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A bilinear extension inequality in two dimensions

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Dedicated to the memory of Terence James Carbery, 1916–2001

Abstract

We provide sharp decay estimates for circular averages of a certain bilinear extension operator on $L^2(\mathbb{S}^1) \times L^2(\mathbb{S}^1)$.

Keywords: Bilinear estimates; Fourier extension operators

1. Introduction

In this paper we establish a certain bilinear estimate for circular averages of the extension operator for the Fourier transform on $\mathbb{S}^1 \subset \mathbb{R}^2$. For $f \in L^2(\mathbb{S}^1)$ let $\tilde{f}(x) = f(-x)$, (so that $f \mapsto \tilde{f}$ represents translation by π when $\mathbb{S}^1$ is thought of as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$); and let the extension operator be given by

$$\widehat{f d\sigma}(x) = \int_{\mathbb{S}^1} e^{-ix\cdot y} f(y)\, d\sigma(y)$$

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for $x \in \mathbb{R}^2$. Here and throughout, $d\sigma$ denotes surface measure on $\mathbb{S}^1$ or, in this introduction only, $\mathbb{S}^{n-1}$, according to context.

Our principal result is:

Theorem 1. *If* $f, g \in L^2(\mathbb{S}^1)$ *with*

$$\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g))$$

and

$$\operatorname{dist}(\operatorname{supp}(\tilde{f}), \operatorname{supp}(g)),$$

bounded below, *then*

$$\int_{\mathbb{S}^1} |\widehat{f d\sigma}(Rx)\widehat{g\, d\sigma}(Rx)|\ d\sigma(x) \leqslant \frac{C}{R^{5/6}} ||f||_2 ||g||_2$$

for all $R > 0$.

Remark. (i) The constant C that appears above depends only upon the lower bound in the support hypothesis.

(ii) Since $\tilde{f}(x) = f(-x)$, the second support condition dictates that the supports of f and g cannot be 'diametrically opposite'.

(iii) The decay rate $R^{-5/6}$ is optimal, as we shall show with some examples in Section 6 below.

Before passing to the proof of Theorem 1, it would seem appropriate to try to place the result in context. In the first instance there is a corresponding linear estimate (which is equivalent to a bilinear estimate without support restrictions). It is the following:

Linear estimate: For $f \in L^2(\mathbb{S}^1)$,

$$\int_{\mathbb{S}^1} |\widehat{f d\sigma}(Rx)|^2\, d\sigma(x) \leqslant \frac{C}{R^{2/3}} ||f||_2^2.$$

Thus, the 'gain' of $R^{1/6}$ in the decay rate of Theorem 1 is attributable to the hypothesis of separation of the supports of f, g.

This linear estimate is a special case of an n-dimensional result:

Linear Estimate (*n-dimensions*): For $f \in L^2(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} |\widehat{f d\sigma}(Rx)|^2\, d\sigma(x) \leqslant \frac{C}{R^{n-1}} R^{1/3} ||f||_2^2.$$

This result was a principal ingredient used by Barceló et al. [2] in their study of radial weighted estimates for solutions to the Helmholtz equation, and, independently, by Carbery and Soria [4] in their study of localisation problems arising in multiple Fourier inversion.

Both these works raised implicitly (see also [5] for further discussion) the possibility that one might consider whether the following inequality holds for arbitrary positive measures μ supported in the unit ball $\mathbb{B}$ of $\mathbb{R}^n$, and $f \in L^2(\mathbb{S}^{n-1})$:

$$\int_{\mathbb{B}} |\widehat{f d\sigma}(Rx)|^2 \, d\mu(x) \leqslant \frac{C}{R^{n-1}} |||\mu|||_R \|f\|_2^2,$$

where

$$|||\mu|||_R = \sup_{T; R^{-1} \leqslant \alpha \leqslant R^{-1/2}} \frac{\mu(T(\alpha, \alpha^2 R))}{\alpha^{n-1}}$$

and $T(\alpha, \beta)$ denotes a tube in $\mathbb{R}^n$ with $n-1$ short sides α and one long side β. This inequality is known to be true (perhaps with some extra logarithmic factors) when $d\mu(x) = w(x)\,dx$ and the weight w is radial [2,4]. In any case, when $d\mu = d\sigma$, it is easily seen that $|||\sigma|||_R$ is realised at $R^{-2/3}$ by $R^{-1/3}$ tubes tangential to $\mathbb{S}^{n-1}$, and that it takes the value approximately $R^{1/3}$, thereby explaining the form of the linear estimate above.

Thus, our Theorem 1 may be seen as a first step in understanding the general two-dimensional bilinear inequality

$$\int_{\mathbb{B}} |\widehat{f d\sigma}(Rx)\widehat{g\, d\sigma}(Rx)| \, d\mu(x) \leqslant \frac{C_\mu}{R^2} \|f\|_2 \|g\|_2$$

under the support conditions of Theorem 1.

Finally, we note that inequalities of this kind, in either their linear or bilinear settings, are likely to prove useful in a variety of problems. (One needs only to point to [8] in the recent literature concerning bilinear extension estimates.)

2. A preliminary reduction

We shall concentrate on proving the following inequality, equivalent to Theorem 1. If $\psi \in L^\infty(\mathbb{S}^1)$ and f and g have separated supports, as in the statement of Theorem 1, then there exists an absolute constant C such that

$$\left| \int_{\mathbb{S}^1} \widehat{f d\sigma}(Rx) \overline{\widehat{g\, d\sigma}(Rx)} \psi(x) \, d\sigma(x) \right| \leqslant \frac{C}{R^{5/6}} \|\psi\|_\infty \|f\|_2 \|g\|_2 \tag{1}$$

for all $R > 0$.

