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Expanding endomorphisms of crystallographic manifolds

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Abstract

Let Γ be a crystallographic group with associated exact sequence $0 \to A \to \Gamma \to G \to 1$, and let M_Γ be the flat crystallographic manifold (i.e., the G-equivariant torus $\mathbb{R}^n/A$) associated to Γ. We construct a new crystallographic group Δ, a quotient of Γ/A_c, where A_c is the sum of all 1-dimensional G-submodules of A. Then we generalize the results of D. Epstein and M. Shub (1968) by showing the existence of equivariant endomorphisms of M_Γ which expand distances in certain directions transverse to the fibers of the map $M_\Gamma \to M_\Delta$. The existence of such expanding maps is of interest to the study of the K-theory as well as the controlled K-theory of Γ.

Keywords: Crystallographic; Expansive maps

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0. Preliminaries

We require some standard facts from group cohomology which may all be found in [1]. We will define and briefly review the properties of crystallographic groups (for details see [5,9]). A crystallographic group Γ, of rank n, is a discrete subgroup of $E(n) = \text{Trans}(n) \rtimes O(n)$, the group of rigid motions of $\mathbb{R}^n$, such that the quotient $E(n)/\Gamma$ is compact. Here Trans(n) denotes the translation group of $\mathbb{R}^n$. $A_\Gamma = \Gamma \cap \text{Trans}(n)$ is called the translation subgroup of Γ; Bieberbach's theorem (see [9, p. 100]) states that A_Γ is a normal free Abelian subgroup of Γ. Moreover A_Γ is equal to its own centralizer. $G_\Gamma = \Gamma/A_\Gamma$ is called the holonomy group of Γ; it is a finite group because Γ is discrete and $O(n)$ is compact. Also Γ acts on $O(n) \backslash E(n) = \mathbb{R}^n$. For any $x \in \mathbb{R}^n = \text{Trans}(n)$, the isotropy group of Γ is $\Gamma_x = \Gamma \cap (xO(n)x^{-1})$, a finite group.

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A crystallographic group Γ is uniquely determined by the resulting exact sequence

$$0 \to A_\Gamma \to \Gamma \to G_\Gamma \to 1.$$

The precise algebraic characterization of crystallographic groups is the following: Every group Γ which contains a normal free Abelian subgroup, which is equal to its own centralizer and has finite index, is isomorphic to a discrete subgroup of $E(n)$ which has compact quotient (see [2, §1.11]). The action of Γ on $\mathbb{R}^n$ is factored into two steps: A_Γ acts on $\mathbb{R}^n$ freely and the orbit space is a flat torus $\mathbb{R}^n/A_\Gamma$; the holonomy group G_Γ acts on $\mathbb{R}^n/A_\Gamma$ as a group of isometries such that $\mathbb{R}^n/\Gamma \simeq G\backslash\mathbb{R}^n/A_\Gamma$. $\mathbb{R}^n/A_\Gamma$ together with the G_Γ-action is called the crystallographic manifold associated to Γ and it is denoted by M_Γ. It should be noted here that the notion of the crystallographic manifold associated to Γ is understood, by several authors, in the following equivalent way: it is the pair $(\tilde{M}_\Gamma, \Gamma)$ where $\tilde{M}_\Gamma$ is the universal cover of M_Γ and Γ acts on $\tilde{M}_\Gamma$ as a cocompact discrete group of isometries.

We will denote by Aff(n) the group of affine transformations of $\mathbb{R}^n$. Recall that $\text{Aff}(n) = \text{Trans}(n) \rtimes \text{GL}_n(\mathbb{R})$. The following theorem was proved by Bieberbach and can be considered as the first rigidity result about crystallographic groups (for a proof see [9, p. 100]).

Theorem 0.1 (Bieberbach). *Every isomorphism $f : \Gamma \to \Delta$ between crystallographic groups of $E(n)$ is of the form $\gamma \to B\gamma B^{-1}$ for some $B \in \text{Aff}(n)$.*

The next lemma follows from Theorem 0.1. It is proved in Farrell and Hsiang [7, §1.2].

