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A cops and robber game in multidimensional grids☆

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ABSTRACT

We theoretically analyze the 'cops and robber' game for the first time in a multidimensional grid. It is shown that in an n-dimensional grid, at least n cops are necessary if one wants to catch the robber for all possible initial configurations. We also present a set of cop strategies for which n cops are provably sufficient to catch the robber. Further, we revisit the game in a two-dimensional grid and provide an independent proof of the fact that the robber can be caught even by a single cop under certain conditions.

1. Introduction

The game of 'cops and robber' is played between a number of cops and a single robber on a predefined graph structure. Each of the cops and the robber starts from some initial node and moves from one node to another as the game proceeds. The cops win if they can 'capture' the robber in a finite time; otherwise the robber wins. In the literature, there are different kinds of movements and various notions of capture (see Section 1.1 for details).

Several attempts have been made to analyze different variants of the cops and robber game in the last two decades. However, there still remain many open problems in this domain, leading to continual research on the topic to date.

1.1. Background

Formal investigation into the problem of cops and robber and its variants dates back to early eighties. The works [12,1,13,11,2] consider discrete movements of the cops and the robber in alternate steps, the cops choosing their initial positions first. The robber is assumed to be captured if her position coincides with that of any cop. In [12], an algorithm for determining whether a given graph is *cop-win* is presented (a graph is *cop-win* if a single cop is sufficient to get hold of the robber). In [1], the notion of the *cop-number* of a graph (the minimum number of cops needed to ensure that the robber is caught under all possible circumstances) is introduced and a detailed analysis is performed for planar graphs. Later, the works [13,11,2] explored the cop-numbers for different graphs and discovered some interesting bounds.

The game of cops and robber can also be generalized to directed graphs. Here the robber moves with an infinite speed, although she is not permitted to run through a cop. The cops move in helicopters from one node to another node directly,

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even if the two nodes are not adjacent. Optionally the robber may be considered to be invisible to the cops, but not vice versa. In [9,3], variations of this scheme have been investigated, the main objective being to determine how may cops are necessary to capture the robber.

Another version of the game deals with a two-dimensional grid [15,5,10,6]. The robber selects her initial position after the cops have chosen theirs and then they keep on moving continuously through the edges of the grid. The robber has complete information about the positions and strategies of all the cops. However, the *visibility* power of each cop is confined to the nodes and edges in her current column (row). In [5,10], the cops win if at some point of time some cop can 'see' the robber, whereas in all other works winning is equivalent to capturing. This form of the game has applications in the motion planning of multiple robots [15]. If the robber moves at most as fast as the cops, then according to [15], two cops are necessary and sufficient to ensure a win for the cops. However, only one cop can always catch the robber if she moves fast enough. Subsequent works [5,10] improved the bound on the minimum speed required by a single cop to ensure the robber's capture. Recently, the work [6] has revisited the study of [15] and presented algorithms for capture using one, two or three pursuers having a constant maximum speed limit. Another recent work [7] considers the minimum number of guards necessary to guarantee the capture of a fugitive who can move with arbitrary speed about the edges of a grid.

The work [4] also considers a two-dimensional grid model, where the cops and the robber choose their initial positions randomly and then move alternately in discrete steps. In addition, the paper discusses applications of this model in the domain of multi-agent systems.

1.2. Our contribution

To our knowledge, this is the first work on the game of cops and robber in a general n-dimensional grid paradigm. Most of the existing works [15,5,10,4] focus only on a two-dimensional grid which is a special case with $n = 2$. As an example of a three-dimensional scenario, one could imagine that the cops are chasing a robber inside a multi-storied apartment complex, and model that with a three-dimensional grid and apply our results with $n = 3$. This may find applications in three-dimensional motion planning for robots.

The works [15,5,10] consider that the cops and the robber move simultaneously in a continuous manner. The focus is mainly on the speed requirements and the notion of capture is defined in terms of visibility. On the other hand, we follow the same model as that of [4]. In our work, it is assumed that the cops and the robber choose their initial positions randomly and their movements take place in discrete steps. The robber is considered to be caught if her position coincides with that of any cop.

