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Envelopes and clutters

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Abstract

In this paper, a set function φ defined on a finite set Ω is said to be an upper envelope if there exists a set $\{p_i\}$ of nonnegative vectors on Ω such that $\varphi(G) = \max\{p_1(G), \dots, p_n(G)\}$ for all $G \subset \Omega$. All upper envelopes form a convex cone. We give a necessary and sufficient condition for an upper envelope to be extremal in the cone of all upper envelopes in terms of its representation. Furthermore we study the upper envelopes represented by clutters. We show that a clutter is extremal in the cone of the upper envelopes if and only if it satisfies some kind of connectivity. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout this paper, let Ω be a nonempty finite ground set.

The study of the upper envelopes of probability measures has been developed in fuzzy theory [3,5]. The upper envelopes of probability measures belong to the class of the set functions that we call the uniform upper envelopes in this paper. But we treat somewhat general class of the set functions, namely, the upper envelopes of nonnegative additive functions. We simply call them the upper envelopes. The set functions that we call upper envelopes are called multiply subadditive in [6].

We exploit nonnegative real vectors on Ω to represent an upper envelope. We denote $p(G) = \sum_{x \in G} p(x)$ for $G \subset \Omega$. By this notation, we can regard a vector p on Ω as an additive set function because $p(A) + p(B) = p(A \cup B)$ for any disjoint sets $A, B \subset \Omega$.

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Definition 1.1. A set function φ on Ω is said to be an upper envelope if there exists a set $\{p_i\}_{i\in I}$ of nonnegative vectors on Ω such that

$$\varphi(G) = \max\{p_i(G) \mid i \in I\}$$

for all $G \subset \Omega$. We call such a set $\{p_i\}$ of vectors a representation of φ.

When we can take a representation $\{p_i\}$ so that $p_i(\Omega)$ has the same value for all $i \in I$, we call φ a uniform upper envelope.

Denote, for a set function φ on Ω,

$$P(\varphi) = \{p : \Omega \to (0,\infty) \mid p(G) \leqslant \varphi(G) \text{ for all } G \subset \Omega\}.$$

Note that if P is a representation of φ, then $P \subset P(\varphi)$ holds.

For a set function φ on Ω, φ is an upper envelope if and only if, for each $G \subset \Omega$, there exists a $p \in P(\varphi)$ with $p(G) = \varphi(G)$. Therefore, for any upper envelope φ, we can take a finite representation $\{p^F\}_{F\subset\Omega}$ where $p^F \in P(\varphi)$ and $p^F(F) = \varphi(F)$.

All upper envelopes form a convex cone because when $\varphi_1(G) = \max\{p_1^F(G) \mid F \subset \Omega\}$ and $\varphi_2(G) = \max\{p_2^F(G) \mid F \subset \Omega\}$, for nonnegative numbers k_1 and k_2, $(k_1\varphi_1 + k_2\varphi_2)(G) = \max\{(k_1 p_1^F + k_2 p_2^F)(G) \mid F \subset \Omega\}$ where $p_i^F \in P(\varphi_i)$ and $p_i^F(F) = \varphi_i(F)$ for $i = 1, 2$. It is also easy to show that all uniform upper envelopes form a convex cone.

In Section 2, we consider a condition for an upper envelope to be extremal in the cone of all upper envelopes in terms of its representation (Theorem 2.1).

In Section 3, we consider the upper envelopes represented by clutters. We call them hypergraphic. While the upper envelope is a generalization of the polymatroid rank function, the hypergraphic upper envelope is a generalization of the matroid rank function.

Nguyen [7] studied a condition for a matroid rank function to be extremal in the cone of polymatroid rank functions. We generalize his results. We study a condition for a hypergraphic upper envelope to be extremal in the cone of all upper envelopes (Theorem 3.6).

2. Extremal condition for envelopes

Kashiwabara [4] studied various problems, for example, the sandwich problem, the representation problem to a binary relation, the domain extension problem, about convex classes of set functions in terms of the set of the extremal elements on a given class of set functions.

In this section, we present a condition for an upper envelope to be extremal in the cone of the upper envelopes in terms of its representations.

A representation P of φ is said to be minimal if there exists no representation of φ which is a proper subset of P.