The support properties of f and g imply that $f\,d\sigma * (g\,d\sigma)^*$ is supported in a closed annulus A centred at the origin and contained in $\{x \in \mathbb{R}^2 : 0 < |x| < 2\}$. (Here, $h^*(x) = \overline{h(-x)}$.) If $Q \in \mathscr{S}(\mathbb{R}^2)$ is real valued, radial, and such that

$\mathrm{supp}(\hat{Q}) \subset \{\xi \in \mathbb{R}^2 : 0 < |\xi| < 2\}$, and $\hat{Q}(\xi) = 1$ on A, then

$$\begin{aligned}\int_{\mathbb{S}^1} \widehat{f\,d\sigma}(Rx)\overline{\widehat{g\,d\sigma}(Rx)}\psi(x)\,d\sigma(x) &= \int_{\mathbb{S}^1} (f\,d\sigma * (g\,d\sigma)^*)^\wedge (Rx)\psi(x)\,d\sigma(x) \\ &= \int_{\mathbb{R}^2} f\,d\sigma * (g\,d\sigma)^*(x)\hat{Q}(x)\widehat{\psi\,d\sigma}(Rx)\,dx \\ &= \int_{\mathbb{R}^2} \widehat{f\,d\sigma}(Rx)\overline{\widehat{g\,d\sigma}(Rx)}Q_{1/R} * \psi\,d\sigma(x)\,dx,\end{aligned}$$

where $Q_{1/R}(x) = R^2 Q(Rx)$. Hence, it suffices to show that

$$\left| \int_{\mathbb{R}^2} \widehat{f\,d\sigma}(Rx)\overline{\widehat{g\,d\sigma}(Rx)}Q_{1/R} * \psi\,d\sigma(x)\,dx \right| \leqslant \frac{C}{R^{5/6}} ||\psi||_\infty ||f||_2 ||g||_2 \tag{2}$$

for all $f, g \in L^2(\mathbb{S}^1)$. Note that the introduction of the function Q has allowed us to work with arbitrary functions $f, g \in L^2(\mathbb{S}^1)$.

If we write f and g as Fourier series,

$$f(x) = \sum_{j \in \mathbb{Z}} \alpha_j e^{ij \arg x}$$

and

$$g(x) = \sum_{k \in \mathbb{Z}} \beta_k e^{ik \arg x},$$

we obtain

$$\widehat{f\,d\sigma}(Rx) = \sum_j \alpha_j J_j(R|x|) e^{ij \arg x}$$

and

$$\widehat{g\,d\sigma}(Rx) = \sum_k \beta_k J_k(R|x|) e^{ik \arg x},$$

where J_n, given by

$$J_n(t) = \int_0^{2\pi} e^{i(n\theta - t\cos\theta)}\,d\theta,$$

denotes the Bessel function of order n. Similarly, if

$$\psi(x) = \sum_{m \in \mathbb{Z}} c_m e^{im \arg x},$$

then

$$\begin{aligned} Q_{1/R}*\psi\, d\sigma(x) &= \sum_m c_m \int_{\mathbb{S}^1} Q_{1/R}(x-y)e^{im \arg y}\, d\sigma(y) \\ &= \sum_m c_m \int_0^{2\pi} R^2 Q(R(te^{i\phi}-e^{i\theta}))e^{im\theta}\, d\theta (x=te^{i\phi}) \\ &= \sum_m c_m \int_0^{2\pi} R^2 Q(R(te^{i(\phi-\theta)}-1))e^{im\theta}\, d\theta \\ &= \sum_m c_m e^{im\phi} \int_0^{2\pi} R^2 Q(R(te^{i\theta}-1))e^{-im\theta}\, d\theta \\ &= \sum_m c_m e^{im\phi} F_m(t,R), \end{aligned}$$

where

$$F_m(t,R) = \int_0^{2\pi} R^2 Q(R(te^{i\theta}-1))e^{-im\theta}\, d\theta. \tag{3}$$

(Here we are identifying $\mathbb{R}^2$ with $\mathbb{C}$ merely for convenience.) Consequently,

$$\begin{aligned} &\int_{\mathbb{R}^2} \widehat{f\, d\sigma}(Rx)\overline{\widehat{g\, d\sigma}(Rx)} Q_{1/R}*\psi\, d\sigma(x)\, dx \\ &= \int_{\mathbb{R}^2} \sum_{j,k,m} c_m \alpha_j J_j(R|x|)\overline{\beta_k J_k(R|x|)} e^{i(j-k+m)\arg x} F_m(|x|,R)\, dx \\ &= 2\pi \int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)} c_{k-j} F_{k-j}(s,R) s\, ds. \end{aligned}$$

We shall prove Theorem 1 by showing that

$$\begin{aligned} &\left| \int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)} c_{k-j} F_{k-j}(s,R) s\, ds \right| \\ &\quad \leqslant \frac{C}{R^{5/6}} ||\psi||_\infty ||\alpha||_{l^2(\mathbb{Z})} ||\beta||_{l^2(\mathbb{Z})} \end{aligned} \tag{4}$$

for all $\alpha, \beta \in l^2(\mathbb{Z})$.

Due to the fact that for each $k \in \mathbb{Z}$, $J_{-k} = J_k$, we will split our analysis of the above into two parts. Part 1 will correspond to summing over $j,k \geqslant 0$, and Part 2 to summing over $j \leqslant 0$ and $k \geqslant 0$. The remaining terms can be understood by symmetry.

Notation. For $X, Y \geqslant 0$, we say that $X \sim Y$ if X lies between two positive absolute constant multiples of Y. The constants may change from line to line, but remain absolute.

3. An overview of the proof

As noted above, we shall prove Theorem 1 by proving inequality (4). In order to understand the left-hand side of (4) we need some estimates on Bessel functions and on the functions F_m. We begin with the Bessel function estimates.

3.1. Behaviour of J_n

Lemma 2. *There exists an absolute constant c such that for each* $k \geqslant 0$,

$$|J_k(s)| \leqslant cs^{-1/2} \min\left\{k^{1/6}, \left|\frac{|s|+k}{|s|-k}\right|^{1/4}\right\}$$

(*so that in particular*, $|J_k(s)| \leqslant cs^{-1/3}$), *and furthermore*,

$$|J_k'(s)| \leqslant cs^{-1/2}.$$

The above lemma can be found for example in [6], and can be used to prove the following:

Lemma 3. *For each* $t \geqslant 1$,

$$\sup_{j,k \in \mathbb{N}; |j-k| \sim t} |J_j(s)J_k(s)|, \quad \sup_{j,k \in \mathbb{N}; |j+k-2s| \sim t} |J_j(s)J_k(s)| \leqslant cs^{-5/6} \begin{cases} 1, & 0 \leqslant s < 3t, \\ (s/t)^{1/4}, & 3t \leqslant s < t^3, \\ s^{1/6}, & s \geqslant t^3. \end{cases}$$

