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The Classification of Hamiltonian Generalized Petersen Graphs*

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The generalized Petersen graph $GP(n, k)$, $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$, has vertex-set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 0 \leqslant i \leqslant n-1$ with subscripts reduced modulo $n\}$. In this paper it is proved that $GP(n, k)$ is hamiltonian if and only if it is neither $GP(n, 2) \cong GP(n, n-2) \cong GP(n, (n-1)/2 \cong GP(n, (n+1)/2)$ when $n \equiv 5 \pmod 6$ nor $GP(n, n/2)$ when $n \equiv 0 \pmod 4$ and $n \geqslant 8$.

1. INTRODUCTION

The *generalized Petersen graph* $GP(n, k)$, $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$, has vertex-set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k}: 0 \leqslant i \leqslant n-1$ with subscripts reduced modulo $n\}$. These graphs were first defined by Watkins in [9]. In this original definition, $GP(n, k)$ was not defined when n is even and $k = n/2$ because the resulting graph is not cubic. However, we do not exclude them because their behavior with regard to Hamiltonian cycles is so easily determined.

In this paper we completely determine which generalized Petersen graphs have a Hamiltonian cycle, thereby settling a modified conjecture of Castagna and Prins [4]. Consequently, a brief but thorough history of the problem is in order.

Watkins posed the question [9] of whether or not every cubic $GP(n, k)$, other than $GP(5, 2) \cong GP(5, 3)$, has a 1-factorization. Meanwhile, Robertson [6] proved that $GP(n, 2)$ is Hamiltonian if and only if $n \not\equiv 5 \pmod 6$. Robertson's result was proved independently by Bondy [3]. In the latter paper, Bondy also proved that $GP(n, 3)$ is Hamiltonian whenever $n \neq 5$. Finally, Castagna and Prins provided an affirmative answer to Watkins' 1-

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factorization question in [4]. In doing so, they observed that they found no non-Hamiltonian cubic generalized Petersen graphs other than those found by Robertson. This led them to conjecture that the Robertson examples were the only non-Hamiltonian examples. The preceding results together with the elementary observation that GP(n, 1) is always Hamiltonian is where the progress on the Castagna–Prins conjecture stood through most of the 1970s.

The first of two important contributions towards the resolution of the conjecture was made by Bannai [2] with the following result:

THEOREM 1. *If n and k are relatively prime, then* GP(n, k) *is Hamiltonian unless* $n \equiv 5 \pmod 6$ *and* GP$(n, k) \cong$ GP$(n, 2)$, *that is,* $k = 2$, $(n-1)/2$, $(n+1)/2$, *or* $n-2$.

The second contribution was the introduction of the lattice diagrams in [1] for the proof of the following result. More will be said about these lattice diagrams in the next section since they are employed extensively throughout this paper.

THEOREM 2. *If* $k \geqslant 3$, *then there exists an* $n(k)$ *such that* GP(n, k) *is Hamiltonian for all* $n \geqslant n(k)$.

It was noticed that it is easy to prove that GP($2k, k$) is Hamiltonian if and only if $k = 2$ or k is odd. This was then incorporated into the Castagna–Prins conjecture to obtain a modified conjecture accordingly. Simmons and Slater have verified the modified conjecture for all $k < 36$ [7, 8]. The purpose of this paper is to prove the following result.

THEOREM 3. *The generalized Petersen graph* GP(n, k) *is Hamiltonian if and only if it is neither*

(i) GP$(n, 2) \cong$ GP$(n, n-2) \cong$ GP$(n, (n-1)/2) \cong$ GP$(n, (n+1)/2$, $n \equiv 5 (\mathrm{mod}\ 6)$, *nor*

(ii) GP$(n, n/2)$, $n \equiv 0 \pmod 4$ *and* $n \geqslant 8$.

2. Lattice diagrams

Label the lattice points in the plane with the integers 0, 1,..., $n-1$ so that if (x, y) is labelled with i, then $(x+1, y)$ is labelled with $i+1$ and $(x, y-1)$ is labelled with $i+k$, all arithmetic being done modulo n. Make it a labelled graph $H(n, k)$ by letting (x, y) and (x', y') be adjacent if and only if $|x-x'| + |y-y'| = 1$. A subgraph $L(n, k)$ of $H(n, k)$ is called a *lattice diagram* for GP(n, k) if $L(n, k)$ has either a closed or an open Eulerian trail such that a traversal of the Eulerian trail obeys the following rules:

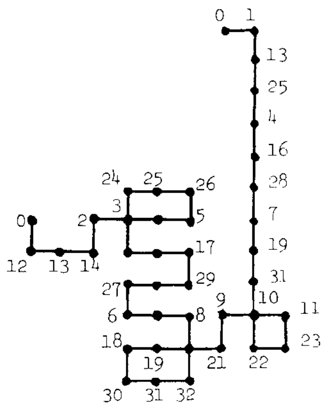

FIGURE 1

(1) If a vertex of degree 4 is entered vertically (horizontally), then it must be departed vertically (horizontally);

(2) Each label $0, 1, \ldots, n-1$ is encountered once in the horizontal direction and once in the vertical direction when traversing $L(n, k)$; and

(3) If $L(n, k)$ has an open Eulerian trail, then the two vertices of odd degree must have the same label and either both have degre 1 or one has degree 1 and the other has degree 3.

