Moments of Dyck paths

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1. Introduction

It is well-known that the Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \) enumerate the Dyck paths of length \( 2n \). These are the paths from \((0,0)\) to \((n,n)\) in the plane having steps \((1,0)\) and \((0,1)\) and lying on or above the main diagonal. By rotating and scaling, instead we can, and shall, consider the paths from \((0,0)\) to \((2n,0)\) with steps \((1,1)\) and \((1,-1)\) which lie on or above the x-axis. We say a Dyck path is strict if its only points on the x-axis are its endpoints. Given a Dyck path one can define its area as the area of the region enclosed by it and the x-axis. The following results are known:

**Theorem 1** (Merlini et al. [3]). The sum of the areas of the Dyck paths of length \( 2n \) is

\[
4^n - \frac{1}{2} \binom{2n+2}{n+1}.
\]

**Corollary 1** (Shapiro et al. [4]). The sum of the areas of the strict Dyck paths of length \( 2n \) is \( 4^{n-1} \).

Effectively these results give information about the mean distance of vertices on Dyck paths from the x-axis averaged out over all Dyck paths or strict Dyck paths of a given length. One might ask for more information about the distribution of these distances. Here we consider the moments of these distances, again averaged over all Dyck paths or strict Dyck paths of a given length.

There are other related results in the literature. Goulden and Jackson ([2] example 5.2.12) given a generating function for Dyck paths with a fixed area. Also Shapiro et al. [5] give moment formulas for generalized Catalan numbers \( B_{n,k} = (k/n) \binom{2n}{n+k} \). Deutsch [1] gives generating functions for many statistics of Dyck paths, from which moment formulae can be deduced, but his examples concentrate on enumerating com-

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binatorial features of Dyck paths such as peaks and valleys, whereas here we look at numerically defined statistics.

2. Dyck paths

Each Dyck path has vertices \((0, a_0), (1, a_1), (2, a_2), \ldots, (2n, a_{2n})\) where \(a_0 = a_{2n} = 0\), \(a_j \geq 0\) for all \(j\), and \(a_j = a_{j-1} \pm 1\) for \(1 \leq j \leq 2n\). We identify a Dyck path with the corresponding sequence \((a_0, a_1, \ldots, a_{2n})\). Let \(P_n\) be the set of Dyck paths of length \(2n\), and let \(P_n\), for \(n \geq 1\), be the set of strict Dyck paths of length \(2n\). Then \(|P_n| = |P_{n+1}| = C_n\). For a Dyck path \(a = (a_0, a_1, \ldots, a_{2n})\), let \(\sigma(a) = (a_1 - a_0, a_2 - a_1, \ldots, a_{2n} - a_{2n-1}) \in \{\pm 1\}^{2n}\). Clearly \(\sigma\) is injective on the set of Dyck paths, and put \(P_n^\sigma = \sigma(P_n)\) and \(P_n^\sigma = \sigma(P_n)\).

Our aim is to consider the distribution of the \(a_j\)s as \(a\) runs through either \(P_n\) or \(P_n\). To this end consider a real-valued function \(\varphi\) defined on the non-negative integers. For \(a \in P_n\) let \(c_\varphi(a) = \sum_{j=0}^{2n} \varphi(a_j)\) where \(a = (a_0, a_1, \ldots, a_{2n})\). If \(\varphi(n) = n\) Theorem 1 and Corollary 1 give

\[
\sum_{a \in P_n} c_\varphi(a) = 4^n - \frac{1}{2} \left( \frac{2n + 2}{n + 1} \right)
\]

and

\[
\sum_{a \in P_n^\sigma} c_\varphi(a) = 4^{n-1}.
\]

We wish to calculate these sums for more general functions \(\varphi\). Of particular interest are the moments of the Dyck paths; these are the above sums with \(\varphi(x) = x^r\) for positive integers \(r\). Given \(n\) we can define a random variable \(A = A_n^\varphi\) (respectively, \(A_n^\sigma\)) as follows; set \(A = a_j\) where \(j\) is selected from \(\{0, 1, \ldots, 2n\}\) with equal probability and \(a\) is independently selected from \(P_n\) (respectively, \(P_n^\sigma\)) with equal probability. The moments of these random variables are related simply to those of the Dyck paths, and so the mean and variance of \(A_n^\varphi\) and \(A_n^\sigma\) can be calculated.