Proof. We sketch the proof of the first estimate only. Firstly, we note that it suffices to establish this estimate with $\sup_{j,k \in \mathbb{N}; |j-k| \sim t}$ replaced by $\sup_{j,k \in \mathbb{N}; j-k=t}$. This is a consequence of a certain scale invariance (in t) of the claimed bound, and the fact that symmetry allows us to suppose that $j \geqslant k$. Hence, it suffices to show that

$$\sup_j |J_j(s)J_{j-t}(s)| \leqslant cs^{-5/6} \begin{cases} 1, & 0 \leqslant s < 3t, \\ (s/t)^{1/4}, & 3t \leqslant s < t^3, \\ s^{1/6}, & s \geqslant t^3. \end{cases}$$

This estimate follows by applying Lemma 2 to the product $|J_j(s)J_{j-t}(s)|$ for each $j \in \mathbb{N}$. $\square$

3.2. Behaviour of F_m

Recall that F_m is defined by (3), where the function Q is smooth, radial, and has $\hat{Q}$ compactly supported in $\{\xi \in \mathbb{R}^2 : 0 < |\xi| < 2\}$. If we pretend temporarily—in violation of the uncertainty principle—that Q itself also has compact support in $\{x \in \mathbb{R}^2 : |x| \leqslant 1\}$, the following properties of F_m are easy to establish:

(i) F_m is essentially supported in $\{|t-1| \leqslant \frac{1}{R}\}$.
(ii) $|F_m(t,R)| \leqslant CR$.
(iii) $|F_m(r,R) - F_{m'}(r,R)| \leqslant C|m-m'|$.
(iv) $\int_0^\infty F_0(r,R) r\, dr = 0$.
(v) $|\int_0^\infty F_{2R}(r,R) r\, dr| \leqslant C/R$, (since $\hat{Q}$ vanishes on $\{|\xi| = 2\}$).

A more rigorous analysis of the functions F_m is contained in the detailed proofs of Sections 4 and 5. For now we only wish to comment that the vanishing of $\hat{Q}$ at the origin is needed for Part 1 terms while the vanishing of $\hat{Q}$ on $\{|\xi| = 2\}$ is needed for Part 2 terms. In particular, see Section 5, analysis of term III$'$, for further details of estimate (v).

3.3. Strategy of the proof—Part 1 terms

The fact that $\hat{Q}$ has compact support means that we should not expect to see any structure in ψ on a scale finer than $1/R$. Thus, we may assume that the Fourier frequencies of ψ of order greater than $2R$ are negligible in comparison with those of order less than $2R$. Therefore, in examining (4) (with $j,k \geqslant 0$) we may assume that the principal contribution arises when $|k-j| \leqslant 2R$, and it is then reasonable to decompose the (j,k) sum into regions where $|k-j| \sim 2^{-p}R$, $1 \leqslant 2^p \leqslant R$. At the expense of incurring at most an extra logarithmic term we may treat each p separately. For each such p, the bilinear form is now 'local' on scale $2^{-p}R$, and so we may assume that for certain j_0, k_0 with $|k_0 - j_0| \sim 2^{-p}R$, the $\{\alpha_j\}$ and $\{\beta_k\}$ are supported in $|j - j_0| \leqslant 2^{-p-2}R$ and $|k - k_0| \leqslant 2^{-p-2}R$, respectively.

For p, j_0, k_0, j and k as above, we may estimate

$$\left| \int_0^\infty J_j(Rs) J_k(Rs) F_{k-j}(s,R) s\, ds \right| \leqslant \left| \int_0^\infty J_j(Rs) J_k(Rs) F_0(s,R) s\, ds \right| + \left| \int_0^\infty J_j(Rs) J_k(Rs) [F_{k-j} - F_0](s,R) s\, ds \right|.$$

For the first term we can use property (iv) of F_0 to allow us to integrate by parts, and then properties (i) and (ii) of F_0 and the relevant Bessel function estimates to control the resulting terms by $O(R^{-5/6})$. For the second term we use property (iii) of F_m to obtain a similar bound.

Of course, it is not merely size of $\int_0^\infty J_j(Rs)J_k(Rs)F_{k-j}(s,R)s\,ds$ which determines the behaviour of the quadratic form in (4); but supposing it were only size that mattered, we would now be finished because, by Plancherel's theorem, $||\sum \alpha_j\overline{\beta_k}c_{k-j}||\leqslant||\psi||_\infty||\alpha||_{l^2(\mathbb{Z})}||\beta||_{l^2(\mathbb{Z})}$.

3.4. Strategy of the proof—Part 2 terms

For Part 2 we need to examine

$$\sum_{j,k\geqslant 0;\ k+j\leqslant 2R}\tilde{\alpha}_j\overline{\beta_k}c_{k+j}\int_0^\infty J_j(Rs)J_k(Rs)F_{k+j}(s,R)s\,ds.$$

where $\tilde{\alpha}_j=\alpha_{-j}$ and it is now more natural to break the (j,k) sum into regions where $|2R-(j+k)|\sim 2^{-p}R,\ 1\leqslant 2^p\leqslant R$. We now estimate the integrated term by adding and subtracting $F_{2R}(s,R)$ (instead of F_0 as in Part 1) and proceed similarly using property (v) of F_{2R} to once again integrate by parts and obtain a suitable estimate.

For technical reasons, the formal proof below in Sections 4 and 5 proceeds along lines slightly different from those described here. Nevertheless, it is hoped that these remarks will provide a useful guide for the reader in following the arguments of the next two sections.

4. The proof of Theorem 1: Part 1

In this section we consider the contribution arising from the indices $j,k\geqslant 0$.

We first set up some further notation. For a 2π-periodic function v on $\mathbb{R}$ we denote by $\hat{v}(n)$ its nth Fourier coefficient. For $N\in\mathbb{N}$, let Φ_N be the Nth Fejér kernel on $\mathbb{R}/\mathbb{Z}=\mathbb{T}$ and let

$$V_N=2\Phi_{2N+1}-\Phi_N$$

be the Nth de la Vallée–Poussin kernel. For $l\in\mathbb{N}$ let

$$W_l=V_{2^{l+1}}-V_{2^l}.$$

We will need the following well-known elementary lemma.