Lemma 0.2. *Let $j : \Gamma \to \Delta$ be an epimorphism between crystallographic groups $\Gamma \subset E(n)$, $\Delta \subset E(m)$. Then there exists a j-equivariant affine surjection $J : \mathbb{R}^n \to \mathbb{R}^m$. Moreover $j(A_\Gamma) \subseteq A_\Delta$, so that j induces an epimorphism $j_+ : G_\Gamma \to G_\Delta$.*

J is the composite of the projection $\mathbb{R}^n \to \mathbb{R}^m$ which has kernel $\langle \ker j \mid_{A_\Gamma} \rangle$ with an invertible affine map $\mathbb{R}^m \to \mathbb{R}^m$ (given by Theorem 0.1). The fiber of J is a translate of $\langle \ker j \mid_{A_\Gamma} \rangle$ (therefore connected), and if F_Γ, F_Δ denote the fundamental domains of $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, $J(F_\Gamma)$ is covered by the union of a finite number, say r, of translates of F_Δ. Here r is the index of $j(A_\Gamma)$ in A_Δ. Therefore, there exists a j_+-equivariant map $p : M_\Gamma \to M_\Delta$, with fiber homeomorphic to r copies of the torus T^{n-m}, where r is the order of $A_\Delta/j(A_\Gamma)$. In particular, if $j(A_\Gamma) = A_\Delta$ then the fiber of p is the torus T^{n-m}.

The rank of Γ is the rank of its translation subgroup A_Γ. For all $s \in \mathbb{Z}$, let Γ_s and A_s denote Γ/sA_Γ and A_Γ/sA_Γ respectively. Γ_s is again an extension of G_Γ by A_s, i.e., the sequence

$$A \to A_s \to \Gamma_s \to G_\Gamma \to 0$$

is exact. If s is prime to $|G_\Gamma|$, then $A_s \rtimes G_\Gamma \approx \Gamma_s$. This is because the cohomology group $H^2(G_\Gamma; A_s)$ vanishes (see [1]). We will write A_Γ and G_Γ without subscripts when it is clear to which crystallographic group we refer.

The following theorem was proved by Farrell and Hsiang (see [7]) and gives the structure of crystallographic groups.

Theorem 0.3. *Let Γ be a crystallographic group with holonomy group G. Then either*

(i) *$\Gamma = \Delta \rtimes \mathbb{Z}$ for some nontrivial crystallographic group Δ, with rank* $\mathrm{rk}(\Delta) = \mathrm{rk}(\Gamma) - 1$, *or*

(ii) *$\Gamma = B *_D C$ where B, C and D are crystallographic groups and D has index* 2 *in both B and C, or*

(iii) *there is an infinite sequence of positive integers s with* $s \equiv 1 \bmod(|G_\Gamma|)$, *such that any hyperelementary subgroup of Γ_s which maps onto G_Γ (via the natural map) is in fact isomorphic to G_Γ.*

Recall that a hyperelementary group is an extension of a p-group by a cyclic group such that $(n, p) = 1$. Here n denotes the order of the cyclic group. The property described in case (iii) of the above theorem will be called in short "hypothesis $\mathscr{H}$".

1. Construction

Let Γ be a crystallographic group with associated exact sequence

$$0 \to A_\Gamma \to \Gamma \to G_\Gamma \to 1.$$

If A_Γ admits an epimorphism to $\mathbb{Z}$, i.e., $H^1(A_\Gamma; \mathbb{Z}) \neq 0$, then Γ splits off a copy of $\mathbb{Z}$, which is in fact a trivial 1-dimensional G-submodule of A_Γ contained in the center of Γ. This is what happens in case (i) of Theorem 0.3. The following lemma shows that in case (ii) of Theorem 0.3 (where Γ admits a homomorphism $\Gamma \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \approx D_\infty$), such a submodule structure exists, except that the G-action is not trivial.

Let D denote the infinite dihedral group D_∞. This is a crystallographic group with translation subgroup $A_D = \mathbb{Z}$, and holonomy group $D/A_D \approx \mathrm{Aut}(\mathbb{Z}) \approx \{\pm 1\}$.

Lemma 1.1. *Let Γ be a crystallographic group admitting an epimorphism $f : \Gamma \to D$. Then*

(i) *$f(A_\Gamma) \subset A_D$,*

(ii) *if $\omega : G_\Gamma \to \mathrm{Aut}(\mathbb{Z})$ is the epimorphism $\Gamma/A_\Gamma \to D/A_D$ induced by f, then there exists a* 1*-dimensional G-submodule of A, on which the G-action is given by ω.*

Proof. First observe that the image of A_Γ under f lies inside the translation subgroup A_D. For if $a \in A_\Gamma$ and $f(a)$ is not in A_D for some a, then, by choosing

$b \in A_\Gamma$ such that $f(b) \in A_D$ and $f(b) \neq 0$, we would have that a and b commute, while $f(a)$ and $f(b)$ do not. Hence $f(A_\Gamma) \subset f(A_D)$.