We introduce generating functions. Let

\[
F_\varphi(X) = \sum_{n=0}^{\infty} \sum_{a \in P_n} c_\varphi(a) X^n
\]

and

\[
G_\varphi(X) = \sum_{n=0}^{\infty} \sum_{a \in P_n^\sigma} c_\varphi(a) X^n.
\]

**Lemma 1.** Suppose that \(\varphi(0) = 0\). Then

\[
F_\varphi(X) = \frac{G_\varphi(X)}{(1 - X C(X))^2} = C(X)^2 G_\varphi(X),
\]
where
\[ C(X) = \sum_{n=0}^{\infty} C_n X^n = \frac{1 - \sqrt{1 - 4X}}{2X} = 1 + XC(X)^2 \]
is the generating function of the Catalan numbers.

**Proof.** For each \( \mathbf{a} \in \mathcal{P}_n \), \( \sigma(\mathbf{a}) \) is obtained, in a unique fashion, by concatenating elements of the \( \mathcal{P}_r \)'s for \( r \leq n \) \( \mathcal{P}_n \) consisting of those \( \mathbf{a} \) where \( \sigma(\mathbf{a}) \) is the concatenation of \( m \) elements of \( \bigcup_{r=1}^{n} \mathcal{P}_r \). Then \( \mathcal{P}_n = \mathcal{P}_n^{(1)} \). Also \( F_\phi(X) = \sum_{m=0}^{\infty} F_\phi^{(m)}(X) \), where
\[ F_\phi^{(m)}(X) = \sum_{n=0}^{\infty} \sum_{\mathbf{a} \in \mathcal{P}_n^{(m)}} c_\phi(\mathbf{a}) X^n. \]
I claim that \( F_\phi^{(m)}(X) = m(XC(X))^{m-1} G_\phi(X) \). If \( \mathbf{a} \in \mathcal{P}_n^{(m)} \) then \( \sigma(\mathbf{a}) \) is the concatenation of \( \sigma(b_1), \sigma(b_2), \ldots, \sigma(b_n) \), in that order, where each \( b_j \in \bigcup_{r=1}^{n} \mathcal{P}_r \). As \( \phi(0) = 0 \) then \( c_\phi(\mathbf{a}) = \sum_{j=1}^{m} c_\phi(b_j) \). The number of \( \mathbf{a} \in \mathcal{P}_n \) where \( b_1 \) has a fixed value \( b_0 \in \mathcal{P}_r \) is the coefficient of \( X^{n-r} \) in \( X^{m-1} C(X)^{m-1} \), as this is the generating function for the number of Dyck paths obtained by adjoining \( m-1 \) strict Dyck paths. Hence
\[ \sum_{n=0}^{\infty} X^n \sum_{\mathbf{a} \in \mathcal{P}_n^{(m)}} c_\phi(\mathbf{b}_1) = G_\phi(X) X^{m-1} C(X)^{m-1}. \]
All the similar sums with \( b_j \) replacing \( b_1 \) are equal, and so \( F_\phi^{(m)}(X) = mG_\phi(X)X^{m-1} \times C(X)^{m-1} \). Adding over all \( m \) gives
\[ F_\phi(X) = \frac{G_\phi(X)}{(1 - XC(X))^2}, \]
and noting that \( C(X) = 1/(1 - XC(X)) \) completes the proof. \( \Box \)

**Theorem 2.** If \( \phi(0) = 0 \) and \( \psi \) is defined by \( \psi(n) = \sum_{j=1}^{n} \phi(j) \) then
\[ G_\psi(X) = \frac{G_\phi(X)}{2} \left( 1 + \frac{1}{\sqrt{1 - 4X}} \right). \]