Lemma 4.

$$||\Phi_N||_1=1$$

and there exists an absolute constant c such that

$$\int_0^{2\pi} |\Phi_N(\theta-\phi) - \Phi_N(\theta)|\, d\theta \leqslant cN|\phi|$$

for all $N \in \mathbb{N}$.

Let l_0 and l_1 be the smallest integers for which $2^{l_0} \geqslant R$, and $2^{l_1} \geqslant (2R)^{1/3}$. Now,

$$\begin{aligned}
&\int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)} c_{k-j} F_{k-j}(s,R) s\, ds \\
&= \int_0^\infty \sum_{l=l_1}^{l_0-1} \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{W_l}(k-j)c_{k-j}(F_{k-j}(s,R) - F_0(s,R))s\, ds \\
&\quad + \int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{V_{2^{l_1}}}(k-j)c_{k-j}(F_{k-j}(s,R) - F_0(s,R))s\, ds \\
&\quad + \int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{V_{2^{l_0}}}(k-j)c_{k-j}F_0(s,R)s\, ds \\
&\quad + \int_0^\infty \sum_{l=l_0}^{\infty} \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{W_l}(k-j)c_{k-j}F_{k-j}(s,R)s\, ds \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Estimating III: Observe that since Q has integral zero on $\mathbb{R}^2$,

$$\int_0^\infty F_0(s,R) s\, ds = 0$$

for all R. Integration by parts in III thus gives,

$$\begin{aligned}
\mathrm{III} = &- R\int_0^\infty \sum_{j,k}\Big(\alpha_j J_j'(Rs)\overline{\beta_k J_k(Rs)} \\
&+ \alpha_j J_j(Rs)\overline{\beta_k J_k'(Rs)}\Big)\left(\int_0^s F_0(t,R)t\, dt\right)\widehat{V_{2^{l_0}}}(k-j)c_{k-j}\, ds.
\end{aligned}$$

Now, for each $s \geqslant 0$,

$$\begin{aligned}
&\left|\sum_{j,k} \alpha_j J_j'(Rs)\overline{\beta_k J_k(Rs)}\widehat{V_{2^{l_0}}}(k-j)c_{k-j}\right| \\
&\quad \leqslant ||f||_2 ||g||_2 \sup_{j,k} |J_j'(Rs)J_k(Rs)| \sup_\theta \left|\sum_m \widehat{V_{2^{l_0}}}(m)c_m e^{im\theta}\right| \\
&\quad \leqslant C||f||_2||g||_2||\psi||_\infty ||V_{2^{l_0}}||_1 |Rs|^{-5/6},
\end{aligned}$$

by Lemma 2. Since, by Lemma 4, V_N is bounded in L^1 uniformly in N,

$$|\mathrm{III}| \leqslant CR^{-5/6}||f||_2||g||_2||\psi||_\infty R\int_0^\infty s^{-5/6}\left|\int_0^s F_0(t,R)t\,dt\right| ds.$$

To obtain the desired estimate for III, it suffices to show that

$$R\int_0^\infty s^{-5/6}\left|\int_0^s F_0(t,R)t\,dt\right| ds$$

is uniformly bounded.

Lemma 5. *For each $N \in \mathbb{N}$ there is a constant C_N such that*

$$\left|\int_0^s F_0(t,R)t\,dt\right| \leqslant \frac{C_N \min\{s,1\}}{(1+R|s-1|)^N}$$

for all $s \geqslant 0$.

Proof.

$$\begin{aligned}\int_0^s F_0(t,R)t\,dt &= \int_0^s\int_0^{2\pi} R^2 Q(R(te^{i\theta}-1))\,d\theta t\,dt\\ &= \int_{|x|\leqslant s} R^2 Q(R(x-(1,0)))\,dx\\ &= -\int_{|x|\geqslant s} R^2 Q(R(x-(1,0)))\,dx,\end{aligned}$$

since $\int_{\mathbb{R}^2} Q = 0$. The lemma now follows from the above two expressions and the fact that Q is rapidly decreasing. $\square$

By Lemma 5, for $N \geqslant 2$,

$$\begin{aligned}R\int_0^\infty s^{-5/6}\left|\int_0^s F_0(t,R)t\,dt\right| ds &\leqslant CR\int_0^\infty \frac{s^{-5/6}\min\{s,1\}}{(1+R|s-1|)^N}\,ds\\ &\leqslant C\int_0^\infty \frac{R\,ds}{(1+R|s-1|)^N}\\ &< \infty\end{aligned}$$

uniformly in R, as required.

Remark. Within the analysis of III lies a proof of the fact that under the hypotheses of Theorem 1,

$$\left|\int_{\mathbb{S}^1} \widehat{f\,d\sigma}(Rx)\overline{\widehat{g\,d\sigma}(Rx)}\,d\sigma(x)\right| \leqslant \frac{C}{R}||f||_2||g||_2 \tag{5}$$

for all $R>0$. However, this requires the additional estimate

$$\sup_{j\in\mathbb{Z}} |J_j(s)J_j'(s)| \leqslant cs^{-1}$$

for some constant $c>0$. This estimate can be found in [1]. (A proof of a result similar to (5) can be found in [5].)

Estimating II: For each $s\geqslant 0$,

$$\left|\sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{V_{2^{l_1}}}(k-j)c_{k-j}(F_{k-j}(s,R)-F_0(s,R))\right|$$

$$\leqslant ||f||_2||g||_2 \sup_{j,k} |J_j(Rs)J_k(Rs)|||\psi||_\infty$$

$$\times \int_0^{2\pi}\left|\sum_m \widehat{V_{2^{l_1}}}(m)(F_m(s,R)-F_0(s,R))e^{im\theta}\right| d\theta.$$