It is not difficult to see that a lattice diagram $L(n, k)$ corresponds to a Hamiltonian cycle in $\mathrm{GP}(n, k)$ and transforms the problem of looking for Hamiltonian cycles to looking for certain labelled Eulerian graphs. For example, in Fig. 1 we show an $L(33, 12)$ lattice diagram. The eulerian traversal starting $0, 1, \ldots,$ corresponds to the Hamiltonian cycle $u_0 u_1 v_1 v_{13} v_{25} \cdots v_{10} v_{22} u_{22} u_{23} v_{23} v_{11} u_{11} u_{10} u_9 \cdots v_{15} v_3 v_{24} u_{24} \cdots u_2 v_2 v_{14} u_{14} u_{13} \cdots v_0 u_0$ in $\mathrm{GP}(33, 12)$. For more details on lattice diagrams see [1].

3. Proof of Theorem 3

We first dispose of the case that n is even and $k = n/2$. Since $\deg(v_i) = 2$ for $i = 0, 1, \ldots, n-1$, if $\mathrm{GP}(n, n/2)$ is Hamiltonian, any Hamiltonian cycle must contain the paths $u_i v_i v_{i+n/2} u_{i+n/2}$ for $i = 0, 1, \ldots, (n-2)/2$. Consequently, it is easy to see that $\mathrm{GP}(n, n/2)$ is hamiltonian if and only if $n = 4$ or $n \equiv 2 \pmod 4$.

For the remainder of the proof, we assume that $k \neq n/2$ so that $\mathrm{GP}(n, k)$ is always cubic. We may also assume that $k \leqslant \lfloor n/2 \rfloor$ because

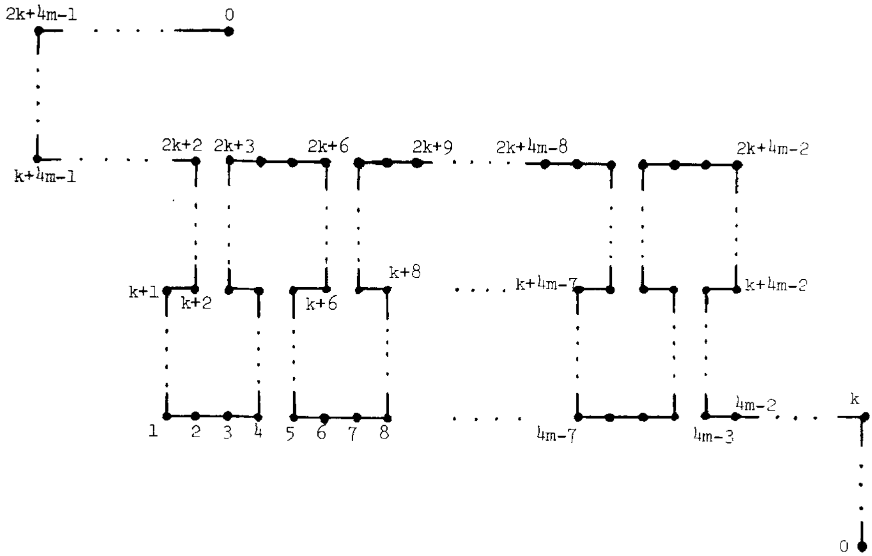

FIGURE 2

$GP(n, k) \cong GP(n, n-k)$. Because of Bannai's theorem, we shall be finished if we show that $GP(n, k)$ is Hamiltonian whenever $\gcd(n, k) \neq 1$.

LEMMA 1. *The cubic generalized Petersen graph* $GP(n, k)$ *is Hamiltonian whenever* $\gcd(n, k)$ *is even.*

Proof. The $L(n, k)$ lattice diagram in Fig. 2 yields a Hamiltonian cycle in $GP(n, k)$ when $\gcd(n, k) = 4m$ for any $m \geqslant 1$ and $n \geqslant 2k + 4m$. Likewise, an $L(n, k)$ lattice diagram is given in Fig. 3 when $\gcd(n, k) = 4m + 2$ for any $m \geqslant 0$ and $n \geqslant 2k + 4m + 2$. ∎

This leaves us with the problem of showing that $GP(dn, dk)$ is Hamiltonian for all odd $d > 1$ when $\gcd(n, k) = 1$. This is considerably more difficult than the even greatest common divisor case covered by Lemma 1. An idea that reduces the problem to a manageable number of cases is presented in Lemma 2. First we need some definitions.

A labelling of the lattice points as described earlier will be called a k-oriented labelling. On occasion we shall need a labelling where we add $n - k$, instead of k, to the label of (x, y) to obtain the label of $(x, y - 1)$. Such a labelling will be called an $(n - k)$-oriented labelling. It is clear that a lattice diagram corresponds to a Hamiltonian cycle as long as every label is encountered once vertically and once horizontally no matter which orientation the labelling may have. If a lattice diagram $L(n, k)$ has a subpath of

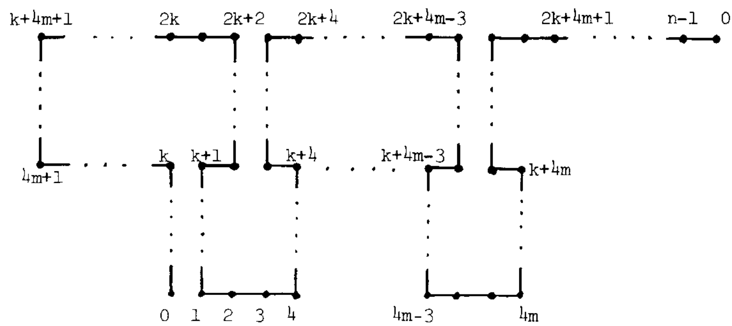

FIGURE 3

the Eulerian traversal of the form $0, 1, k+1, 2k+1, \ldots, n+1-k, n+2-k$ when the labelling is k-oriented or $0, 1, n+1-k, n+1-2k, \ldots, k+1, k+2$ when then labelling is $(n-k)$-oriented, then it is called an *extendible* lattice diagram.