**Proof.** Note that each \( \mathbf{a} \in \mathcal{P}_n \) has the form \( \mathbf{a} = (0, 1 + b_0, 1 + b_1, \ldots, 1 + b_{2n-2}, 0) \) where \( \mathbf{b} = (b_0, b_1, \ldots, b_{2n-2}) \in \mathcal{P}_{n-1} \). Conversely, each \( \mathbf{b} \in \mathcal{P}_{n-1} \) gives rise to such a \( \mathbf{a} \in \mathcal{P}_n \). With this notation
\[ c_\phi(\mathbf{a}) = \sum_{j=0}^{2n-2} \psi(1 + b_j) = \sum_{j=0}^{2n-2} \psi(b_j) + \sum_{j=0}^{2n-2} \phi(1 + b_j) = c_\psi(\mathbf{b}) + c_\phi(\mathbf{a}). \]
On summing, we get \( G_\phi(X) = XF_\phi(X) + G_\phi(X) \). From Lemma 1 we get \( F_\phi(X) = C(X)^2 G_\phi(X) \), and the equation
\[ G_\psi(X) = \frac{G_\phi(X)}{2} \left( 1 + \frac{1}{\sqrt{1 - 4X}} \right) \]
readily follows. \( \square \)
3. Examples

For completeness we reprove Theorem 1 and Corollary 1. If we define \( \varphi(0) = 0 \) and \( \varphi(x) = 1 \) for \( x > 0 \), then, in the notation of Theorem 2, \( \psi(x) = x \). We have

\[
G_{\varphi}(X) = \sum_{n=1}^{\infty} (2n-1) C_n X^n = \sum_{n=1}^{\infty} \frac{1}{2} \binom{2n}{n} X^n = \frac{1}{2} \left( \frac{X}{\sqrt{1-4X}} - 1 \right).
\]

By Theorem 2 then

\[
G_{\varphi}(X) = \frac{1}{4} \left( 1 - \frac{1}{1-4X} \right) = \frac{X}{1-4X}
\]
and then by Lemma 1

\[
F_{\varphi}(X) = \frac{1 - 2X - \sqrt{1 - 4X}}{2X(1 - 4X)}.
\]

Calculating the coefficients of \( X^n \) in these equations gives Theorem 1 and Corollary 1.

With the notation of Theorem 2 if

\[
\varphi(x) = \begin{pmatrix} x + j - 1 \\ j \end{pmatrix}, \quad \text{then} \quad \psi(x) = \begin{pmatrix} x + j \\ j + 1 \end{pmatrix}.
\]

Set \( \varphi_j(x) = \binom{x+j-1}{j} \). Then by induction on \( j \), Theorem 2 immediately gives

\[
G_{\varphi_j}(X) = \frac{X(1 + \sqrt{1 - 4X})^{j-1}}{2^{j-1}(1 - 4X)^{j+1/2}}.
\]

For small values of \( j \) we can read off the coefficients, and obtain identities for \( \sum_{a \in \mathbb{Z}_+} c_{\varphi}(a) \). For \( \varphi_2(x) = \binom{x+1}{2} = x(x+1)/2 \) we get

\[
\sum_{a \in \mathbb{Z}_+} c_{\varphi_2}(a) = \frac{n}{2} \binom{2n}{n} + \frac{4^{n-1}}{2}.
\]

For \( \varphi_3(x) = \binom{x+2}{3} = (x+1)(x+2)/6 \) we get

\[
\sum_{a \in \mathbb{Z}_+} c_{\varphi_3}(a) = \frac{n}{4} \binom{2n}{n} + (1+n)4^{n-2}.
\]

Taking linear combinations of the above identities produces more, including examples of moments: if \( \varphi(x) = x^2 = 2\varphi_2(x) - \varphi_1(x) \) then

\[
\sum_{a \in \mathbb{Z}_+} c_{\varphi}(a) = \frac{n}{2} \binom{2n}{n}.
\]
if \( \varphi(x) = x^3 = 6\varphi_3(x) - 6\varphi_2(x) + \varphi_1(x) \) then

\[
\sum_{a \in \mathbb{Z}_+} c_{\varphi}(a) = \frac{3n-1}{2} 4^{n-1}.
\]
and if \( \varphi(x) = x(2x^2 + 1)/3 = 4\varphi_3(x) − 4\varphi_2(x) + \varphi_1(x) \) then
\[
\sum_{a \in A_n} c_\varphi(a) = n4^{n-1}.
\]

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**References**