Now,

$$\sum_m F_m(s,R)e^{im\theta} = R^2 Q(R(se^{i\theta}-1))$$

and so

$$\sum_m \widehat{V_{2^{l_1}}}(m)(F_m(s,R)-F_0(s,R))e^{im\theta}$$

$$= \int_0^{2\pi} [V_{2^{l_1}}(\theta-\phi) - V_{2^{l_1}}(\theta)]R^2 Q(R(se^{i\phi}-1))\, d\phi$$

and hence, by Lemma 4,

$$\int_0^{2\pi}\left|\sum_m \widehat{V_{2^{l_1}}}(m)(F_m(s,R)-F_0(s,R))e^{im\theta}\right| d\theta$$

$$\leqslant \int_0^{2\pi}\left(\int_0^{2\pi} |V_{2^{l_1}}(\theta-\phi)-V_{2^{l_1}}(\theta)|\, d\theta\right)R^2|Q(R(se^{i\phi}-1))|\, d\phi$$

$$\leqslant C\int_0^{2\pi} 2^{l_1}|\phi|R^2|Q(R(se^{i\phi}-1))|\, d\phi \tag{6}$$

and so,

$$|\mathrm{II}| \leqslant C2^{l_1} \|f\|_2 \|g\|_2 \|\psi\|_\infty \int_0^\infty (Rs)^{-2/3} \int_0^{2\pi} |\phi| R^2 |Q(R(se^{i\phi} - 1))| \, d\phi s \, ds.$$

Now,

$$\begin{aligned}
&\int_0^\infty (Rs)^{-2/3} \int_0^{2\pi} |\phi| R^2 |Q(R(se^{i\phi} - 1))| \, d\phi s \, ds \\
&= \int_{\mathbb{R}^2} |Rx|^{-2/3} |\arg x| R^2 |Q(R(x - (1,0)))| \, dx \\
&= \int_{\mathbb{R}^2} |(y_1 + R, y_2)|^{-2/3} |\arg(y_1 + R, y_2)| |Q(y)| \, dy \\
&\leqslant \left\{ \int_{|y| \leqslant R/2} |(y_1 + R, y_2)|^{-2/3} |\arg(y_1 + R, y_2)| |Q(y)| \, dy \right. \\
&\quad \left. + \int_{|y| > R/2} |(y_1 + R, y_2)|^{-2/3} |\arg(y_1 + R, y_2)| |Q(y)| \, dy \right\}. \qquad (7)
\end{aligned}$$

Since the first term in the above sum is dominated by

$$R^{-2/3} \int_{|y| \leqslant R/2} \frac{|y_2|}{y_1 + R} |Q(y)| \, dy \leqslant 2R^{-5/3} \int_{\mathbb{R}^2} |y| |Q(y)| \, dy$$

and the second term is rapidly decreasing in R,

$$|\mathrm{II}| \leqslant C2^{l_1} R^{-5/3} \|f\|_2 \|g\|_2 \|\psi\|_\infty \leqslant CR^{-4/3} \|f\|_2 \|g\|_2 \|\psi\|_\infty,$$

since $2^{l_1} \leqslant (2R)^{1/3}$. Evidently, we could have chosen l_1 to be much larger; however, this turns out to be of no advantage to us in the analysis that remains.

Estimating I: Fix $l \in \mathbb{Z}$ such that $l_1 \leqslant l < l_0$ and $s \geqslant 0$, and consider

$$\sum_{j,k} \alpha_j J_j(Rs) \overline{\beta_k J_k(Rs)} \widehat{W_l}(k - j) c_{k-j} (F_{k-j}(s,R) - F_0(s,R)).$$

Now, $\widehat{W_l}(k-j) \neq 0 \Leftrightarrow |k-j| \sim 2^l$, and so we may restrict our attention to sequences α and β such that $\alpha = \{\alpha_j\}_{|j-j_0| \leqslant 2^l/10}$, and $\beta = \{\beta_k\}_{|k-k_0| \leqslant 2^l/10}$, for some j_0 and k_0 satisfying $|k_0 - j_0| \sim 2^l$.

Now,

$$\left|\sum_{j,k}\alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{W_l}(k-j)c_{k-j}(F_{k-j}(s,R)-F_0(s,R))\right|$$
$$\leqslant\left(\sum_{|j-j_0|\leqslant 2^l/10}|\alpha_j J_j(Rs)|^2\right)^{1/2}\left(\sum_{|k-k_0|\leqslant 2^l/10}|\beta_k J_k(Rs)|^2\right)^{1/2}$$
$$\times\sup_\theta\left|\sum_m \widehat{W_l}(m)c_m[F_m(s,R)-F_0(s,R)]e^{im\theta}\right|$$
$$\leqslant C||f||_2||g||_2||\psi||_\infty \sup_{|j-k|\sim 2^l;j,k\geqslant 0}|J_j(Rs)J_k(Rs)|$$
$$\times\int_0^{2\pi}\left|\sum_m \widehat{W_l}(m)[F_m(s,R)-F_0(s,R)]e^{im\theta}\right|d\theta$$
$$\leqslant C||f||_2||g||_2||\psi||_\infty \sup_{|j-k|\sim 2^l;j,k\geqslant 0}|J_j(Rs)J_k(Rs)|$$
$$\times\int_0^{2\pi}2^l|\phi|R^2|Q(R(se^{i\phi}-1))|\,d\phi.$$

The last inequality above follows as in (6). Now, by Lemma 3,

$$\int_0^\infty \sup_{|k-j|\sim 2^l}|J_j(Rs)J_k(Rs)|\int_0^{2\pi}2^l|\phi|R^2|Q(R(se^{i\phi}-1))|\,d\phi s\,ds$$
$$\leqslant CR^{-5/6}\int_{|x|\leqslant 3.2^l/R}|x|^{-5/6}2^l|\arg x|R^2|Q(R(x-(1,0)))|\,dx$$
$$+\,CR^{-5/6}\int_{3.2^l/R<|x|\leqslant 2^{3l}/R}|x|^{-5/6}(R|x|/2^l)^{1/4}2^l|\arg x|R^2|Q(R(x-(1,0)))|\,dx$$
$$+\,C\int_{|x|\geqslant 2^{3l}/R}2^lR^2|Q(R(x-(1,0)))|\,dx.$$

The first and second terms in the above sum are bounded above by $C\frac{2^l}{R}R^{-5/6}$ and $C(\frac{2^l}{R})^{3/4}R^{-5/6}$, respectively, by arguing as in (7). The third term decays rapidly in R since $2^l\geqslant 2^{l_1}\geqslant(2R)^{1/3}$. Summing in $l_1\leqslant l<l_0$ gives the desired estimate for I.