LEMMA 2. *Let n and k be relatively prime with $1 \leqslant k < n/2$. If there exist an extendible lattice diagram $L(3n, 3k)$, then $\mathrm{GP}(dn, dk)$ is Hamiltonian for all odd $d > 1$.*

Proof. Suppose there is an extendible lattice diagram $L(3n, 3k)$. Suppose the labelling is $3k$-oriented. Notice that the graph $\mathrm{GP}(3n, 3k)$ has three disjoint cycles of length n for the subgraph induced on the set of vertices $\{v_0, v_1, \ldots, v_{3n-1}\}$. The extendible lattice diagram $L(3n, 3k)$ yields a Hamiltonian cycle C in $\mathrm{GP}(3n, 3k)$ containing the path $P = u_0 u_1 v_1 v_{3k+1} v_{6k+1} \cdots v_{-3k+1} u_{-3k+1} u_{-3k+2}$, that is, all the vertices v_i with $i \equiv 1 \pmod 3$ appear as a Hamiltonian path in the cycle C. Now consider the graph $\mathrm{GP}(dn, dk)$ for odd $d > 3$. Notice that the subgraph induced on the set of vertices $\{v_0, v_1, \ldots, v_{dn-1}\}$ consists of d disjoint cycles of length n. We use the cycle C in $\mathrm{GP}(3n, 3k)$ to construct a Hamiltonian cycle C' in $\mathrm{GP}(dn, dk)$ as follows: Any vertex of $\mathrm{GP}(3n, 3k)$ with subscript $3i$, $0 \leqslant i \leqslant n-1$, corresponds to the same type of vertex with subscript di, $0 \leqslant i \leqslant n-1$, in $\mathrm{GP}(dn, dk)$. Those with subscripts of the form $3i+1$ correspond to those in $\mathrm{GP}(dn, dk)$ with subscripts $di+1$ and those with subscripts $3i+2$ in $\mathrm{GP}(3n, 3k)$ correspond to those with subscripts $di + (d-1)$ in $\mathrm{GP}(dn, dk)$. The path P in C becomes the path P' given by

$$\begin{aligned} P' = {} & u_0 u_1 v_1 v_{dk+1} \cdots v_{-dk+1} u_{-dk+1} u_{-dk+2} v_{-dk+2} v_{-2dk+2} \cdots v_{dk+2} v_2 u_2 \\ & u_3 v_3 v_{dk+3} v_{2dk+3} \cdots v_{d-3} u_{d-3} u_{d-2} v_{d-2} v_{dk+d-2} \cdots v_{-dk+d-2} \\ & u_{-dk+d-2} u_{-dk+d-1}. \end{aligned}$$

Notice that P' uses the vertices $u_0, u_1, \ldots, u_{d-2}, u_{-dk+1}, u_{-dk+2}, \ldots, u_{-dk+d-1}$ and all vertices v_i with $i \not\equiv 0 \pmod d$ and $i \not\equiv d-1 \pmod d$. Now let $\bar{P}$ be the path in C from u_0 to u_{-3k+2} such that $\bar{P} \cup P = C$. Now any occurrence of u_{3i+1} on $\bar{P}$ must have u_{3i} and u_{3i+2} as its immediate neighbors because v_{3i+1} lies on P. So in building the path $\bar{P}'$ of C' corresponding to $\bar{P}$, replace the 2-path $u_{3i}u_{3i+1}u_{3i+2}$ by the $(d-1)$-path $u_{di}u_{di+1}u_{di+2} \cdots u_{di+d-1}$. Any vertices v_{3i} and v_{3i+2} on $\bar{P}$ are simply replaced by v_{di} and v_{di+d-1}, respectively. Then $P' \cup \bar{P}'$ is a Hamiltonian cycle of $\mathrm{GP}(dn, dk)$.

A similar argument works if the labelling is $(3n-3k)$-oriented. ∎

It is easy to see how the proof of Lemma 2 works by considering the corresponding lattice diagrams. As an illustration, Fig. 4 contains an $L(55, 20)$ lattice diagram obtained from the $L(33, 12)$ lattice diagram of Fig. 1 according to the proof of Lemma 2.

The proof of Theorem 3 will be complete if we show that there is an extendible $L(3n, 3k)$ lattice diagram for all relatively prime n and k, $1 \leqslant k < n/2$. There are two main cases to consider, namely, n even and n odd. We first consider the case that n is odd.

The length of the path $3ik, 3(i+1)k, \ldots, 3jk$ will be called the $3k$-distance from $3ik$ to $3jk$ while the length of the path $3ik, 3(i-1)k, \ldots, 3jk$ will be called the $(3n-3k)$-distance between them where length refers to the number of edges. Since n is odd, either the $3k$-distance from 0 to 3 is odd and the $(3n-3k)$-distance is even or vice-versa. We shall orient the lattice diagram so that the distance from 0 down to 3 is odd.

