td>
<td>60</td>
<td>$A_7$</td>
<td>2,520</td>
<td>$L_3(4)$</td>
<td>20,160</td>
</tr>
<tr>
<td>$A_5 \times A_5$</td>
<td>3,600</td>
<td>$L_3(3)$</td>
<td>5,616</td>
<td>$Sz(8)$</td>
<td>29,120</td>
</tr>
<tr>
<td>$L_3(7)$</td>
<td>168</td>
<td>$U_3(3)$</td>
<td>6,048</td>
<td>$U_3(4)$</td>
<td>62,400</td>
</tr>
<tr>
<td>$A_6$</td>
<td>360</td>
<td>$L_2(25)$</td>
<td>7,800</td>
<td>$L_2(64)$</td>
<td>262,080</td>
</tr>
<tr>
<td>$L_3(8)$</td>
<td>504</td>
<td>$L_2(27)$</td>
<td>9,828</td>
<td>$J_2$</td>
<td>604,800</td>
</tr>
<tr>
<td>$L_2(13)$</td>
<td>1,092</td>
<td>$A_8$</td>
<td>20,160</td>
<td>Tits</td>
<td>17,971,200</td>
</tr>
</tbody>
</table>

Of these, $L_2(8)$, $L_2(27)$, $L_2(64)$, $L_3(3)$, $L_3(4)$, $Sz(8)$, and the Tits group do not have six-dimensional quaternionic representations, and nor do any double covers [3], so they cannot be contained in $G_2(4)$. Furthermore, although $2L_2(25)$ has such a representation, it has elements of order 24.

Now any element of order 3 in $A_6$ is contained in an $S_4$, so must be of class $3B$ (see Table II). But we know from the form of the Sylow 3-subgroup (see [7] or Sections 8(1) and 10(1) below) that there is no elementary Abelian group of order 9 in $G$ with eight elements of class $3B$, so there can be no $A_6$ contained in $G$, and thus no $A_7$ or $A_8$ either. We investigate the remaining groups in the above order, and find their normalizers when they exist.

(1) $A_5$ and $A_5 \times A_5$

**Uniqueness of $A_5 \times A_5$.** The Sylow three-group of $G$ is extra-special of order 27 and exponent 3, in which the central elements are of class $3A$ and all the other elements are of class $3B$ (see [7]), and $A_5 \times A_5$ contains an elementary Abelian group of order 9, so we can suppose that the first factor contains an element of class $3B$. Then the second factor must be the $A_5$ in the centralizer of the corresponding cyclic subgroup, so contains elements of classes $2B$, $3A$, $5C$ and $5D$ (N.R., the square of an element of class $5C$ is of class $5D$, and vice versa), and we say the $A_5$ has type $(2B, 3A, 5C/D)$. Similarly, the first factor is the $A_5$ in the centralizer of any cyclic subgroup of order 5 in the second factor, and so has type $(2A, 3B, 5A/B)$. Thus every $A_5 \times A_5$ in $G$ can be obtained in the same way, and there is at most one conjugacy class of subgroup of this type.

**Existence of $A_5 \times A_5$.** The first factor may be taken as the monomial group $R_1$ generated by $(01234)$ and $e_{23}(\infty0)(14)$, which we will be considering again later on, and the second factor as the group generated by $i$, $\omega$, and the matrix $a$ given below.
The only non-trivial check we need to make is that this matrix commutes with \(e_{2,3}(\infty 0)(14)\).

The factorization of \(A_5 \times A_5\) is unique, the factors are not conjugate to each other, and no element of order 5 is conjugate to its square in \(G\), so the \(A_5 \times A_5\) is self-normalizing.

Non-existence of other maximal \(A_d\)-normalizers. The two classes of diagonal \(A_d\) in \(A_5 \times A_5\) are of types \((2B, 3B, SC/D)\) and \((2B, 3B, 5A/B)\). For the first case we know that \(C(3B) = \langle 3B \rangle \times \langle 2B, 3A, SC/D \rangle\) and \(C(5C) = \langle 5C \rangle \times \langle 2B, 3B, SC/D \rangle\) so \(A_d\) is trivial. For the second case we can check from Table I that no element of \(G\) has a centralizer containing an \(A_d\) of the required type. Furthermore, since no element of order 5 is conjugate to its square, these \(A_d\)'s are self-normalizing and hence not maximal.