Estimating IV: Fix $l\geqslant l_0$, and let the sequences α and β be localised as before. Now,

$$\left|\sum_{j,k}\alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{W_l}(k-j)c_{k-j}F_{k-j}(s,R)\right|$$
$$\leqslant||f||_2||g||_2||\psi||_\infty \sup_{|k-j|\sim 2^l}|J_j(Rs)J_k(Rs)|\int_0^{2\pi}\left|\sum_m \widehat{W_l}(m)F_m(s,R)e^{im\theta}\right|d\theta.$$

On integrating by parts we see that

$$\begin{aligned}
&\sum_m \widehat{W_l}(m)F_m(s,R)e^{im\theta}\\
&= \int_0^{2\pi} W_l(\phi)R^2 Q(R(se^{i(\theta-\phi)}-1))\,d\phi\\
&= -\int_0^{2\pi} \sum_n \frac{1}{in}\widehat{W_l}(n)e^{-in\phi}\frac{d}{d\phi}\Big(R^2 Q(R(se^{i(\theta-\phi)}-1))\Big)\,d\phi\\
&= \frac{R}{2^l}\int_0^{2\pi} \widetilde{W_l}(\phi)\frac{d}{dt}(RQ(R(se^{it}-1)))(\theta-\phi)\,d\phi,
\end{aligned}$$

where

$$\widehat{\widetilde{W_l}}(n) = \frac{2^l}{in}\widehat{W_l}(n)$$

for each $n\in\mathbb{Z}$.

Remark. Since, like W_l, $\widetilde{W_l}$ has mean value zero on $[0,2\pi]$, the above integration by parts argument can be iterated yielding rapid decay in $2^l/R$. However, this is not necessary for our purposes.

Observe that

$$\begin{aligned}
\frac{d}{dt}(RQ(R(se^{it}-1))) &= \frac{d}{dt}(RQ(R(s\cos t-1, s\sin t)))\\
&= R^2 x^{\perp}\cdot\nabla Q(R(x-(1,0))),
\end{aligned}$$

where $x=(s\cos t, s\sin t)$, and $x^{\perp}=(-s\sin t, s\cos t)$. Hence,

$$\begin{aligned}
&\int_0^{2\pi}\left|\sum_m \widehat{W_l}(m)F_m(s,R)e^{im\theta}\right|d\theta\\
&\leqslant \frac{R}{2^l}\|\widetilde{W_l}\|_1\int_0^{2\pi}\big|(-s\sin\theta, s\cos\theta)\cdot R^2\nabla Q(R(s\cos\theta-1, s\sin\theta))\big|\,d\theta
\end{aligned}$$

and since $\sup_l \|\widetilde{W_l}\|_1<\infty$,

$$\begin{aligned}
&\int_0^{\infty}\sup_{|k-j|\sim 2^l}|J_j(Rs)J_k(Rs)|\int_0^{2\pi}\left|\sum_m \widehat{W_l}(m)F_m(s,R)e^{im\theta}\right|d\theta s\,ds\\
&\leqslant C\frac{R}{2^l}\int_{\mathbb{R}^2}\sup_{|k-j|\sim 2^l}|J_j(R|x|)J_k(R|x|)||x|R^2|\nabla Q(R(x-(1,0)))|\,dx
\end{aligned}$$

$$= C\frac{R}{2^l}\int_{|x|\leqslant 3.2^l/R} \sup_{|k-j|\sim 2^l} |J_j(R|x|)J_k(R|x|)||x|R^2|\nabla Q(R(x-(1,0)))|\,dx$$
$$+ C\frac{R}{2^l}\int_{|x|>3.2^l/R} \sup_{|k-j|\sim 2^l} |J_j(R|x|)J_k(R|x|)||x|R^2|\nabla Q(R(x-(1,0)))|\,dx.$$

Now, by Lemma 3, for $|x|\leqslant 3.2^l/R$,

$$\sup_{|k-j|\sim 2^l} |J_j(R|x|)J_k(R|x|)|\leqslant C|Rx|^{-5/6}$$

and so,

$$\int_{|x|\leqslant 3.2^l/R} \sup_{|k-j|\sim 2^l} |J_j(R|x|)J_k(R|x|)||x|R^2|\nabla Q(R(x-(1,0)))|\,dx$$
$$\leqslant CR^{-5/6}\int_{\mathbb{R}^2} |x|^{1/6}R^2|\nabla Q(R(x-(1,0)))|\,dx$$
$$\leqslant CR^{-5/6}.$$

Since for all $l\geqslant l_0$,

$$\int_{|x|>3.2^l/R} \sup_{|k-j|\sim 2^l} |J_j(R|x|)J_k(R|x|)||x|R^2|\nabla Q(R(x-(1,0)))|\,dx$$
$$\leqslant C\int_{|x|>3} |x|R^2|\nabla Q(R(x-(1,0)))|\,dx$$

is rapidly decreasing in R, we conclude that

$$\int_0^\infty \left|\sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{W_l}(k-j)c_{k-j}F_{k-j}(s,R)\right| s\,ds$$
$$\leqslant C\frac{R}{2^l}R^{-5/6}\,||f||_2||g||_2||\psi||_\infty.$$

Summing in $l\geqslant l_0$, we obtain

$$|\mathrm{IV}|\leqslant CR^{-5/6}||f||_2||g||_2||\psi||_\infty$$

as required.

5. The proof of Theorem 1: Part 2

In this section we consider the contribution arising from the remaining indices; $j\leqslant 0$ and $k\geqslant 0$.