The first subcase is that the distance from 0 down to 3 is $n-2$. If this were $(3n-3k)$-distance, then $3-6k \equiv 0 \pmod{3n}$ would hold which would

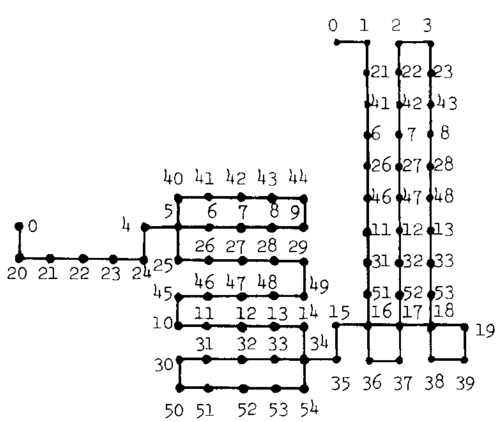

FIGURE 4

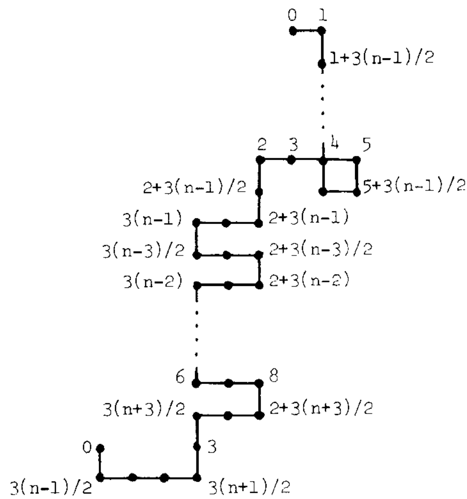

FIGURE 5

force $6k = 3n + 3$ to hold. This is a contradiction because $k < n/2$. Hence, the distance must be $3k$-distance (that is, the lattice diagram must have $3k$-orientation; this will be used over and over). This implies that $3 + 6k \equiv 0 \pmod{3n}$ holds which in turn implies that $k = (n-1)/2$. So we need a $3k$-oriented $L(3n, 3(n-1)/2)$ extendible lattice diagram. One is shown in Fig. 5. It works for all $n \geqslant 5$. When $n = 3$ and $(n-1)/2 = 1$, the diagram of Fig. 6 works since it covers all cases when k is odd and k divides n.

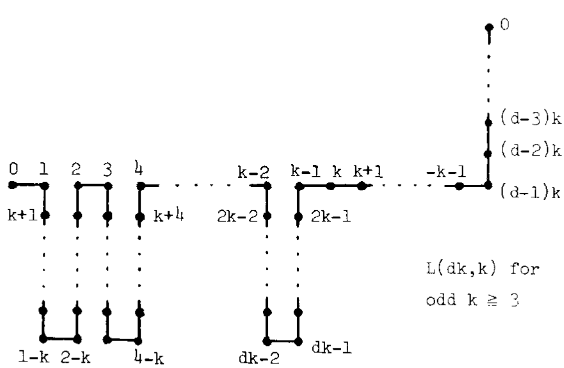

FIGURE 6

If the distance from 0 to 3 is 1, then either $-3k \equiv 3 \pmod{3n}$ or $3k \equiv 3 \pmod{3n}$. The first is impossible because $k < n/2$ while the second implies that $k = 1$. The latter situation has been disposed of by Fig. 6. Hence, we may assume that the distance from 0 to 3 is at least 3 and at most $n-4$. In subsequent diagrams, if there are two signs in front of a term, the upper sign refers to a $3k$-oriented labeling and the lower sign to a $(3n-3k)$-oriented labelling. So the vertex immediately below 4, for example, will be labelled $4 \pm 3k$.

In the rest of the subcases for n odd, the location of the label $\mp 6k - 3$ in the column beneath 0 is crucial. Notice that $\mp 6k - 3$ lies the same distance above $\mp 6k$ that 0 lies above 3. Since $\mp 6k$ has even distance at least 2 below 3, $\mp 6k - 3$ has even distance at least 2 below 0. The extendible lattice diagram in Fig. 7 works for all the cases that $\mp 6k - 3$ lies between $\pm 6k$ and $3 \mp 9k$. This is because the left part of the diagram from $\pm 6k$ down through $\pm 3k + 3$ needs length at least 3 from $\mp 6k - 3$ down to 3 in order to work.

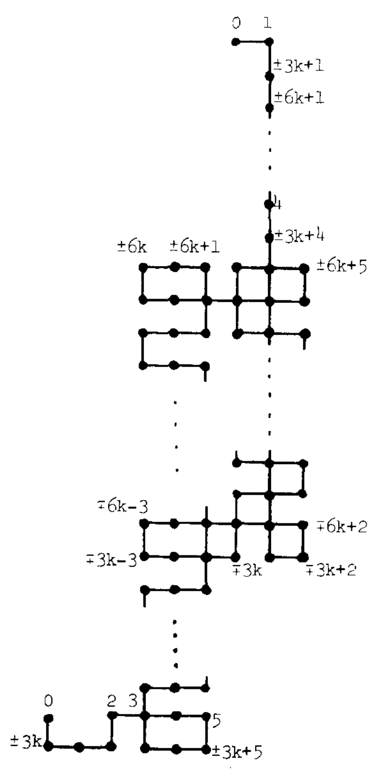

FIGURE 7

Also, in the special case that the length from 3 to $\mp 3k$ is precisely 3, the diagram still works because all the switchbacks above the horizontal part containing the vertex $\mp 6k$ are gone and the edge between $\mp 6k - 1$ and $\mp 6k$ is also deleted so that both have degree 2.