There are three more groups of type \(A_d\) that we wish to consider, namely, the groups \(R_2, R_3\) and \(R_4\) generated by \((01234)\) and the respective involutions \((\infty 0)(14), e_{\infty 0}(\infty 0)(14)\) and \(e_{14}(\infty 0)(14)\). To show that \(R_1, R_2, R_3\) and \(R_4\) are representatives of distinct classes of \(A_d\), notice that all pairs of elements of orders 2 and 5 which generate an \(A_d\) are conjugate therein provided only that the elements of order 5 are conjugate, so it is only necessary to check that the given generating involutions are not conjugate in \(C(01234)\).

Now the centralizer of each of \(R_2, R_3\) and \(R_4\) is contained in the centralizer of \((01234)\), which is of shape \(\langle(01234)\rangle \times \langle i, \omega, a \rangle\) and contains the \(A_d\) generated by \(i\) and \(\omega\), but contains no element of order 5, since we have already accounted for all \(A_d\)'s centralizing an element of order 5. Thus the centralizer of each of \(R_2, R_3\) and \(R_4\) is exactly this \(A_d\), and we have accounted for the remainder \(\frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}\) of the structure constant. But the normalizer of each of these \(A_d\)'s is contained in the normalizer of \(\langle i, \omega \rangle\), which is contained in the 2-local subgroup \(N(i, j)\) which we shall consider in Section 9(2).

(2) \(L_2(7)\)

\(L_2(7)\) is generated by an element of order 2 and an element of order 3 whose product is of order 7, and all elements of order 2 in \(L_2(7)\) are squares and so must be of class \(2A\) in \(G\). Thus we can see from Table II that any
L₂(7) in G must be of type (2A, 3B, 7A). Now it is known that L₂(4) contains three conjugacy classes of L₂(7), which fuse in 3L₂(4) . 2 to give a single class of 3 × L₂(7) together with a single class of (3 × L₂(7)) · 2. In each case, the normalizer of L₂(7) is contained in the normalizer of an element of class 3A, and so is not maximal. Furthermore, the centralizer in G of an L₂(7) is contained in C(7A) = ⟨3A⟩ × ⟨7A⟩, so we have accounted for the figure 1 = 3 + 3 given in Table II, and thus there is no other L₂(7) in G.

We have incidentally proved that all subgroups of G which are generated by elements of class 2A and 3B whose product is of order 7 are in fact isomorphic to L₂(7).

(3) L₂(13)

Existence. A computer search found that the two matrices

\[
t = \begin{bmatrix}
0 & i & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & j & 0 \\
0 & 0 & 0 & k & 0 & 0 \\
0 & 0 & 0 & 0 & k & 0 \\
0 & 0 & j & 0 & 0 & 0 \\
\end{bmatrix}
\]

and

\[
u = \frac{1}{2}
\begin{bmatrix}
-1 & \omega & 0 & 0 & \omega & -1 \\
-\bar{\omega} & 0 & -\omega & -\omega & 0 & \omega \\
0 & -\bar{\omega} & 1 & -1 & \bar{\omega} & 0 \\
0 & \bar{\omega} & 1 & -1 & -\bar{\omega} & 0 \\
-\bar{\omega} & 0 & \omega & \omega & 0 & -\bar{\omega} \\
1 & \omega & 0 & 0 & \omega & 1 \\
\end{bmatrix}
\]

generate a group 2L₂(13) in 2G₂(4), satisfying the presentation ⟨t, u | t², u³, (tu)¹³, ((tu)² (ut)²)³⟩ modulo its centre.

Uniqueness. L₂(13) has no six-dimensional complex or quaternionic representation, so must lift to 2L₂(13) in 2G₂(4). We consider the six-dimensional quaternionic representation of 2G as a twelve-dimensional complex representation, and let α be a primitive 13th root of unity. Then given any element z of order 13 in 2G, it fixes a complex six-space on which its eigenvalues are αⁿ, for n a quadratic residue mod 13. This six-space is then fixed by any L₂(13) containing z, as is seen by looking at the character table. Since 2L₂(13) is maximal as a six-dimensional complex group (see [5]), the stabilizer of this six-space in 2G is 2L₂(13), and since z is not
conjugate to its inverse in $G$, there is no $L_2(13) \cdot 2$ contained in $G$, i.e., $L_2(13)$ is maximal in $G$. We have thus defined every $L_2(13)$ as the stabilizer of a unique concept, so there is only one conjugacy class of $L_2(13)$ in $G$.