We recall the notation established at the beginning of Section 4. Let $N \in \mathbb{N}$ be such that $2R \leqslant N < 2R+1$. Now,

$$\int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)} c_{k-j} F_{k-j}(s,R) s\, ds$$

$$= \int_0^\infty \sum_{l=l_1}^{l_0-1} \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)} \widehat{W_l}(k-j-N) c_{k-j}(F_{k-j}(s,R) - F_N(s,R)) s\, ds$$

$$+ \int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)} \widehat{V_{2^{l_1}}}(k-j-N) c_{k-j}(F_{k-j}(s,R) - F_N(s,R)) s\, ds$$

$$+ \int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)} \widehat{V_{2^{l_0}}}(k-j-N) c_{k-j} F_N(s,R) s\, ds$$

$$+ \int_0^\infty \sum_{l=l_0}^{\infty} \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)} \widehat{W_l}(k-j-N) c_{k-j} F_{k-j}(s,R) s\, ds$$

$$+ \mathrm{I}' + \mathrm{II}' + \mathrm{III}' + \mathrm{IV}'.$$

Estimating III$'$: We begin by observing that

$$\int_0^\infty F_N(s,R) s\, ds = O(R^{-1}).$$

To see this we write

$$\left| \int_0^\infty F_N(s,R) s\, ds \right|$$

$$= \left| \int_0^\infty \int_0^{2\pi} R^2 Q(R(se^{i\theta} - 1)) e^{-iN\theta}\, d\theta s\, ds \right|$$

$$= \left| \int_{\mathbb{R}^2} R^2 Q(R(x - (1,0))) e^{-iN \arg x}\, dx \right|$$

$$= \left| \int_{\mathbb{R}^2} R^2 Q(R(x - (1,0))) e^{-i(x-(1,0))\cdot(0,N)} \left(e^{iN(x_2 - \arg x)} - 1 \right) dx \right|$$

(since $\hat{Q}$ vanishes on $|\xi| = N/R$)

$$\leqslant \int_{\mathbb{R}^2} R^2 |Q(R(x - (1,0)))| R |x_2 - \arg x|\, dx$$

$$= \int_{\mathbb{R}^2} |Q(y)| |y_2 - R \arg (y + (R,0))|\, dy.$$

By decomposing the range of integration in the final expression above and using the fact that Q is rapidly decreasing, $\mathrm{O}(R^{-1})$ follows.

Let $\eta \in C_c^\infty(\mathbb{R})$ have integral 1, and let $\eta_{1/R} = R\eta(R\cdot)$.

Now, for

$$\lambda = \int_0^\infty F_N(s,R)s\,ds,$$

we write

$$\mathrm{III}' = \int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{V_{2^{l_0}}}(k-j-N)c_{k-j}[F_N(s,R)s - \lambda\eta_{1/R}(s-1)]\,ds$$
$$+\lambda\int_0^\infty \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{V_{2^{l_0}}}(k-j-N)c_{k-j}\eta_{1/R}(s-1)\,ds.$$

To the first term in the above we may apply the same integration by parts argument that we used to control III, since by construction,

$$\int_0^\infty [F_N(s,R)s - \lambda\eta_{1/R}(s-1)]\,ds = 0.$$

The remaining term we trivially control by $cR^{-1}||f||_2||g||_2||\psi||_\infty$.

Observation: In all of our estimates it is enough to restrict our attention to s satisfying $R|s-1|<R^\varepsilon$, for any fixed $\varepsilon>0$. This is a consequence of the rapid decay of Q, and can be seen as follows:

$$\left|\int_{s\geqslant 0;R|s-1|>R^\varepsilon} \sum_{j,k} \alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}c_{k-j}F_{k-j}(s,R)s\,ds\right|$$
$$\leqslant ||f||_2||g||_2||\psi||_\infty \int_{R||x|-1|>R^\varepsilon} R^2|Q(R(x-(1,0)))|\,dx$$
$$\leqslant C_K R^{-K}||f||_2||g||_2||\psi||_\infty$$

for some constant C_K (depending on ε), for all $K\in\mathbb{N}$.

In the following estimates it will be convenient to make such a restriction on s.

Estimating I': We begin as in the estimation of I. Fix $l\in\mathbb{Z}$ such that $l_1\leqslant l\leqslant l_0$, and $s>0$ such that $R|s-1|<R^\varepsilon$ for some $0<\varepsilon<1/3$. As before, we restrict our attention to sequences $\alpha=\{\alpha_j\}_{|j-j_0|\leqslant 2^l/10}$, and $\beta=\{\beta_k\}_{|k-k_0|\leqslant 2^l/10}$, for some j_0 and k_0 satisfying $|k_0-j_0-N|\sim 2^l$.

Now,

$$\left|\sum_{j,k}\alpha_j J_j(Rs)\overline{\beta_k J_k(Rs)}\widehat{W_l}(k-j-N)c_{k-j}(F_{k-j}(s,R)-F_N(s,R))\right|$$
$$\leqslant C||f||_2||g||_2||\psi||_\infty \sup_{|k-j-N|\sim 2^l}|J_j(Rs)J_k(Rs)|$$
$$\times\int_0^{2\pi} 2^l|\phi|R^2|Q(R(se^{i\phi}-1))|\,d\phi, \qquad (8)$$

where the supremum is restricted to $j \leqslant 0$ and $k \geqslant 0$. Since $2^l \geqslant 2^{l_1} \geqslant (2R)^{1/3}$,

$$\{(j,k) : |k-j-N| \sim 2^l\} \subset \{(j,k) : |k-j-2Rs| \sim 2^l\}$$

and so (8) is less than or equal to

$$C\|f\|_2\|g\|_2\|\psi\|_\infty \sup_{|k-j-2Rs|\sim 2^l; j\leqslant 0, k\geqslant 0} |J_j(Rs)J_k(Rs)|$$
$$\times \int_0^{2\pi} 2^l|\phi|R^2|Q(R(se^{i\phi}-1))|\,d\phi.$$

We now proceed as in the analysis of I, but using the second estimate from Lemma 3 rather than the first.

Estimating II$'$ *and* IV$'$: We may estimate II$'$ and IV$'$ as we did II and IV, respectively, with little complication.

6. Optimality of the decay rate in Theorem 1

We give two examples which demonstrate the sharpness of the decay rate in Theorem 1.

We will say that a function $f : \mathbb{S}^1 \to \mathbb{C}$ is an '$\mathbb{S}^1$-modulated cap' if

$$f(x) = \chi_C(x)e^{ik \arg x}$$

for some cap $C \subset \mathbb{S}^1$ and $k \in \mathbb{Z}$. We will say that f is an '$\mathbb{R}^2$-modulated cap' if

$$f(x) = \chi_C(x)e^{ia\cdot x}$$

for some $C \subset \mathbb{S}^1$ and $a \in \mathbb{R}^2$.