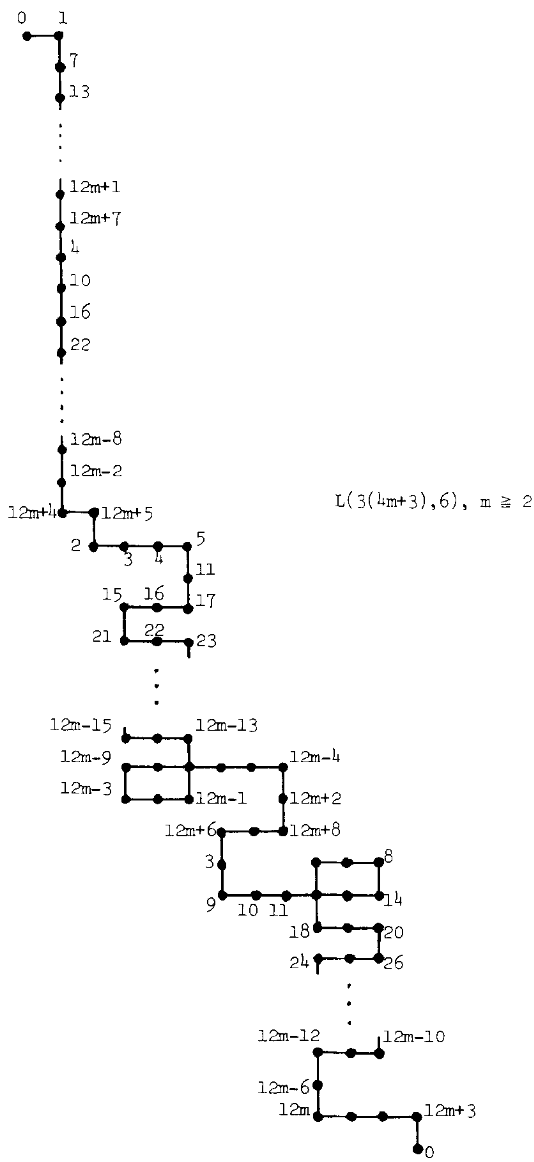

FIGURE 8

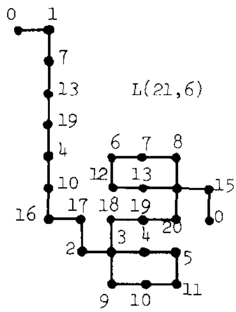

FIGURE 9

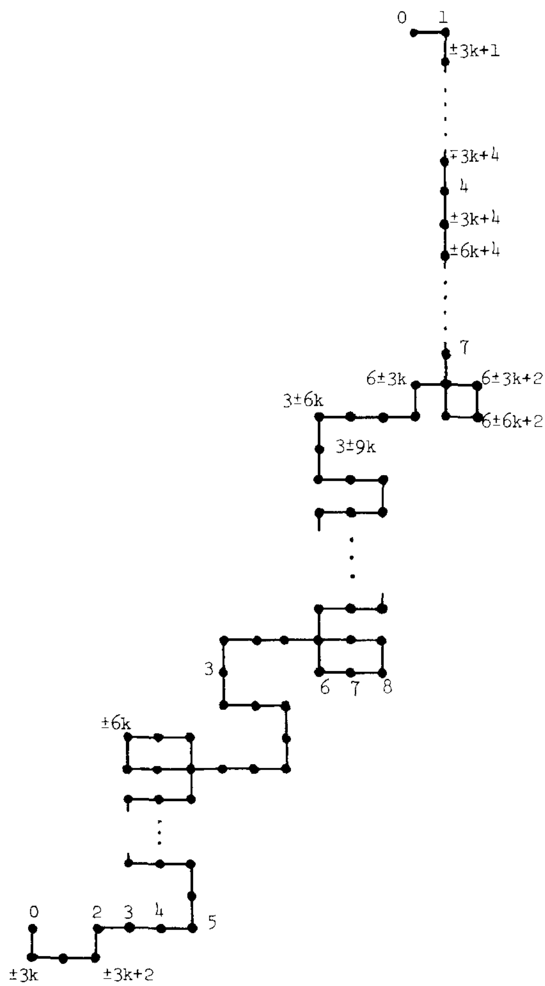

FIGURE 10

Next suppose that $\mp 6k - 3 \equiv 3 \mp 3k \pmod{3n}$. If we are in the $3k$-distance situation, then $-6k - 3 \equiv 3 - 3k \pmod{3n}$ must hold. But this contradicts $k < n/2$ for $n \geqslant 5$ and yields $n = 3$, $k = 1$ otherwise. The latter has already been dealt with. Thus, we know that $6k - 3 \equiv 3 + 3k \pmod{3n}$ is the only possible situation. Solving this congruence yields $k = 2$ and since we must be in the $(3n - 3k)$-distance situation, $n \equiv 3 \pmod 4$ must hold. The extendible lattice diagram in Fig. 8 covers all cases with $n \geqslant 11$ and the extendible lattice diagram in Fig. 9 covers the case that $n = 7$. Notice that the diagrams are drawn with 6-orientation rather than $(3n - 6)$-orientation.

The next subcase to consider is that $\mp 6k - 3 \equiv 3 \pm 3k \pmod{3n}$. This forces $k = (n \mp 2)/3$ to hold. Thus k cannot be even or else n is even. The case that $k \equiv 3 \pmod 4$ cannot arise either since the appropriate distances from 0 to 3 are then even. When $k \equiv 1 \pmod 4$, the distances from 0 to 3 are odd and $k \geqslant 5$ may be assumed because $k = 1$ has already been done. Figure 10 contains an extendible lattice diagram that covers all such cases.