Let $\omega : \Gamma/A_\Gamma \to D/A_D$ be the map induced by f. So f/A_Γ is a homomorphism $A_\Gamma \to A_D$, which is equivariant with respect to the induced homomorphism $\omega : G \to G_D \approx \mathrm{Aut}(A_D)$. Therefore, $\mathrm{Hom}_G(A_\Gamma, \mathbb{Z}^\omega) \neq 0$, where $\mathbb{Z}^\omega$ denotes A_D with the G-action induced by ω.

For any nontrivial homomorphism $h : A_\Gamma \to \mathbb{Z}^\omega$, after tensoring with $\mathbb{Q}$, we get a nontrivial homomorphism $h : A_\Gamma \otimes \mathbb{Q} \to \mathbb{Z}^\omega \otimes \mathbb{Q}$ of $\mathbb{Q}G$-modules. Since $\mathbb{Q}G$ is a semisimple ring, there exists a splitting $g : \mathbb{Z}^\omega \otimes \mathbb{Q} \to A_\Gamma \otimes \mathbb{Q}$. After replacing g by some multiple ng, $n \in \mathbb{N}$, if necessary, we may assume that $\mathbb{Z}^\omega$ is taken into A_Γ. Then we have that the composition

$$\mathbb{Z}^\omega \xrightarrow{ng} A_\Gamma \xrightarrow{f} \mathbb{Z}^\omega$$

is just multiplication by n. So A_Γ contains a copy of $\mathbb{Z}^\omega$. $\square$

Let C denote the image of $\mathrm{Hom}(\Gamma, D)$ in $\mathrm{Hom}(G, \{\pm 1\})$ under the map given by Lemma 1.1. For each $\omega \in C$ let $A^\omega = \{a \in A_\Gamma \mid g \cdot a = \omega(g) \cdot a \ \forall g \in G\}$. Note that for ω trivial, A^ω is just A_Γ^G, the center of Γ. Set $A_C = \Sigma_{\omega \in C} A^\omega$. By definition of A^ω, and since A_Γ is normal in Γ, it is obvious that A_C is normal in Γ.

Let $\hat{\Delta} = \Gamma/A_C$, $\hat{A} = A/A_C$ and $\hat{j} : \hat{\Delta} \to G$ be the map induced by $j : \Gamma \to G$. Let $p_1 : G \to \mathrm{Aut}(A_C)$ and $p_2 : G \to \mathrm{Aut}(A/A_C)$ be the natural maps. Let K denote the kernel of p_2. $p_1|_K$ is injective because the action of G on A_Γ is faithful. But

$$p_1(G) \subset \prod_{\omega \in C} \{\pm \mathrm{id}_{A^\omega}\} \subset \mathrm{Aut}(A_C).$$

Therefore K is an elementary Abelian 2-group.

Set $\hat{\Delta}_K = \hat{j}^{-1}(K)$. Then we have the following short exact sequence

$$0 \to \hat{A} \to \hat{\Delta}_K \to K \to 1 \tag{$*$}$$

where the action of K on $\hat{A}$ is trivial. We conclude that $\hat{\Delta}_K$ is a central extension of the elementary Abelian 2-group K by the free Abelian group $\hat{A}$.

Lemma 1.2. *$\hat{\Delta}_K$ is a finitely generated Abelian group.*

Proof. Let $\{x_k \mid k \in K\}$ be a set of representatives of K in $\hat{\Delta}_K$ so that $\hat{j}(x_k) = k$. Let $\chi(*)$ be the element of $H^2(K; \hat{A})$ which corresponds to exact sequence $(*)$ and let f be a representative of $\chi(*)$, i.e., f is a map $f : K \times K \to \hat{A}$ such that $f(k, n) + f(kn, m) = f(n, m) + f(k, nm)$.