Note. An alternative existence proof is due to S. P. Norton and runs as follows. From the 2-modular character table of $L_2(13)$ we see that it is contained in $Sp_6(4)$ which in turn is contained in $D_4(4)$. The three eight-dimensional 2-modular representations of $D_4(4)$, which are interchanged by triality, all split as $1 + 6 + 1$ for $L_2(13)$; in other words $L_2(13)$ fixes a vector in all three representations. But the stabilizer of a pair of vectors in two distinct eight-dimensional $p$-modular representations of $D_4(p^n)$ is $G_2(p^n)$, so $L_2(13)$ is contained in $G_2(4)$.

(4) $U_3(3)$

Existence. $U_3(3) \cdot 2$ is isomorphic to $G_2(2)$, so it is contained in $G_2(4)$. Alternatively it is the edge stabilizer in the 416-point graph.

Uniqueness. This 416-point representation decomposes for any subgroup isomorphic to $U_3(3)$ as the character sum $6 \times 1a + (7a + 7b) + 5 \times 14a + 5 \times 21a + 2 \times (21b + 21c) + 3 \times 27a + (28a + 28b)$, and so there are exactly six orbits of $U_3(3)$ on these 416 points. Now the maximal subgroups of $U_3(3)$ have indices 28, 36, and 63, the subgroup of index 36 being $L_2(7)$. From this we see that any orbit length not divisible by 7 is either 1, 36 or 288, all of which are congruent to 1 modulo 7, and the remaining orbit lengths are divisible by 28 or 63. Since $416 \equiv 3 \pmod{7}$ there must be three orbit lengths from 1, 36 and 288, and so there is a fixed point, since otherwise we would have at least two orbits of length 36, which is impossible since the 36-character splits as $1a + (7a + 7b) + 21a$. Since any $U_3(3)$ fixes a point in this 416-point representation, it must be contained in some $J_2$ in $G$, and thus there is a unique conjugacy class of subgroup $U_3(3)$ contained in $G$, whose normalizer $U_3(3) \cdot 2$ is a maximal subgroup. Note that since $U_3(3) \cdot 2 \cong G_2(2)$ is the centralizer of a field automorphism of $G_2(4)$, its maximality follows from a general result of Burgoyne et al. (see [1]).

(5) $U_3(4)$

Existence. $U_3(q)$ is contained in $G_2(q)$ for any $q$.

Uniqueness. Given any $U$ isomorphic to $U_3(4)$ contained in $G$, let $H$ be the centralizer in $G$ of any fixed element of class $5C$ in $U$. Then $H$ is isomorphic to $5 \times A_4$ and is contained in $U$, and inside $H$ there is a unique conjugacy class of diagonal cyclic subgroup of order 5 containing elements of classes $5C/D$, whose centralizer $K$ in $G$ is another $5 \times A_5$ contained in $U$. The groups $H$ and $K$ then generate $U$ since they are each maximal in $U$, and since all groups $\langle 5C \rangle \times A_5$ in $G$ are conjugate, so are all groups $U_3(4)$. 
Now the centralizer in $G$ of $U_3(4)$ is trivial, and any involution in $G$ which inverts our original $5C$-element must preserve the above construction, and so must normalize the $U_3(4)$. Thus the normalizer of any $U_3(4)$ in $G$ is $U_3(4).2$.

**Note.** The following alternative argument for the existence of $U_3(4).2$ was suggested by the referee. $G_2(4)$ is contained in $PO_7(4)$, which acts on 2016 hyperplanes of type $O_6^-(4)$. But by Lagrange's Theorem, neither $G_2(4)$ nor $J_2$ is contained in $O_6^-(4)$, so from independent results of this paper the only possible orbit length strictly less than 2016 is 1365, which is clearly impossible. Thus $G_2(4)$ is transitive on the hyperplanes of this type, and so has a subgroup of index 2016, which, again from independent results, can only be $U_3(4).2$.