The last special subcase to consider is that $\mp 6k - 3 \equiv 3 \pm 9k \pmod{3n}$. In Fig. 11 we show a diagram that covers all cases for which the distance from 0 to $3 \mp 6k$ is at least 7, that is, n is at least 21. This leaves the special cases

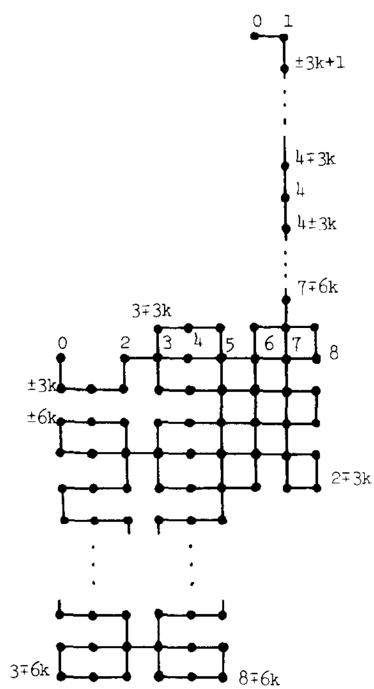

FIGURE 11

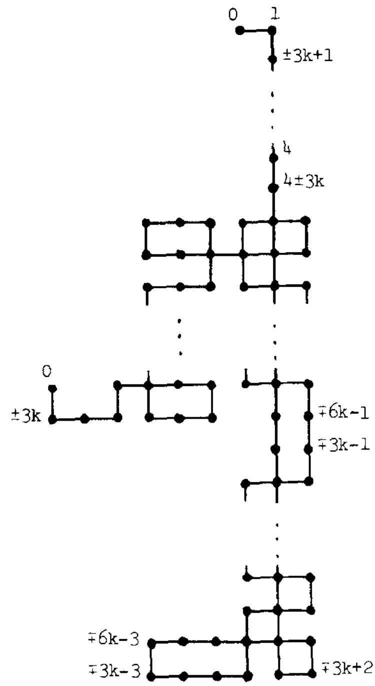

FIGURE 12

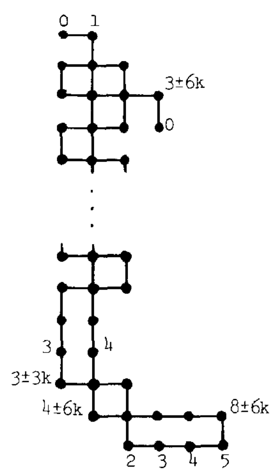

FIGURE 13

of $n = 3$, $k = 1$ and $n = 11$, $k = 4$. The first one is covered by the diagram of Fig. 6 and the last special case is covered by the extendible $L(33, 12)$ lattice diagram of Fig. 1.

To complete the case of n odd, the extendible lattice diagram of Fig. 12 takes care of all the situations for which $\mp 6k - 3$ lies distance 5 or more below 3.

This now leaves us with the case that n is even. Thus, k is odd and both the $3k$-distance and $(3n - 3k)$-distance from 0 to 3 are odd. We shall work with whichever of the two distances is greater. Hence, the distance from 0 to 3 is at least $n/2$.

If the distance from 0 to 3 is $n - 1$, then $k = 1$ and we are done because of Fig. 6. So we may assume the distance from 0 to 3 is at most $n - 3$. If the distance is exactly $n - 3$, then the diagram of Fig. 13 covers all such cases.

We now may assume the distance from 0 to 3 is $n - 5$ or less. Let the

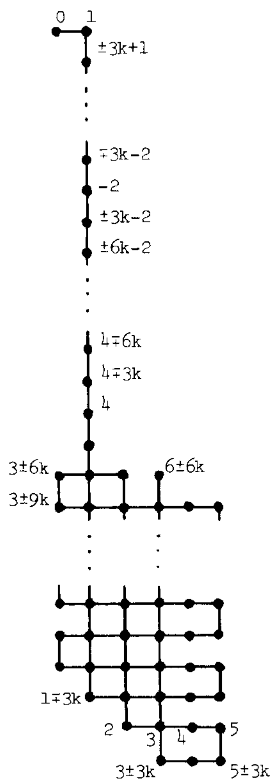

FIGURE 14

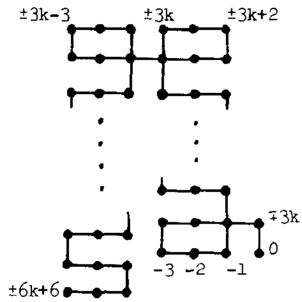

FIGURE 15

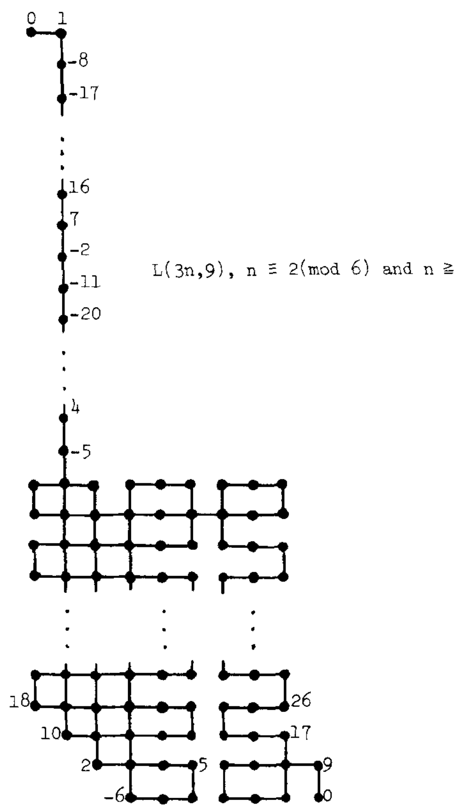

FIGURE 16

distance from 4 to $1 \mp 3k$ be d. Since the distance from 0 to 3 is $n - d - 1$, d is even and $d \geqslant 4$. Consider the partial lattice diagram shown in Fig. 14. The location of $6 \pm 6k$ in the column under 0 is crucial. First, $6 \pm 6k$ lies between 0 and $3 \mp 9k$ because the distance from $6 \pm 6k$ to $3 \pm 6k$ is $d + 1$ which is at least 5. Also, notice that $d + 1$ is odd while the distance from -3 to $3 \pm 6k$ is even so that $6 \pm 6k$ cannot be -3.