In [4], four predator agents (i.e., cops) chase a target agent (i.e. robber) in a square grid. Three related convergence metrics are introduced and an algorithm is presented based on one of them. Applying our general result in two dimensions, only two predator agents can successfully capture the target agent. Thus our work may be considered to be a major improvement over [4].

A few works [8,16] have considered variants of the problem in three dimensions. A topological framework is considered in [8] that covers the case of a cube. The setting in [16] is a three-dimensional grid that requires at least five pursuers to ensure capture of the evader. Our analysis and results for the n-dimensional grid are different from these works.

In Section 2, we rigorously formulate the problem and introduce some terminologies that will be used throughout this paper. Section 3 shows that capture of the robber can never be guaranteed with less than n cops under certain constraints on the initial starting positions. We also propose a set of cop strategies and prove that n cops operating in accordance with these strategies will always be able to nab the robber.

Section 4 presents a strategy for a single cop in two dimensions. This strategy ensures a win for the cop with a probability of 0.5, provided that the initial positions of the cop and the robber are determined uniformly at random. The main result of this section (Theorem 3) can be deduced from the work of Sugihara and Suzuki [14, Corollary 2]. However, we present an independent proof in this paper.

2. Mathematical formulation

Let there be m *cops* $C_0, C_1, \ldots, C_{m-1}$ chasing a *robber* R. The term *agent* represents either a cop or the robber. Each agent occupies some node of a given undirected graph G. A node may contain more than one agent. However, no agent can simultaneously occupy more than one node. Like [4], we also assume that the initial positions of the cops and the robber are decided arbitrarily. Whenever an agent moves from one node to an adjacent node, the movement is called a *jump*.

This paper considers the situation when the game is being played on an *n-dimensional grid*. Any node in an n-dimensional $d_0 \times d_1 \times \cdots \times d_{n-1}$ grid can be expressed as an n-tuple $(u_0, u_1, \ldots, u_{n-1})$, where each u_i is an integer belonging to the closed interval $[0, d_i - 1]$. Two distinct nodes are adjacent if and only if exactly one of their n coordinates differs by 1, all other coordinates remaining the same. In other words, the nodes $(u_0, u_1, \ldots, u_{n-1})$ and $(v_0, v_1, \ldots, v_{n-1})$ are adjacent if and only if $\exists i \in \{0, 1, \ldots, n-1\}$ such that

(a) $|u_i - v_i| = 1$ and
(b) $\forall j \in \{0, 1, \ldots, n-1\} \setminus \{i\},\ u_j = v_j$.

We also assume that $d_i > 1$ for $i = 0, \ldots, n-1$. Otherwise, an n-dimensional grid may degenerate into a lower dimensional grid and some of the results discussed in subsequent sections may no longer be valid.

If an agent occupies some node $(u_0, u_1, \ldots, u_{n-1})$, then u_j would be referred to as the *coordinate j* of the current position of that agent, $0 \le j \le n-1$. Let $R_j^{(t)}$ denote the value of coordinate j of the robber after she completes t jumps and $C_{i,j}^{(t)}$ denote the value of coordinate j of cop C_i after her tth jump, $0 \le j \le n-1$, $0 \le i \le m-1$, $t \ge 0$. The vector $R^{(t)} = (R_0^{(t)}, R_1^{(t)}, \ldots, R_{n-1}^{(t)})$ and the vector $C_i^{(t)} = (C_{i,0}^{(t)}, C_{i,1}^{(t)}, \ldots, C_{i,n-1}^{(t)})$ denote the nodes occupied by the robber R and the cop C_i after their t jumps, respectively. Thus, $R^{(0)}$ and $C_i^{(0)}$ denote their initial positions. Whenever the number of jumps is not important, for the sake of simplicity we omit the superscript t and use the notation R, R_j, C_i and $C_{i,j}$ instead of $R^{(t)}$, $R_j^{(t)}$, $C_i^{(t)}$ and $C_{i,j}^{(t)}$ respectively. It will be clear from the context whether the symbols R and C_i denote a particular agent or her position. The game continues in the following steps.