Theorem 2.1. *An upper envelope φ is extremal in the cone of the upper envelopes if and only if for all minimal representations $\{p_i\}_{1\leqslant i\leqslant n}$ of φ there exist no $\{q_i\}_{1\leqslant i\leqslant n}$ and ψ satisfying the following conditions where q_i is a vector on Ω for each i with $1\leqslant i\leqslant n$, and ψ is a set function on Ω.*
- *ψ is not proportional to φ.*
- *For all $G\subset\Omega$ and i with $1\leqslant i\leqslant n$, $\varphi(G)=p_i(G)$ implies $\psi(G)=q_i(G)$.*
- *For all $x\in\Omega$ and i with $1\leqslant i\leqslant n$, $p_i(x)=0$ implies $q_i(x)=0$.*

Proof. *Sufficiency*: Let $\{p_i\}$ be a minimal representation of φ. Assume on the contrary that there are ψ and $\{q_i\}$ satisfying the above conditions for $\{p_i\}$.
We show that $(\varphi\pm\varepsilon\psi)(G)=\max\{(p_1\pm\varepsilon q_1)(G),\dots,(p_n\pm\varepsilon q_n)(G)\}$ for a sufficiently small $\varepsilon>0$.
For $G\subset\Omega$ such that $\varphi(G)=p_i(G)$, we have $(\varphi\pm\varepsilon\psi)(G)=(p_i\pm\varepsilon q_i)(G)$ because of $\psi(G)=q_i(G)$.
For $G\subset\Omega$ such that $\varphi(G)>p_i(G)$, we have $(\varphi\pm\varepsilon\psi)(G)>(p_i\pm\varepsilon q_i)(G)$ for a sufficiently small $\varepsilon>0$.
Since $p_i(x)=0$ implies $q_i(x)=0$, $p_i\pm\varepsilon q_i$ is nonnegative.

Necessity: Assume that an upper envelope φ is not extremal in the cone of the upper envelopes. Then there are upper envelopes φ' and φ'' such that $\varphi=\varphi'+\varphi''$ where φ' is not proportional to φ.

For each $G\subset\Omega$, there exist $p'_G\in P(\varphi')$ and $p''_G\in P(\varphi'')$ such that $\varphi(G)=\varphi'(G)+\varphi''(G)=p'_G(G)+p''_G(G)$ because φ' and φ'' are upper envelopes. Since $p'_G(H)+p''_G(H)\leqslant\varphi'(H)+\varphi''(H)=\varphi(H)$ for all $H\subset\Omega$, we have $p'_G+p''_G\in P(\varphi)$.

Therefore, $\{p'_G+p''_G\}_{G\subset\Omega}$ is a finite representation of φ. So there exists a minimal representation $\{p_1,\dots,p_n\}$ of φ such that, for all i, $p_i=p'_i+p''_i$ for some $p'_i\in P(\varphi')$ and $p''_i\in P(\varphi'')$.

Let $q_i=p'_i-p_i$ and $\psi=\varphi'-\varphi$.

For each $G\subset\Omega$ and i with $p_i(G)=\varphi(G)$, we have $p'_i(G)=\varphi'(G)$ and $p''_i(G)=\varphi''(G)$ since $p'_i(G)+p''_i(G)=p_i(G)$, $\varphi'(G)+\varphi''(G)=\varphi(G)$, $p'_i(G)\leqslant\varphi'(G)$ and $p''_i(G)\leqslant\varphi''(G)$. So the first condition is satisfied.

When $p_i(x)=0$, $p'_i(x)=p''_i(x)=0$ since $p_i=p'_i+p''_i$. So $q_i(x)=0$ when $p_i(x)=0$. □

Corollary 2.2. *A uniform upper envelope φ is extremal in the cone of the uniform upper envelopes if and only if for all minimal representations $\{p_i\}_{1\leqslant i\leqslant n}$ of φ there exist no $\{q_i\}_{1\leqslant i\leqslant n}$ and ψ satisfying the following conditions where q_i is a vector on Ω for each i with $1\leqslant i\leqslant n$, and ψ is a set function on Ω.*
- *ψ is not proportional to φ.*
- *For all $G\subset\Omega$ and i with $1\leqslant i\leqslant n$, $\varphi(G)=p_i(G)$ implies $\psi(G)=q_i(G)$.*
- *For all $G\subset\Omega$, $\varphi(G)=\varphi(\Omega)$ implies $\psi(G)=\psi(\Omega)$.*

Proof. *Sufficiency*: Let $\{p_i\}$ be a minimal representation of φ. Assume on the contrary that there are ψ and $\{q_i\}$ satisfying the above conditions for $\{p_i\}$.