First, suppose that $6 \pm 6k$ lies below -3 and has distance 3 or more from -3. Then attach the diagram of Fig. 15 to that of Fig. 14 at $6 \pm 6k$ to give an extendible $(3n, 3k)$ lattice diagram.

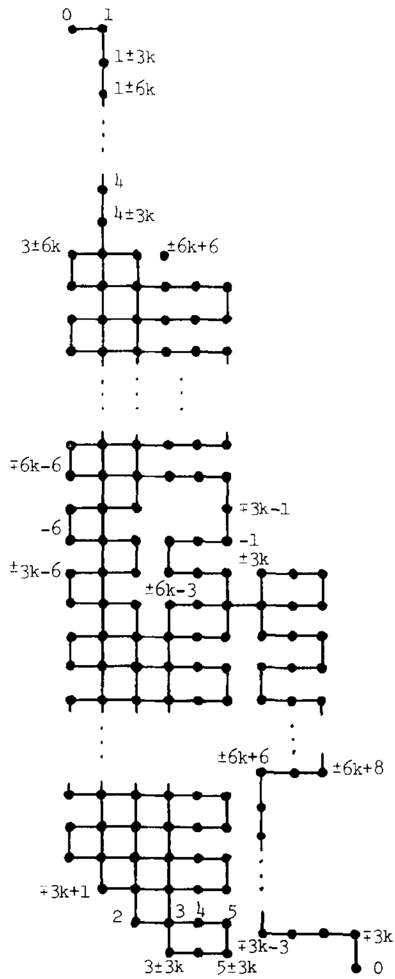

FIGURE 17

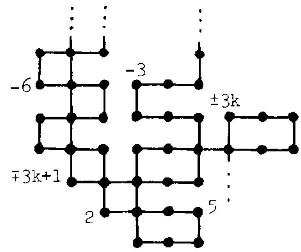

FIGURE 18

Now suppose that $6 \pm 6k \equiv -3 \pm 3k \pmod{3n}$. If we are working with a $3k$-orientation situation, we have $6 + 6k \equiv -3 + 3k \pmod{3n}$ which implies $3n = 3k + 9$. This is impossible because we may assume $n \geqslant 7$ since the smaller values of n are covered by earlier cases. The only possibility left is that we are working with a $(3n - 3k)$-orientation situation. Thus, $6 - 6k \equiv -3 - 3k \pmod{3n}$ holds which implies that $k = 3$. The diagram of Fig. 16 covers this case because it is easy to verify that $n \equiv 2 \pmod 6$ and $n \geqslant 14$ are the only values of n that have not been previously covered for this case.

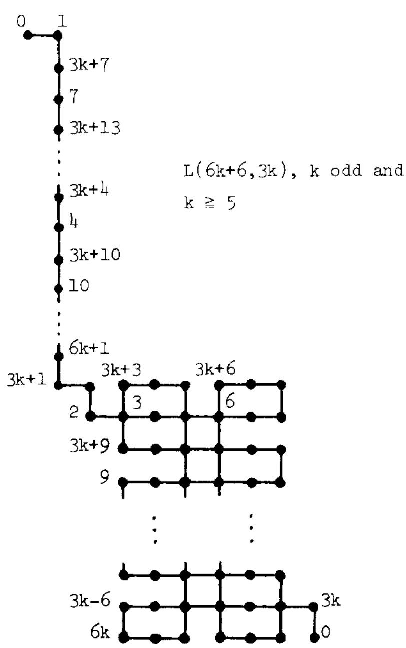

FIGURE 19

We are now left with the situation that $6 \pm 6k$ occurs above -3. We know that the distance from $6 \pm 6k$ to -3 is odd. The distance from 0 to 3 is $n - d - 1$ so that the distance from 0 to $3 \pm 6k$ is $n - d + 1$. The distance from 3 to 0 is $d + 1$ which implies the distance from $6 \pm 3k$ to $3 \pm 3k$ is also $d + 1$. Since -3 lies above 3, $n - d - 1 > d + 1$. Hence, the distance from 0 to $6 \pm 6k$ is at least 3 and because it is even, it is at least 4. Thus, $6 \pm 6k$ lies between $\pm 12k$ and $-3 \mp 3k$ inclusive.