(1) $t = 0$.
(2) The robber jumps to $R^{(t+1)}$.
(3) Each cop jumps simultaneously. The new node occupied by cop C_i is $C_i^{(t+1)}$, $i = 0, 1, \ldots, m-1$.
(4) $t = t + 1$. Go to Step 2.

Definition 1. A *configuration* is defined as the $(m+1)$-tuple $(C_0, C_1, \ldots, C_{m-1}, R)$ and it specifies the position of each cop and the robber at an instant.

Definition 2. A configuration is *terminating* if some cop occupies the same node as the robber.

Definition 3. A *strategy* for an agent is an algorithm that takes the current configuration as input and returns a node to which the agent will take the next jump.

While taking a jump, each agent selects an adjacent node by applying her own strategy. Our basic objective is to develop strategies for the cops so that eventually some terminating configuration is attained with minimum possible number of cops.

Definition 4. A set of strategies for the cops is *winning* if and only if for all possible initial configurations and robber strategies, some terminating configuration is achieved after a finite number of jumps.

3. Analysis of the game in an *n*-dimensional grid

In this section, we formally analyze the minimum number of cops required to ensure capture of the robber in an n-dimensional grid. This number is independent of any cop or robber strategy. We also investigate a relevant question: that of how to construct a set of cop strategies that uses exactly this minimum number of cops and therefore is optimal.

3.1. The necessary number of cops

We are going to prove that at least n cops are necessary, under certain constraints on the initial starting positions, if one wants to guarantee the capture of the robber.

Definition 5. $D_{i,j}^{0(t)} \triangleq |C_{i,j}^{(t)} - R_j^{(t)}|$, and $D_{i,j}^{1(t)} \triangleq |C_{i,j}^{(t)} - R_j^{(t+1)}|$. Moreover, $D_i^{0(t)} \triangleq \sum_{j=0}^{n-1} D_{i,j}^{0(t)}$, and $D_i^{1(t)} \triangleq \sum_{j=0}^{n-1} D_{i,j}^{1(t)}$.

Note that $D_i^{0(t)}$ is the Manhattan distance between cop C_i and the robber after each of them has taken t jumps. Similarly, $D_i^{1(t)}$ is the Manhattan distance between cop C_i and the robber after the robber has taken $(t+1)$ jumps and the cop has taken t jumps. During each jump, an agent (the robber or some cop) changes exactly one of its coordinates by exactly one unit. We thus have the following result.

Proposition 1. *For each cop C_i and $t \ge 0$,*

(a) $D_i^{1(t)} = D_i^{0(t)} \pm 1$.
(b) $D_i^{0(t+1)} = D_i^{1(t)} \pm 1$.

Lemma 1. *For each cop C_i and $0 \le t_1 < t_2 < \infty$; $D_i^{0(t_1)}$ and $D_i^{0(t_2)}$ are of the same parity.*

Proof. From Proposition 1(a), $D_i^{0(t_1)}$ and $D_i^{1(t_1)}$ are of the opposite parity. From Proposition 1(b), $D_i^{1(t_1)}$ and $D_i^{0(t_1+1)}$ are of the opposite parity. So $D_i^{0(t_1)}$ and $D_i^{0(t_1+1)}$ are of the same parity. Similarly we may prove that $D_i^{0(t_1+1)}$ and $D_i^{0(t_1+2)}$ are of the same parity. Continuing in this manner, we reach the desired result. □

Next, we are going to show (Theorem 1) that in an n-dimensional grid, n cops are necessary to guarantee capture of the robber for certain initial configurations.

Theorem 1. *In an n-dimensional grid, if the initial configuration is such that $D_i^{0(0)}$ is of odd parity for each cop C_i, then there exists a robber strategy for which the robber can never be caught with less than n cops.*

Proof. Lemma 1 implies that $\forall t > 0$ and $\forall i \in \{0, 1, \ldots, m-1\}$, $D_i^{0(t)}$ is of odd parity, m being the number of cops. The configuration attained after the robber and all the cops have each taken t jumps can never be terminating; otherwise, there would be some cop C_j occupying the same node as the robber, implying that $D_j^{0(t)} = 0$, an integer with even parity.