We can show similarly to the proof of the sufficiency of Theorem 2.1 that $(\varphi \pm \varepsilon\psi)(G) = \max\{(p_1 \pm \varepsilon q_1)(G),\dots,(p_n \pm \varepsilon q_n)(G)\}$ for a sufficiently small $\varepsilon > 0$.

We show that $p_i(x) = 0$ implies $q_i(x) = 0$ when $\psi(G) = \psi(\Omega)$ for all $G \subset \Omega$ with $\varphi(G) = \varphi(\Omega)$. Then we have $p_i \pm \varepsilon q_i$ is nonnegative for a sufficiently small $\varepsilon > 0$. When $p_i(x) = 0$, $\varphi(\Omega - \{x\}) = p_i(\Omega - \{x\}) = p_i(\Omega) = \varphi(\Omega)$ since φ is uniform. So $\psi(\Omega - \{x\}) = q_i(\Omega - \{x\}) = q_i(\Omega) = \psi(\Omega)$ by the second and the third assumptions. So $q_i(x) = 0$ when $p_i(x) = 0$.

Since $(p_i \pm \varepsilon q_i)(\Omega) = (\varphi \pm \varepsilon\psi)(\Omega)$ for all i, $\varphi \pm \varepsilon\psi$ are uniform upper envelopes. So φ is not extremal in the cone of the uniform upper envelopes.

Necessity: The proof is similar to the necessity part of Theorem 2.1. Assume that a uniform upper envelope φ is not extremal in the cone of the uniform upper envelopes. The first and second conditions can be proved similarly to the necessity part of Theorem 2.1. We show the third condition.

For each $G \subset \Omega$ with $p_i(G) = p_i(\Omega) = \varphi(\Omega)$, we have $p_i'(G) = p_i'(\Omega)$ and $p_i''(G) = p_i''(\Omega)$ since $p_i'(G) + p_i''(G) = p_i(G)$, $p_i'(\Omega) + p_i''(\Omega) = p_i(\Omega)$, $p_i'(G) \leqslant p_i'(\Omega)$ and $p_i''(G) \leqslant p_i''(\Omega)$. So $q_i(G) = p_i'(G) - p_i(G) = p_i'(\Omega) - p_i(\Omega) = q_i(\Omega) = \psi(\Omega)$. □

Corollary 2.3. *Let a set function φ be extremal in the cone of the uniform upper envelopes. Then $\varphi(\Omega - \{a\}) = 0$ or $\varphi(\Omega - \{a\}) = \varphi(\Omega)$ for all $a \in \Omega$.*

Proof. Let $\{p_1,\dots,p_n\}$ be a minimal representation of φ. Assume there exists an $a \in \Omega$ such that $0 < \varphi(\Omega - \{a\}) < \varphi(\Omega)$ and fix such a. Then $p(a) \geqslant \varphi(\Omega) - \varphi(\Omega - \{a\}) > 0$. We define, for a sufficiently small $\varepsilon > 0$, a vector q_i on Ω by

$$q_i(a) = \varepsilon p_i(a) - \varepsilon\varphi(\Omega),$$

$$q_i(x) = \varepsilon p_i(x) \quad \text{for } x \neq a.$$

Note that $q_i(\Omega) = 0$ for all i, and that $0 \leqslant (p_i + q_i)(G) \leqslant \varphi(\Omega)$ for all $G \subset \Omega$ for sufficiently small $\varepsilon > 0$ since $0 < p_i(a) \leqslant \varphi(\Omega)$.

We define $\psi(G) = q_i(G)$ for $G \subset \Omega$ such that $p_i(G) = \varphi(G)$. We show that these satisfy the third condition of Corollary 2.2. Assume that $\varphi(\Omega) = \varphi(G)$. Then we have $a \in G$ since $\varphi(\Omega - \{a\}) < \varphi(\Omega)$. When $p_i(G) = \varphi(G)$,

$$\begin{aligned}\psi(G) &= q_i(G) = \varepsilon p_i(G) - \varepsilon\varphi(\Omega) = \varepsilon\varphi(G) - \varepsilon\varphi(\Omega) = \varepsilon\varphi(\Omega) - \varepsilon\varphi(\Omega)\\ &= \varepsilon p_i(\Omega) - \varepsilon\varphi(\Omega) = q_i(\Omega) = \psi(\Omega).\end{aligned}$$

So by Corollary 2.2, we obtain the desired result.