Let F be the subgroup $(1/|K|) \cdot \hat{A}$ of $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{A}$. So we have that F is a free Abelian group isomorphic to $\hat{A}$ so that $\hat{A}$ can be identified with the image of the map $\times |K| : F \to F$. Extend the trivial action of K on $\hat{A}$ to a trivial action of K on F and define an extension of K by F by the rule $(ax_k)(bx_n) = [a + b + f(k, n)]x_{kn}$ for $a, b \in F$. So we get a group D and a short exact sequence

$$0 \to F \to D \to K \to 1 \tag{$**$}$$

such that, by construction, $\hat{\Delta}_K$ is isomorphic to a subgroup of D and the action of

K on F is trivial. Moreover, after normalizing by the map $\sigma: K \to F$ given by $\sigma(d) = (|K| \prod_{k \in K} f(d, k))^{-1}$, the cocycle f satisfies $f(k, n) = 1$ for all $k, n \in K$. This means that the element $\chi(**) \in H^2(K; F)$ is trivial. Since $(**)$ is also a central extension we conclude that $\hat{\Delta}_K$ can be viewed as a subgroup of $F \times K$. □

Let T denote the subgroup of $\hat{\Delta}_K$ which consists of all the elements of finite order in $\hat{\Delta}_K$. T is normal in $\hat{\Delta}$ because $\hat{\Delta}_K$ is normal in $\hat{\Delta}$.

Definition 1.3. Let Δ be the quotient group $\hat{\Delta}/T$.

We proceed to show that Δ is crystallographic and examine its properties. Set $A_\Delta = \hat{\Delta}_K/T$, a free Abelian group, and $G_\Delta = G/K$. Note that G_Δ acts faithfully on A_Δ because it does so on $\hat{A}$.

Proposition 1.4. *The group Δ, as defined in Definition* 1.3, *has the following properties*:

(i) *Δ is a crystallographic group.*

(ii) *The translation subgroup of Δ is $\hat{\Delta}_K/T$ and it is isomorphic to A/A_C.*

(iii) *The rank of Δ is* $\text{rk}(\Delta) = \text{rk}(\Gamma) - \text{rk}(A_C)$.

(iv) *The holonomy group G_Δ fits into an exact sequence* $1 \to K \to G \to G_\Delta \to 1$, *where K is an elementary Abelian* 2-*group.*

(v) *If Γ has odd-order holonomy, then $\Delta \approx \Gamma/A_\Gamma^G$ and $G \to G_\Delta$ is an isomorphism.*

(vi) *A_Δ does not contain any* 1-*dimensional G_Δ-submodules.*

Proof. The group Δ contains, by construction, a free Abelian subgroup, namely $\hat{\Delta}_K/T$. The group G/K acts by conjugation on $\hat{\Delta}_K$/T. Moreover, the action is faithful because K is the kernel of the G-action on A/A_C. Therefore an element $x \in \Delta$ acts trivially on A_Δ if and only if $x \in A_\Delta$. Hence A_Δ is equal to its own centralizer. It follows that Δ is a crystallographic group with translation subgroup $\hat{\Delta}_K/T$ and holonomy subgroup G/K.

If the order of the holonomy group G is odd, then all G-submodules A^ω are trivial, so that A_C is just the center of Γ. Moreover, G acts faithfully on A/A_C, i.e., K is trivial and hence A/A_C is crystallographic. □

In view of the fact that the groups $K_{-i}(\Gamma)$ and the controlled groups $K_{-i}(\Gamma)_c$ are computable from the class of hyperelementary subgroups of G (see [2,3]), we show that Δ satisfies "hypothesis $\mathscr{H}$" so that hyperelementary induction can be carried out over the group Δ.

Proposition 1.5. *The crystallographic group Δ, as constructed above, satisfies hypothesis $\mathscr{H}$ (for hypothesis $\mathscr{H}$ see Theorem* 0.3).

Proof. Combine Theorem 0.3, Lemma 1.1 and Proposition 1.4. □

2. Construction of expanding endomorphisms

If s is an integer such that $s \equiv 1 \bmod(|G|)$ then, an expansive map, for a crystallographic group Γ, is an endomorphism $f: \Gamma \to \Gamma$, which induces the following commutative diagram:

$$\begin{array}{ccccccccc} O & \longrightarrow & A_\Gamma & \longrightarrow & \Gamma & \longrightarrow & G_\Gamma & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \times s} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_G} & & \\ O & \longrightarrow & A_\Gamma & \longrightarrow & \Gamma & \longrightarrow & G_\Gamma & \longrightarrow & 1 \end{array}$$

The class of crystallographic groups enjoys the existence of a lot of expansive maps. Epstein and Shub showed in [4] that compact manifolds with a flat Riemannian metric admit expanding endomorphisms, i.e., Bieberbach [1] groups admit expansive maps. The same result is true for crystallographic groups, by a verbatim extension of their argument. We show here the existence of endomorphisms of Γ which are expanding only "in the Δ direction" and behave as the identity on the kernel of $\Gamma \to \Delta$.