Since $R^{(t)}$ has at least n adjacent nodes and there are less than n cops, the robber may easily jump to a 'free' node v which is not occupied by any cop. Thus, the configuration after the robber has taken $t+1$ jumps and each cop has taken t jumps is also non-terminating. Further, by the assumption of the oddness of the distances at time t, there is no cop within distance 1 of v, resulting in a non-terminating configuration at time $t+2$. □

3.2. Winning sets of strategies and the sufficient number of cops

Analysis in the previous section poses a natural question for an n-dimensional grid: does there exist a winning set of strategies for exactly n cops? Before addressing this issue, we present some general results that hold for all cop strategies. These results will be required for subsequent analysis in this section.

Definition 6. $J_R^{(t)} \triangleq$ the coordinate along which the robber makes a move in her tth jump. $J_{C_i}^{(t)} \triangleq$ the coordinate along which cop C_i makes a move in her tth jump.

We use the notation $a \bullet b$ to denote $(a+b) \bmod n$, where a and b are integers.

Definition 7. Let $j_1 \in \{0, 1, \ldots, n-1\}$ be the smallest integer such that $C_{i,i\bullet j_1}^{(t)} \neq R_{i\bullet j_1}^{(t)}$. Then, for all $t \geq 0$, we say that the $(t+1)$th jump of the robber is *favorable* to cop C_i if and only if $J_R^{(t+1)} \notin \{i, i\bullet 1, \ldots, i\bullet(j_1-1)\}$.

The word *favorable*, as will be clear from the subsequent analysis, signifies that such a jump "favors" reaching the terminating configuration.

Observe that if $C_{i,i}^{(t)} \neq R_i^{(t)}$, then $j_1 = 0$ and $\{i, i\bullet 1, \ldots, i\bullet(j_1-1)\}$ reduces to the empty set. In such cases the next jump of the robber will always be favorable to cop C_i.

Lemma 2. *For $p = 0, 1, \ldots, n-1$, if the robber moves along coordinate p in her $(t+1)$th jump, then that jump is favorable to cop $C_{p\bullet 1}$.*

Proof. We can safely assume that $C_{p\bullet 1}^{(t)} \neq R^{(t)}$. (Otherwise the robber has already been captured.) Let $j_1 \in \{0, 1, \ldots, n-1\}$ be the smallest integer such that $C_{p\bullet 1,(p\bullet 1)\bullet j_1}^{(t)} \neq R_{(p\bullet 1)\bullet j_1}^{(t)}$. Now, $J_R^{(t+1)} = p = (p\bullet 1)\bullet(n-1) \notin \{(p\bullet 1), (p\bullet 1)\bullet 1, \ldots, (p\bullet 1)\bullet(j_1-1)\}$, since $j_1 - 1 \leq n-2$. By Definition 7, the $(t+1)$th jump of the robber is favorable to cop $C_{p\bullet 1}$. □

Algorithm 1 shows a set of cop strategies that will later be proved to be winning. The strategy for each cop C_i is denoted by S_i. The main idea is that each cop C_i tries to match as many coordinates as possible to those of the robber, starting from her i-th coordinate.

A major advantage of this set of strategies is that a cop need not know the positions of other cops. S_i determines $C_i^{(t+1)}$ based on $C_i^{(t)}$ and $R^{(t+1)}$ only. The purpose of the loop in Step 2 is to find the smallest integer $j \in \{0, 1, \ldots, n-1\}$ such that $C_{i,i}^{(t)} = R_i^{(t+1)}, C_{i,i\bullet 1}^{(t)} = R_{i\bullet 1}^{(t+1)}, \ldots, C_{i,i\bullet(j-1)}^{(t)} = R_{i\bullet(j-1)}^{(t+1)}$ and $C_{i,i\bullet j}^{(t)} \neq R_{i\bullet j}^{(t+1)}$. Either such an integer j exists, or $C_i^{(t)} = R^{(t+1)}$. In Step 3, $j = n$ if and only if $C_i^{(t)} = R^{(t+1)}$, indicating that a terminating configuration has already been achieved. Step 4 determines the node to which cop C_i is going to jump if the present configuration is not a terminating one.