It remains to show that the above definition is well defined.

In the case of $a \notin G$, for i and j such that $p_i(G) = p_j(G)$,

$$q_i(G) = \varepsilon p_i(G) = \varepsilon p_j(G) = q_j(G).$$

In the case of $a \in G$, for i and j such that $p_i(G) = p_j(G)$,

$$\begin{aligned} q_i(G) &= q_i(a) + q_i(G - \{a\}) = \varepsilon p_i(a) - \varepsilon\varphi(\Omega) + \varepsilon p_i(G - \{a\}) \\ &= \varepsilon p_i(G) - \varepsilon\varphi(\Omega) = \varepsilon p_j(G) - \varepsilon\varphi(\Omega) = \varepsilon p_j(a) - \varepsilon\varphi(\Omega) + \varepsilon p_j(G - \{a\}) \\ &= q_j(a) + q_j(G - \{a\}) = q_j(G). \end{aligned}$$

So we have $q_i(G) = q_j(G)$ when $p_i(G) = p_j(G)$. $\square$

3. Extremal condition for clutters

A family of subsets of Ω is called a hypergraph on Ω. We call $\mathscr{H}$ a hypergraph only if $\mathscr{H}$ is nonempty. A hypergraph $\mathscr{H}$ is called a clutter if $A \not\subset B$ and $B \not\subset A$ for all distinct $A, B \in \mathscr{H}$. Given a clutter $\mathscr{H}$ on Ω, define

$$\varphi_{\mathscr{H}}(G) = \max_{A \in \mathscr{H}} \chi_A(G)$$

where $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$.

So a clutter induces an upper envelope. An upper envelope induced by a clutter is called hypergraphic in this paper.

For a hypergraph $\mathscr{H}$, if the all elements of $\mathscr{H}$ have the same cardinality, it is called uniform. Obviously, a uniform hypergraph is a clutter. Note that a uniform hypergraph induces a uniform upper envelope.

Denote by $B(\mathscr{H})$ the set of minimal elements of $\{G \subset \Omega\colon A \cap G \neq \emptyset$ for all $A \in \mathscr{H}\}$ with respect to set inclusion. $B(\mathscr{H})$ is so called the blocking set of $\mathscr{H}$. It is well known that $B(B(\mathscr{H})) = \mathscr{H}$ holds for a clutter $\mathscr{H}$(e.g. [2]).

Lemma 3.1. *Let φ be a hypergraphic upper envelope. Then φ has a unique minimal representation.*

Proof. Let P be a minimal representation of φ and $\mathscr{H}$ be the set of maximal elements of $\{H \subset \Omega \mid \varphi(H) = |H|\}$ with respect to set inclusion. Since $\varphi(\{x\}) \leqslant 1$ for all $x \in \Omega$, $p(x) \leqslant 1$ for all $p \in P$ and $x \in \Omega$. For $H \in \mathscr{H}$, there exists a $p^H \in P$ such that $\varphi(H) = |H| = p^H(H)$. But since $p^H(x) \leqslant 1$ for all $x \in H$ and $\sum_{x \in H} p^H(x) = |H|$, we have $p^H(x) = 1$ for all $x \in H$. If there exists an $x \notin H$ such that $p^H(x) \neq 0$, $p^H(x) = 1$ since φ is integral. In that case, $\varphi(H \cup x) = p^H(H \cup \{x\}) = |H \cup \{x\}|$, a contradiction to the maximality of H. So $p^H(x) = 0$ for all $x \notin H$. We have $p^H = \chi_H$ for all $H \in \mathscr{H}$. So $\{\chi_H\}_{H \in \mathscr{H}} \subset P$.