(6) $J_2$

**Existence.** $J_2$ is the point stabilizer in the 416-point representation (see [9]). Alternatively, we may use the argument of Section 6.

**Uniqueness.** Given any subgroup $J$ isomorphic to $J_2$ of $G$, let $H$ be the normalizer in $G$ of a fixed element of class $5A$ in $J$. Then $H$ is of shape $D_{10} \times A_5$ and is contained in $J$, and inside $H$ there is a unique class of diagonal $5A$ element, whose normalizer $K$ in $G$ is another group $D_{10} \times A_5$ contained in $J$. The groups $H$ and $K$ then generate $J$, as they are both maximal in $J$, and just as above we see that there is a unique conjugacy class of $J_2$ in $G$. Also, the centralizer of $J$ in $G$ is trivial, and no outer automorphisms of $J$ are realizable in $G$, since $2J_2.2$ has elements of order 24, and so $J$ is self-normalizing.

9. **2-Local Subgroups**

Since the product of two commuting elements of class $2A$ is again of class $2A$ (see Table II), we see that the $2A$-part (if non-empty) of any elementary Abelian 2-group is normal in the normalizer of that group, so we may restrict our attention to $2A$-pure and $2B$-pure subgroups.

(1) **$2A$-Pure Subgroups**

We have shown in Section 5 above that $C(e_{A_0})$ has the shape $2^2 \cdot 2^8 : A_5$, and is strictly contained in $N(P_1)$ which has the shape $2^2 \cdot 2^8 : (3 \times A_5)$, where $P_1$ is the centre of $C(e_{2A})$. Now consider the groups $P_2$ generated by $e_{A_0}$ and $e_{12}$, $P_3$ generated by $e_{A_0}$ and $e_{01}$, and $P_4$ generated by $e_{A_0}$ and $e_{02}$. It is clear that $P_1$, $P_2$, $P_3$ and $P_4$ are pairwise non-conjugate, and so, since $C(5C)$ is of shape $5 \times A_5$ and contains only one class of four-group, none of $P_2$, $P_3$ or $P_4$ is centralized by an element of order 5.
Now by looking at the involution centralizer $2^2 \cdot 2^8 : A_4$, we see that the centralizers of $P_2$, $P_3$ and $P_4$ are all equal to the group $2^2 \cdot 2^6 : A_4$ generated by matrices of the form $g_1 \oplus g_2 \oplus g_3$, where each of the $g_i$ is the product of an even number of the matrices $H, I, J$ and $K$, together with matrices of the form $h_1 \oplus h_2 \oplus h_3$ for $i = 1$ or $2$. We have thus accounted for the complete structure constant of $62/61 = 2/61 + 1/3072 + 1/3072 + 1/3072$, so $P_1$, $P_2$, $P_3$ and $P_4$ are the only conjugacy classes of $2A$-pure four-groups in $G$. Furthermore, the centralizer of $P_2$, $P_3$ and $P_4$ centralizes all sign changes and so is equal to the centralizer of the group $Q$ consisting of all even sign changes, modulo the central involution, and thus the normalizers of these three groups are strictly contained in the normalizer of $Q$, whose structure was elucidated in Section 3.

From the structure of $C(P_1)$ we see that any elementary Abelian $2A$-pure subgroup of order 8 or 16 is contained in $Q$, and that $G$ contains no elementary Abelian $2A$-pure subgroup of order 32, so this completes the investigation of the $2A$-pure subgroups.

(2) $2B$-Pure Subgroups

To find the structure of $C(2B)$, notice that the element $i$ of class $2B$ is centralized by $j$, by the full $2^4$-group of sign changes, and by the full permutation group $L_3(5)$, so $C(2B)$ has the shape $2^2 \times 2^4 : A_4$. The centre of $C(2B)$ is therefore the group $T_1$ generated by $i$ and $j$, whose normalizer, of shape $A_4 \times 2^4 : A_4$, strictly contains $C(2B)$.