Theorem 2.1. *For each integer s such that* $s \equiv 1 \bmod(|G|^2)$ *there exists a monomorphism* $f_s: \Gamma \to \Gamma$ *sending* A_Γ *to* A_Γ, *inducing multiplication by s on* A_Γ / A_C *and such that* $f_s|_{A_C} = \mathrm{id}|_{A_C}$. *Moreover, there exist an* f_s*-equivariant diffeomorphism* $\phi_s^\Gamma: \tilde{M}_\Gamma \to \tilde{M}_\Gamma$ *and a* g_s*-equivariant diffeomorphism* $\phi_s^\Delta: \tilde{M}_\Delta \to \tilde{M}_\Delta$ *such that the following diagram commutes*

$$\begin{array}{ccc} \tilde{M}_\Gamma & \xrightarrow{\phi_s^\Gamma} & \tilde{M}_\Gamma \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J} \\ \tilde{M}_\Delta & \xrightarrow{\phi_s^\Delta} & \tilde{M}_\Delta \end{array} \qquad (***)$$

and $\| D\phi_s^\Delta(x) \| = s \| x \|$ *for all* $x \in T\tilde{M}_\Delta$.

Here $\tilde{M}_\Gamma$ and $\tilde{M}_\Delta$ denote universal covers, and $g_s: \Delta \to \Delta$ is the map induced by f_s; this map is explained in the course of the proof. The map J is provided by Lemma 0.2.

Note that if A_C is trivial then the maps of the above proposition reduce to the expansive maps provided by Epstein and Shub. But if $A_C \neq 0$ then f_s expands distances in certain directions transverse to the fibers of the map $p: M_\Gamma \to M_\Delta$.

Proof. We will use cohomology arguments involving the Hochschild–Serre spectral sequence (for details see [8]).

For the group extension $0 \to A_\Gamma \to \Gamma \to G \to 0$ and the G-module A_Γ, consider the Hochschild–Serre spectral sequence converging to $H^{p+q}(\Gamma;\ A_\Gamma)$, where the

[1] A Bieberbach group is a torsion-free crystallographic group.

E_2-term is given by: $E_2^{p,q}=H^p(G;\, H^q(A_\Gamma;\, A_\Gamma))$. This gives rise to a five-term exact sequence of the form:

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(\Gamma;\, A_\Gamma) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2(\Gamma;\, A_\Gamma).$$

We have the following identifications:

$$E_2^{1,0}=H^1\big(G;\, H^0(A_\Gamma;\, A_\Gamma)\big)\approx H^1(G;\, A_\Gamma),$$

$$E_2^{0,1}=H^0\big(G;\, H^1(A_\Gamma;\, A_\Gamma)\big)\approx H^0(G;\, \mathrm{Hom}(A_\Gamma,\, A_\Gamma))\approx \mathrm{Hom}_G(A_\Gamma,\, A_\Gamma),$$

$E_2^{2,0}=H^2(G;\, A_\Gamma)$, and this group has exponent $|G|$.

Let I_A be the identity element in $\mathrm{Hom}_G(A_\Gamma,\, A_\Gamma)$ and let σ be the element

$$\sigma=\sum_{\omega\in C}\sigma_\omega\in \mathrm{Hom}_G(A_\Gamma,\, A_\Gamma), \quad \text{where } \sigma_\omega=\sum_{g\in G}\omega(g)^{-1}g.$$

For any integer $m\geqslant 0$ let $\psi=m(|G|I_A-\sigma)$. Then, since $|G|$ annihilates $H^2(G;\, A_\Gamma)$, $d_2(|G|\psi)=0$. By exactness, there exists a crossed homomorphism $\nu:\Gamma\to A_\Gamma$ representing an element of $H^1(\Gamma;\, A_\Gamma)$ such that $\nu|_{A_\Gamma}=|G|\psi$. Consider the homomorphism $f_s:\Gamma\to\Gamma$ defined by $f_s(\gamma)=\nu(\gamma)\cdot\gamma$, where $s=1+m|G|^2$ and m is any nonpositive integer. f_s is an endomorphism of Γ sending A_Γ to A_Γ and f_s induces $\mathrm{id}_G:\Gamma/A_\Gamma\to\Gamma/A_\Gamma$.