It is easy to show that, for any $G \subset \Omega$, $\varphi(G) = \varphi_{\mathscr{H}}(G)$. Therefore, $\{\chi_H\}_{H \in \mathscr{H}}$ is a unique minimal representation of φ. $\square$

Lemma 3.2. *Let $\mathscr{H}$ be a clutter on Ω and let $G \in B(\mathscr{H})$. For each $x \in G$, there exists an $A \in \mathscr{H}$ such that $G \cap A = \{x\}$.*

Proof. If $|G \cap A| \geqslant 2$ for any $A \in \mathscr{H}$ with $x \in A$, then $(G - \{x\}) \cap H \neq \emptyset$ for all $H \in \mathscr{H}$. This contradicts the minimality of G. If there exists no $A \in \mathscr{H}$ with $x \in A$, this contradicts the minimality of G because $(G - \{x\}) \cap H \neq \emptyset$ for all $H \in \mathscr{H}$. □

For a clutter $\mathscr{H}$, we introduce an equivalence relation on Ω. For $x, y \in \Omega$, let x and y belong to the same equivalence class if and only if there exists a $G \in B(\mathscr{H})$ such that $x, y \in G$. This relation generates an equivalence relation on Ω. When there exists only one equivalence class, we call $\mathscr{H}$ connected.

Note that this definition of connectedness is different from the usual definition of connectedness to the hypergraphs.

Let Ω_1 and Ω_2 be nonempty disjoint sets. For a clutter $\mathscr{H}_1$ on Ω_1 and a clutter $\mathscr{H}_2$ on Ω_2. We define the clutter $\mathscr{H}_1 \oplus \mathscr{H}_2$ on $\Omega_1 \cup \Omega_2$ as follows:

$$\mathscr{H}_1 \oplus \mathscr{H}_2 = \{H_1 \cup H_2 \mid H_1 \in \mathscr{H}_1, H_2 \in \mathscr{H}_2\}.$$

We can naturally define $\oplus_i \mathscr{H}_i$.

Lemma 3.3. *For clutters $\mathscr{H}_1$ and $\mathscr{H}_2$, $\mathscr{H}_1 \oplus \mathscr{H}_2$ is not connected.*

Proof. We show that, for $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, there exist no elements $x \in \Omega_1$ and $y \in \Omega_2$ such that x and y belong to the same equivalence class. Assume, on the contrary, that there exist $x \in \Omega_1$, $y \in \Omega_2$ and $G \in B(\mathscr{H})$ with $x, y \in G$. Then there exists an $H' \in \mathscr{H}$ such that $H' \cap G = \{x\}$ by the minimality of G and Lemma 3.2. Similarly, there exists an $H'' \in \mathscr{H}$ such that $H'' \cap G = \{y\}$. Let $H = (H' \cap \Omega_2) \cup (H'' \cap \Omega_1)$. Then $H \in \mathscr{H}$ by the definition of $\mathscr{H}$. But $H \cap G = \emptyset$, a contradiction. □

Denote $\mathscr{H}|\Omega_i = \{H \cap \Omega_i | H \in \mathscr{H}\}$.

Lemma 3.4. *Let $\{\Omega_i\}$ be the set of equivalence classes of a clutter $\mathscr{H}$. Then $\mathscr{H} = \bigoplus_{i \in I}(\mathscr{H} \mid \Omega_i)$. Moreover when $\mathscr{H}$ is uniform, $\mathscr{H} \mid \Omega_i$ is uniform for all i.*

Proof. Denote by B_i the same operation as B defined on Ω_i.

For Ω_i such that $\Omega_i \cap H = \emptyset$ for all $H \in \mathscr{H}$, obviously $\mathscr{H} = (\mathscr{H}|\Omega_i) \oplus (\mathscr{H}|\Omega_i^{\mathrm{c}})$. So we assume that $\Omega_i \cap H \neq \emptyset$ for some $H \in \mathscr{H}$ in the sequel of this proof.

It is obvious from the definition of Ω that $B_i(\mathscr{H}|\Omega_i) = B(\mathscr{H})|\Omega_i$.