Our first example is in terms of $\mathbb{S}^1$-modulated caps, and is the natural example given our proof of Theorem 1. Our second example will be in terms of $\mathbb{R}^2$-modulated caps.

6.1. Example 1

The important observation here is that, in a very precise way 'the operator

$$g \mapsto \widehat{g\,d\sigma}(R\cdot),$$

restricted to $\mathbb{S}^1$, rotates $\mathbb{S}^1$-modulated caps, and the angle of rotation depends on the frequency of the modulation'. So, by choosing the frequencies appropriately, we can 'run the caps into each other'.

Notation. For a cap $C \subset \mathbb{S}^1$ we denote by C^* the cap with the same centre as C but with half the angular length (the 'concentric half' of C).

Let C_1 and C_2 be 1–caps on $\mathbb{S}^1$ centred at $(1,0)$ and $(0,1)$, respectively. Let

$$f(x) = \chi_{C_1}(x) e^{iR \arg x}$$

and

$$g(x) = \chi_{C_2 \cup (-C_2)}(x).$$

Now,

$$\widehat{f\,d\sigma}(R\xi) = \int_{C_1} e^{iR(\theta - \cos(\theta - \arg \xi))}\, d\theta$$

and

$$\widehat{g\,d\sigma}(R\xi) = \int_{C_2 \cup (-C_2)} e^{-iR\cos(\theta - \arg \xi)}\, d\theta$$

for $\xi \in \mathbb{S}^1$. Let $h_1(\theta) = \cos(\theta - \arg \xi) - \theta$, and $h_2(\theta) = \cos(\theta - \arg \xi)$. We observe that

$$h_1'(\theta) = 0 \Leftrightarrow \theta = \arg \xi - \pi/2$$

and

$$h_2'(\theta) = 0 \Leftrightarrow \theta = \arg \xi, \quad \arg \xi + \pi$$

It is now easy to see that

$$\widehat{f\,d\sigma}(R\xi) = e^{iR \arg \xi} J_R(R) + O(R^{-1})$$

on $C_1^* + \{\pi/2\}$ (C_1^* rotated anticlockwise through $\pi/2$). Similarly,

$$\widehat{g\,d\sigma}(R\xi) = J_0(R) + O(R^{-1})$$

on $C_2^* \cup (-C_2^*)$. From the optimal[4] asymptotic estimates,

$$|J_R(R)| \sim R^{-1/3}, \text{and} |J_0(R)| \lesssim R^{-1/2},$$

[4] Optimal from the point of view of the decay exponents; in particular, we refer the reader to Watson ([9, p. 260]) for the estimate for $J_R(R)$.

we may now conclude that since $C_1^* + \{\pi/2\} = C_2^*$, the estimate

$$\int_{\mathbb{S}^1} |\widehat{f\,d\sigma}(R\xi)\widehat{g\,d\sigma}(R\xi)|\,d\sigma(\xi) \lesssim R^{-5/6}\|f\|_2\|g\|_2,$$

is also optimal.

Remark. If f is an $\mathbb{S}^1$-modulated cap with frequency k, $(0<k<R)$, then we get some intermediate rotation (between 0 and $\pi/2$) given by the critical points of the phase $R\cos(\theta - \arg\xi) + k\theta$.

6.2. Example 2

Let $\mathfrak{C} \subset \mathbb{S}^1$ be a cap of angular length 1, centred at the north pole $(0,1)$. Let $c \subset \mathbb{S}^1$ be a cap of angular length $R^{-1/3}$, centred at the point $(1,0)$. We now choose

$$g(x) = \chi_c(x)e^{iRx_2}$$

and

$$f(x) = \chi_{\mathfrak{C}\cup(-\mathfrak{C})}(x)e^{iRa\cdot x},$$

where $a = (0, 1 - 2R^{-1/3})$. By easy considerations, there exists an absolute constant $c>0$ such that

$$|\widehat{g\,d\sigma}(R\xi)| \geqslant cR^{-1/3}\chi_T(\xi), \tag{9}$$

where T is the rectangle of dimensions $R^{-2/3} \times R^{-1/3}$, centred at $(0,1)$ with long side pointing in the direction $(1,0)$. By arguments similar to those in Example 1, $(\chi_{\mathfrak{C}\cup(-\mathfrak{C})}\,d\sigma)^\wedge(R\xi)$ is well approximated by $\widehat{d\sigma}(R\xi)$ on the cone

$$\Gamma = \{\xi \in \mathbb{R}^2 : |\xi_2| \geqslant 2|\xi_1|\},$$

with an error of order $(1 + R|\xi|)^{-1}$. Hence, for $R|\xi - a| \gg 1$,

$$\widehat{f\,d\sigma}(R\xi) = \widehat{d\sigma}(R|\xi - a|) + O((R|\xi - a|)^{-1})$$

on $\Gamma + \{a\}$. By stationary phase (see [7]) we have the asymptotic estimate

$$\widehat{d\sigma}(X) = c|X|^{-1/2}\cos(|X| - \pi/4) + O(|X|^{-3/2}), \quad \text{as } |X| \to \infty$$

and so, for $R|\xi - a| \gg 1$,

$$\widehat{f\,d\sigma}(R\xi) = c(R|\xi - a|)^{-1/2}\cos(R|\xi - a| - \pi/4) + O((R|\xi - a|)^{-1})$$

on $\Gamma + \{a\}$.

Merely to avoid irrelevant technicalities, let us suppose that

$$|\widehat{f\,d\sigma}(R\xi)| \geqslant c(1 + R|\xi - a|)^{-1/2}$$

on $\Gamma + \{a\}$.

Now, by construction, $|T \cap \mathbb{S}^1| \sim R^{-1/3}$, and on $T \subset \Gamma + \{a\}$, $|\widehat{f\,d\sigma}(R\xi)| \geqslant cR^{-1/3}$. Since $||f||_2 \sim 1$ and $||g||_2 \sim R^{-1/6}$,

$$\int_{\mathbb{S}^1} |\widehat{f\,d\sigma}(R\xi)\widehat{g\,d\sigma}(R\xi)|\, d\sigma(\xi) \geqslant cR^{-1} \sim R^{-5/6}||f||_2||g||_2,$$

as required.

Finally, we remark that Example 1 has very much in common with the example in [3, Section 3]

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