Now consider the groups $T_2$ generated by $i$ and $e_{10}j$, and $T_3$ generated by $i$ and $(14)(23)j$. It is clear from the structure of $C(i)$ that $C(T_2)$ has order 256 and $C(T_3)$ has order 64. Thus we have accounted for the full structure constant of $152/3840 = 2(1/3840 + 1/256 + 1/64)$. Now the normalizers of $T_1$ and $T_3$ each contain a characteristic subgroup of shape $2^4$ all of whose involutions are of class $2A$, and so are contained in the normalizer of this $2^4$-group and are not maximal. Similarly, the normalizer of $T_2$ has a characteristic subgroup of shape $2^3$ all of whose involutions are of class $2A$, so $N(T_3)$ is contained in the normalizer of this four-group and is not maximal. Finally we note that there can be no elementary Abelian $2B$-pure subgroup of order 8, since $C(2B)$ contains only a four-group of $2B$-elements.

10. Other $p$-Local Subgroups

(1) $3$-Local Subgroups

The centralizer of an element of class $3A$ in $G$ has shape $3 \cdot L_3(4)$, and every element of class $3A$ is conjugate to its inverse in $G$, so the normalizer of a cyclic group of type $3A$ is $3 \cdot L_3(4) \cdot 2$. (Note that this is the stabilizer of a hyperplane of type $O_6^-(4)$ in the embedding of $G_2(4)$ in $PO_7(4)$.)
The centralizer of an element of class 3 in \( G \) has shape \( 3 \times A_5 \), so its normalizer has shape \( S_3 \times A_5 \) and is contained in \( A_5 \times A_5 \), and thus is not maximal.

Finally, from the structure of the Sylow 3-group (see Section 8(1)) we see that any elementary Abelian group of order 9 must contain the centre of any Sylow 3-group it is contained in, so its normalizer is contained in the normalizer of some element of order 3.

(2) 5-Local Subgroups

The centralizer of any element of order 5 has shape \( 5 \times A_5 \), so the normalizer of a cyclic group of order 5 is contained in the normalizer of the corresponding \( A_5 \), which we have considered in Section 8(1).

The Sylow 5-group is self-centralizing, so its index in its normalizer divides the order of \( GL_2(5) \), which is 480, and so divides 96. But its index in its normalizer in \( J_2 \) is 12, so its index in its normalizer in \( G \) is \( 12 \times 2^n \), where \( n \) is 0, 1, 2, or 3. Now by Sylow's theorem, the index in \( G \) of the normalizer of a Sylow 5-group is congruent to 1 modulo 5, so \( a = 0 \) and the Sylow 5-normalizer is contained in \( J_2 \).

(3) 7-Local Subgroups

The centralizer of an element of order 7 in \( G \) has shape \( \langle 3A \rangle \times \langle 7A \rangle \), so the normalizer of a Sylow 7-group is contained in \( 3 \cdot L_3(4) \cdot 2 \).

(4) 13-Local Subgroups

The Sylow 13-group is self-centralizing, and there are just two classes of element of order 13 in \( G \), so the Sylow 13-group has normalizer of shape \( 13 \cdot 6 \), which is contained in \( L_2(13) \).

11. Conclusion

It remains only to show that all the groups found in Sections 8, 9 and 10 as candidates for maximal subgroups really are maximal, i.e., that no one is contained in any other. Considering the orders, the only possible inclusions are

1. \( N(2^3) = N(2^4) \),
2. \( U_3(3) \cdot 2 \) contained in \( 3 \cdot L_3(4) \cdot 2 \) or \( J_2 \),
3. \( A_5 \times A_5 \) or \( 3L_3(4) \cdot 2 \) contained in \( J_2 \).

But [7] tells us that \( J_2 \) does not contain \( U_3(3) \cdot 2 \), \( 3L_3(4) \cdot 2 \) or \( A_5 \times A_5 \), and \( N(2^3) \) and \( N(2^4) \) contain 5-elements of different classes so they cannot be conjugate. Furthermore, \( U_3(3) \) does not have a proper triple cover, so if it is
contained in $3 \cdot L_3(4) \cdot 2$ then it must be contained in $L_3(4) \cdot 2$, which contradicts Lagrange's theorem.

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Note added in proof. After this article was written, the same result was announced by G. Butler (see [13]), though without proof.

REFERENCES