$B(\bigcup_i B_i(\mathscr{H}|\Omega_i)) = \oplus_i B_i(B_i(\mathscr{H}|\Omega_i))$ since $B \in B(\bigcup_i B_i(\mathscr{H}|\Omega_i))$ if and only if $B \cap \Omega_i \in B_i(B_i(\mathscr{H} \mid \Omega_i))$ for all i.

$$\mathscr{H} = B(B(\mathscr{H})) = B\left(\bigcup_i B(\mathscr{H}) \mid \Omega_i\right)$$

$$= B\left(\bigcup_i B_i(\mathscr{H} \mid \Omega_i)\right) = \bigoplus_i B_i(B_i(\mathscr{H}|\Omega_i)) = \bigoplus_i (\mathscr{H}|\Omega_i).$$

When $\mathscr{H}_i$ is not uniform for some i, $\mathscr{H}$ is not uniform. □

Define the dual of a clutter $\mathscr{H}$ as

$$\mathscr{H}^{\mathrm{d}} = \{H^{\mathrm{c}} \mid H \in \mathscr{H}\}.$$

This definition of the dual operation is different from the usual definition of the dual operation to hypergraphs. It is easy to show that the dual of a clutter becomes a clutter.

Lemma 3.5. *A clutter $\mathscr{H}$ is connected if and only if $\mathscr{H}^{\mathrm{d}}$ is connected.*

Proof.

$$\begin{aligned}(\mathscr{H}_1 \oplus \mathscr{H}_2)^{\mathrm{d}} &= \{(H_1 \cup H_2)^{\mathrm{c}} \mid H_1 \in \mathscr{H}_1, H_2 \in \mathscr{H}_2\}\\ &= \{H_1^{\mathrm{c}} \cap H_2^{\mathrm{c}} \mid H_1 \in \mathscr{H}_1, H_2 \in \mathscr{H}_2\}\\ &= \{(\Omega_1 - H_1) \cup (\Omega_2 - H_2) \mid H_1 \in \mathscr{H}_1, H_2 \in \mathscr{H}_2\}\\ &= \mathscr{H}_1^{\mathrm{d}} \oplus \mathscr{H}_2^{\mathrm{d}}.\end{aligned}$$

So the dual of a disconnected clutter is disconnected by Lemma 3.3. So the dual of a connected clutter is connected. □

Theorem 3.6. *Let $\mathscr{H}$ be a clutter on Ω* with $\bigcup \mathscr{H} = \Omega$. *Then $\varphi_{\mathscr{H}}(G) = \max_{A \in \mathscr{H}} \chi_A(G)$ is extremal in the cone of the upper envelopes if and only if $\mathscr{H}$ is connected.*

Proof. *Necessity*: By Lemma 3.1, $\{\chi_A\}_{A \in \mathscr{H}}$ is a unique representation of φ. Assume that $\mathscr{H}$ is connected and there exist ψ and $\{q_A\}_{A \in \mathscr{H}}$ satisfying the conditions in Theorem 2.1. Since $\mathscr{H}$ is connected, $\varphi(\{x\}) = 1$ for all $x \in \Omega$. For $A \in \mathscr{H}$ and $x \in \Omega$ with $x \in A$, $1 = \varphi(\{x\}) = \chi_A(x)$ and $q_A(x) = \psi(\{x\})$ by the second condition in Theorem 2.1. For $A \in \mathscr{H}$ and $x \notin A$, $\chi_A(x) = 0$ and $q_A(x) = 0$ by the third condition in Theorem 2.1.

We show that, for each $G \in B(\mathscr{H}^{\mathrm{d}})$, $\psi(\{x\})$ takes a constant value for all $x \in G$.

By Lemma 3.2, for $G \in B(\mathscr{H}^{\mathrm{d}})$ and $x \in G$, there exists an $A^{\mathrm{c}} \in \mathscr{H}^{\mathrm{d}}$ such that $G \cap A^{\mathrm{c}} = \{x\}$. Then $G \cap A = G - \{x\}$. Since $G \cap B^{\mathrm{c}} \neq \emptyset$ for all $B^{\mathrm{c}} \in \mathscr{H}^{\mathrm{d}}$, $|G \cap B| \leqslant |G| - 1$ for all $B \in \mathscr{H}$. So $\varphi(G) = \chi_A(G) = |A \cap G| = |G - \{x\}| = \varphi(G - \{x\}) = \chi_A(G - \{x\})$. So $\psi(G - \{x\}) = q_A(G - \{x\}) = \sum_{y \in G - \{x\}} \psi(\{y\})$ since $\varphi(\{y\}) = \chi_A(y) = 1$ for all $y \in G - \{x\}$. Because of $q_A(x) = 0$, $\psi(G) = q_A(G) = q_A(x) + q_A(G - \{x\}) = \psi(G - \{x\})$. Since $\psi(G) = \sum_{y \in G - \{x\}} \psi(\{y\})$ for all $x \in G$, $\psi(\{x\})$ takes a constant value for all $x \in G$.