We first dispose of the general case that the distance from $6 \pm 6k$ to -3 is 3 or more, that is, $6 \pm 6k \not\equiv -3 \mp 3k \pmod{3n}$. Figure 17 contains an ex-

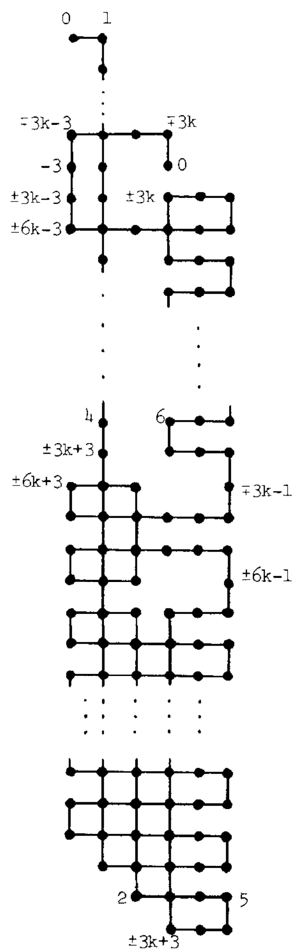

FIGURE 20

tendible lattice diagram that works as we now verify. First, the switchover in the diagram at the $4 \pm 6k$ and $4 \pm 9k$ level requires that the distance from $3 \pm 6k$ to -6 is at least 3. This is precisely the case we are working in. Next, the part of the diagram from $\pm 3k$ to $\pm 6k + 6$ works as long as the distance from $\pm 3k$ to $\pm 6k + 6$ is odd and at least 3. This is the case since it is the same as the distance from 0 to $\pm 3k + 6$. Finally, the only potential special cases will arise if the distance from -6 to $\mp 3k$ is insufficient. This disance is odd and the diagram works as shown if it is 5 or more. If the distance is 3, the modification depicted in Fig. 18 will work. If this distance is 1, then $3k$-orientation leads to $k = 1$ which has already been covered. A $(3n - 3k)$-orientation leads to $n = 2k + 2$, k odd and $k \geqslant 5$ (smaller odd values are covered by the diagrams of Figs. 6 and 13) which is covered by the extendible lattice diagram of Fig. 19.

We are at last left with the special case that $6 \pm 6k \equiv -3 \mp 3k \pmod{3n}$. The extendible lattice diagram of Fig. 20 covers this case as long as the distance from 3 to $\mp 3k$ is at least 6. The only other possibility is that this distance is precisely 4 since Fig. 13 covers the situation that the distance is 2. In the k-orientation case with the distance 4, we have $3 + 15k \equiv 0 \pmod{3n}$ and $9k + 9 \equiv 0 \pmod{3n}$ which imply that $6k + 6 \equiv 0 \pmod{3n}$. This forces $k = 1$ to hold and this has already been done. In the $(3n - 3k)$-orientation case, we have $3 - 15k \equiv 0 \pmod{3n}$ and $9 - 9k \equiv 0 \pmod{3n}$. This implies that $6k \equiv -6 \pmod{3n}$ or that $k = (n-2)/2$. But these are all covered by the extendible lattice diagram of Fig. 20.

This completes the proof of Theorem 3. ∎

4. Conclusion

It is now easy to obtain a slight generalization of the main result of Castagna and Prins in [4].

Corollary 1. *The chromatic index of every generalized Petersen graph, other than the Petersen graph itself and* GP(2, 1), *is three. In particular, every cubic generalized Petersen graph, other than the Petersen graph, has a 1-factorization.*

Proof. It is easy to 3-color the edges of all GP$(2k, k)$, $k \geqslant 2$, by starting with one color for all of the edges of the form $u_i v_i$, $i = 0, 1, \ldots, 2k - 1$. Clearly, GP(2, 1) has chromatic index two and it is well known that the Petersen graph has chromatic index four. It is easy to see that GP(n, k) has a 1-factorization when it has a Hamiltonian cycle. This leaves only GP$(n, 2)$, $n \equiv 5 \pmod 6$ and $n \geqslant 11$, to consider. They are easy to do by observing that it suffices to find a 2-factor all of whose components are even length

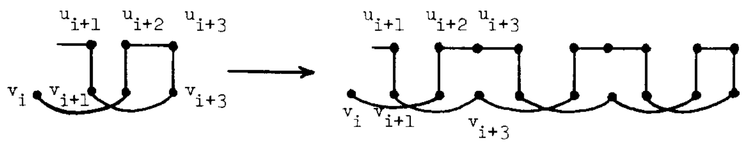

FIGURE 21

cycles. Such a 2-factor is easy to obtain by induction. The two cycles $u_0u_1v_1v_3u_3u_2v_2v_0u_0$ and $u_4u_5v_5v_7v_9u_9u_{10}v_{10}v_8u_8u_7u_6v_6v_4u_4$ form such a 2-factor in GP(11, 2). The induction step works by noticing that if an even length cycle contains a path of the form $v_iv_{i+2}u_{i+2}u_{i+3}v_{i+3}v_{i+1}u_{i+1}$, then 12 vertices can be incorporated to allow us to go from GP$(6m+5, 2)$ to GP$(6(m+1)+5, 2)$ and still have a 2-factor whose components are even length cycles. This is depicted in Fig. 21. The above 2-factor for GP(11, 2) has the required property. ∎

Klee [5] has asked a related question about what he calls H-prisms. Take two vertex disjoint n-cycles $u_0u_1u_2 \cdots u_{n-1}u_0$ and $v_0v_1v_2 \cdots v_{n-1}v_0$ together with some permutation σ of $\{0, 1,..., n-1\}$. Form a cubic graph $H(\sigma)$ by adjoining the edges $u_iv_{\sigma(i)}$, $i = 0, 1,..., n-1$. The question is to determine for which σ the graph $H(\sigma)$ is Hamiltonian. Bannai's result mentioned earlier answers this question for the permutations of the form $\sigma(i) = ai + b$ when $\gcd(a, n) = 1$ and $0 \leqslant b \leqslant n-1$. One could also ask the same question for various special 2-regular graphs other than n-cycles.

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