Now if ω, ω_0 are any elements of C and $a\in A^{\omega_0}$ then

$$\sigma_\omega(a)=\begin{cases}|G|a & \text{if } \omega=\omega_0,\\ 0 & \text{if } \omega\neq\omega_0.\end{cases}$$

Also $\sigma_\omega(A_\Gamma)\subset A^\omega$ for all $\omega\in C$. Therefore $\sigma(a)=|G|a$ for all $a\in A_C$ and $\sigma(A_\Gamma)\subset A_C$. Hence the homomorphism $(|G|I_A-\sigma)$ sends A_C to 0 and induces the map

$$\times|G|:\Gamma/A_\Gamma\to\Gamma/A_\Gamma.$$

So $f_s|_{A_\Gamma}:A_\Gamma\to A_\Gamma$ is a monomorphism, $f_s|_{A_C}=\mathrm{id}|_{A_c}$ and the map $A_\Gamma/A_C\to A_\Gamma/A_C$ induced by f_s is multiplication by s, where $s=1+m|G|^2$. Hence, f_s induces a map $\hat{g}_s:\Gamma/A_C=\hat{\Delta}\to\hat{\Delta}=\Gamma/A_C$. The induced map $G_\Gamma\to G_\Gamma$ is the identity and therefore, $\hat{g}_s$ takes $\hat{\Delta}_K$ into $\hat{\Delta}_K$ (cf. discussion before Definition 1.3). Hence $\hat{g}_s(T)=T$ since T is the torsion subgroup. Therefore, f_s induces a map $g_s:\hat{\Delta}/T\to\hat{\Delta}/T$. So g_s is an expanding endomorphism of Δ and $g_s(\Delta)\subset\Delta\subseteq E(n-k)$ is again a crystallographic group isomorphic to Δ. By Theorem 0.1, g_s is conjugation by an element $\phi_s^\Delta\in\mathrm{Aff}(n-k)=\mathbb{R}^{n-k}\times\mathrm{GL}_n(\mathbb{R})$. This affine transformation is actually a dilation because g_s is m_s (here m_s denotes multiplication by s). Moreover, $m_s^{-1}g_s$ fixes A_Δ because $\mathrm{Dil}(n-k)=Z_{\mathrm{Aff}(n-k)}(\mathrm{Trans}(n))$. Therefore, we get a map $\phi_s^\Delta:\mathbb{R}^{n-k}\to\mathbb{R}^{n-k}$ which is g_s-equivariant and the differential is an isometry up to a constant. Again by Theorem 0.1 (Bieberbach), $f_s:\Gamma\to\Gamma$ gives rise to an f_s-equivariant diffeomorphism $\phi_s^\Gamma:\tilde{M}_\Gamma\to\tilde{M}_\Gamma$.

Next we describe the map $J:\tilde{M}_\Gamma\to\tilde{M}_\Gamma$ (c.f. [7, §1.2]), in order to establish commutativity of diagram $(***)$. Identify $\mathrm{Trans}(n)$ with $\mathbb{R}^n$ and G_Γ with its image in $O(n)$, so that $\Gamma \subseteq \mathbb{R}^n \rtimes G_\Gamma$ and $\Delta \subseteq \mathbb{R}^{n-k} \rtimes G_\Delta$. Let W be the subspace of $\mathbb{R}^n$ spanned by A_C and $W^\perp$ its orthogonal complement. Identify $W^\perp$ with $\mathbb{R}^{n-k}$ by an isometry and let $J_1 : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be the orthogonal projection with ker $J_1 = W$. Let G' be the image of G in $O(n-k)$. Extend J_1, in the canonical way, to a group homomorphism $\mathbb{R}^n \rtimes G_\Gamma \to \mathbb{R}^{n-k} \rtimes G' \subseteq E(n-k)$, also denoted by J_1. It follows that $J_1(\Gamma)$ is a crystallographic group isomorphic to Δ (note that ker j = ker θ where j is the map $\Gamma \to \Delta$ and $\theta = J_1 |_\Gamma$). By Theorem 0.1 (Bieberbach) there exists an invertible, θ-equivariant, affine map $J_2 : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$. The map J is defined as the composite $J_2 \circ J_1$, and so J collapses W to a point. The effect of ϕ_s^Γ on $W^\perp$ is the same, after the identification $W^\perp \approx \mathbb{R}^{n-k}$, as the effect of ϕ_s^Δ on $\mathbb{R}^{n-k}$ (note that g_s and the map $A_\Gamma/A_C \to A_\Gamma/A_C$ induced by f_s are multiplication by s). Moreover, since $f_s |_{A_C} = \mathrm{id} |_{A_C}$, ϕ_s^Γ fixes W, up to a translation. Therefore, since J collapses W to a point, diagram $(***)$ is commutative. □

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