Then by Lemma 3.5, $B(\mathscr{H}^{\mathrm{d}})$ is connected. So together with the above claim, $\psi(\{x\})$ takes a constant value for all $x \in \Omega$. Therefore, $q_A(x) = l\chi_A(x)$ for all $x \in \Omega$ and $A \in \mathscr{H}$ for some $l \in \mathbf{R}$. So $\psi(G) = l\varphi(G)$ for all $G \subset \Omega$, a contradiction to the first condition of Theorem 2.1.

Sufficiency: Let $\Omega_1, \ldots, \Omega_r \subset \Omega$ be the set of equivalence classes induced by $\mathscr{H}$. We first show that, for each $G \subset \Omega$ and each i with $1 \leqslant i \leqslant r$, $|\Omega_i \cap A_1 \cap G| = |\Omega_i \cap A_2 \cap G|$ for any $A_1, A_2 \in \mathscr{H}$ such that $\varphi(G) = \chi_{A_1}(G) = \chi_{A_2}(G)$.

Assume on the contrary that, for some A_1, A_2 and $G \subset \Omega$ such that $\varphi(G) = \chi_{A_1}(G) = \chi_{A_2}(G)$, there exists an i such that $|\Omega_i \cap A_1 \cap G| > |\Omega_i \cap A_2 \cap G|$. Then $|\Omega_i^{\mathrm{c}} \cap A_1 \cap$

$G| < |\Omega_i^c \cap A_2 \cap G|$ since $|A_1 \cap G| = |A_2 \cap G|$. By Lemma 3.4, $(\Omega_i \cap A_1) \cup (\Omega_i^c \cap A_2) \in \mathscr{H}$. We have $\varphi(G) = \chi_{A_1}(G) = |A_1 \cap G| < |((\Omega_i \cap A_1) \cup (\Omega_i^c \cap A_2)) \cap G|$. But by the definition of φ, $\varphi(G) \geqslant |H \cap G|$ for all $H \in \mathscr{H}$, a contradiction.

For $H \subset \mathscr{H}$, let $q_H(x) = 1$ for $x \in \Omega_1 \cap H$ and $q_H(x) = 0$ otherwise. Then $q_H(A) = |\Omega_1 \cap H \cap A|$. $\chi_H(x) = 0$ implies $q_H(x) = 0$. Let $\psi(G) = |\Omega_1 \cap H \cap G|$ where $\varphi(G) = \chi_H(G)$. By the above claim, if $\chi_H(G) = \chi_{H'}(G) = \varphi(G)$, then $q_H(G) = q_{H'}(G)$. So ψ is well defined. Moreover $q_H(G) = \psi(G)$ when $\chi_H(G) = \varphi(G)$. Because of $\Omega_1 \neq \Omega$ and the assumption $\bigcup \mathscr{H} = \Omega$, ψ is not proportional to φ. Therefore, the conditions of Theorem 2.1 are satisfied. So $\varphi_{\mathscr{H}}$ is not extremal. □

Corollary 3.7. *Let $\mathscr{H}$ be a uniform hypergraph on Ω with $\bigcup \mathscr{H} = \Omega$. Then $\varphi_{\mathscr{H}}(G) = \max_{A \in \mathscr{H}} \chi_A(G)$ is extremal in the cone of the uniform upper envelopes if and only if $\mathscr{H}$ is connected.*

Proof. *Necessity*: When $\mathscr{H}$ is connected, $\varphi_{\mathscr{H}}$ is extremal in the cone of the upper envelopes by Theorem 3.6. So $\varphi_{\mathscr{H}}$ is extremal in the cone of uniform upper envelopes.

Sufficiency: The proof is similar to Theorem 3.6. To apply Corollary 2.2 instead of Theorem 2.1, we modify the last part of the proof of sufficiency in Theorem 3.6.

Note that $|H| = \varphi(\Omega)$ for all $H \in \mathscr{H}$ since $\mathscr{H}$ is uniform. When $\varphi(\Omega) = \varphi(G)$, $|H| = \varphi(\Omega) = \varphi(G) = \chi_H(G) = |H \cap G|$ for all $H \in \mathscr{H}$. So $H \subset G$ for all $H \in \mathscr{H}$ when $\varphi(\Omega) = \varphi(G)$. In that case, $\psi(G) = |\Omega_1 \cap H \cap G| = |\Omega_1 \cap H| = |\Omega_1 \cap H \cap \Omega| = \psi(\Omega)$. So the assumptions of Corollary 2.2 are satisfied. □

Example 3.8. Let $\Omega = \{a, b, c, d\}$. Let

$$p_1 = (1,1,0,0), \quad p_2 = (0,1,1,0), \quad p_3 = (0,0,1,1), \quad p_4 = (1,0,0,1).$$

$B(\mathscr{H}) = \{\{a,c\},\{b,d\}\}$. Because $\mathscr{H}$ is not connected, $\varphi_{\mathscr{H}}$ is not extremal in the cone of the uniform upper envelopes.

Let

$$p_1 = (1,1,0,0), \quad p_2 = (0,1,1,0), \quad p_3 = (0,0,1,1).$$

Then $\mathscr{H} = \{\{a,b\},\{b,c\},\{c,d\}\}$. $B(\mathscr{H}) = \{\{a,c\},\{b,d\},\{b,c\}\}$. Because $\mathscr{H}$ is connected, $\varphi_{\mathscr{H}}$ is extremal in the cone of the uniform upper envelopes.

For a uniform hypergraph $\mathscr{H}$, $\mathscr{H}$ is called (the basis of) a matroid if, for any $B_1, B_2 \in \mathscr{H}$ and any $x \in B_1$, there exists a $y \in B_2$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathscr{H}$.

The rank function of a matroid $\mathscr{H}$ is $\varphi_{\mathscr{H}}$. We define the circuits of a matroid as usual. For a matroid $\mathscr{H}$ and $x \in \Omega$, we define the relation γ on Ω by $e \gamma f$ if and only if the matroid has a circuit containing both e and f. It is known that this relation is an equivalence relation on Ω (e.g., [8]). A matroid is called connected when there exists only one equivalence class.

Lemma 3.9. *A matroid $\mathscr{H}$ is connected if and only if $\mathscr{H}$ is connected as a clutter.*

Proof. $\mathscr{H}$ is connected as a clutter if and only if $B(\mathscr{H}^{\mathrm{d}})$ generates only one equivalence class by Lemma 3.5.

$$\{G \subset \Omega \mid A \cap G \neq \emptyset \text{ for all } A \in \mathscr{H}^{\mathrm{d}}\} = \{G \subset \Omega \mid A^{\mathrm{c}} \cap G \neq \emptyset \text{for all } A \in \mathscr{H}\}$$
$$= \{G \subset \Omega \mid G \not\subset A \text{ for all } A \in \mathscr{H}\}.$$

So $B(\mathscr{H}^{\mathrm{d}})$ is the set of the circuits of $\mathscr{H}$. $\square$

A set function φ on Ω is a polymatroid rank function if the following three conditions are satisfied.
1. $\varphi(\emptyset) = 0$.
2. $\varphi(A) \leqslant \varphi(B)$ for all $A \subset B \subset \Omega$.
3. $\varphi(A) + \varphi(B) \geqslant \varphi(A \cup B) + \varphi(A \cap B)$ for all $A, B \subset \Omega$.

It is easy to show that a polymatroid rank function is a uniform upper envelope (e.g. [1]).

The next corollary is also shown in [7].

Corollary 3.10. *For a matroid $\mathscr{H}$, it is connected if and only if the rank function $\varphi_{\mathscr{H}}$ is extremal in the cone of the polymatroid rank functions.*

Proof. *Necessity*: When a matroid $\mathscr{H}$ is disconnected, we can write $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ for some matroids $\mathscr{H}_1$ and $\mathscr{H}_2$. Let r_1 and r_2 be rank functions of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Then $\varphi_{\mathscr{H}} = r_1 + r_2$.

Sufficiency: The cone of the polymatroid rank functions is included in the cone of the uniform upper envelopes. So if a polymatroid rank function is extremal in the cone of the uniform upper envelopes, it is extremal in the cone of the polymatroid rank functions. By Corollary 3.7 and Lemma 3.9, a connected matroid is extremal in the cone of the upper envelopes.

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