Algorithm 1: Strategy S_i of cop C_i for the $(t+1)$-th jump	
1.	$j \leftarrow 0$;
2.	While $(j \neq n)$
2.1.	If $R_{i\bullet j}^{(t+1)} \neq C_{i,i\bullet j}^{(t)}$ then go to Step 3;
2.2.	$j \leftarrow j+1$;
3.	If $j = n$ then terminate the game;
4.	Else
4.1.	$C_i^{(t+1)} \leftarrow C_i^{(t)}$;
4.2.	If $R_{i\bullet j}^{(t+1)} < C_{i,i\bullet j}^{(t)}$ then $C_{i,i\bullet j}^{(t+1)} \leftarrow C_{i,i\bullet j}^{(t+1)} - 1$;
4.3.	Else $C_{i,i\bullet j}^{(t+1)} \leftarrow C_{i,i\bullet j}^{(t+1)} + 1$;

Lemma 3. *In an n-dimensional $d_0 \times d_1 \times \cdots \times d_{n-1}$ grid, if the robber has taken $\sum_{j=0}^{n-1} d_j$ jumps favorable to cop C_i and the cop follows strategy S_i, then the robber must have already been captured.*

Table 1
Performance comparison of three different robber strategies.

Grid size	Average No. of jumps by robber		
	Strategy 1	Strategy 2	Strategy 3
10 × 10	8	11	13
15 × 15	14	17	23
20 × 20	19	24	32
25 × 25	24	31	41
30 × 30	30	37	50
35 × 35	35	44	59
40 × 40	40	51	69
45 × 45	46	57	78
50 × 50	51	64	87

Proof. We are going to show that if the robber has taken $(d_i + d_{i\bullet 1} + \cdots + d_{i\bullet k})$ jumps favorable to cop C_i, then the configuration $R_i = C_{i,i}, R_{i\bullet 1} = C_{i,i\bullet 1}, \ldots, R_{i\bullet k} = C_{i,i\bullet k}$ has been reached. We use induction on k.

Base step ($k = 0$). Without any loss of generality, let $R_i^{(0)} > C_{i,i}^{(0)}$. Until $R_i = C_{i,i}$, every jump of the robber is favorable to cop C_i. At each jump, the value of $(R_i - C_{i,i})$ changes by at most 1. Initially $(R_i - C_{i,i})$ is positive. It cannot become negative without touching 0 at some stage. And it cannot remain positive for more than d_i jumps of the robber. Otherwise, $C_{i,j}$ is constantly incremented by 1 more than d_i times, a contradiction.

Induction step. Let the statement be true for $k = l$. We start from the configuration $R_i = C_{i,i}, R_{i\bullet 1} = C_{i,i\bullet 1}, \ldots, R_{i\bullet l} = C_{i,i\bullet l}$. Let $R_{i\bullet(l+1)} > C_{i,i\bullet(l+1)}$. If a jump of the robber is not favorable to cop C_i, then the subsequent jump of the cop is used to maintain the above mentioned equalities. Otherwise $C_{i,i\bullet(l+1)}$ is adjusted so as to get it closer to $R_{i,i\bullet(l+1)}$. Like for the *Base step*, it may be shown that the configuration $R_i = C_{i,i}, R_{i\bullet 1} = C_{i,i\bullet 1}, \ldots, R_{i\bullet l} = C_{i,i\bullet l}, R_{i\bullet(l+1)} = C_{i,i\bullet(l+1)}$ will be attained within $(d_i + d_{i\bullet 1} + \cdots + d_{i\bullet l} + d_{i\bullet(l+1)})$ favorable robber-jumps.

Putting $k = n - 1$, the result follows. □

Theorem 2. *In an n-dimensional grid, n cops are sufficient to ensure capture of the robber.*

Proof. Suppose each cop C_i follows strategy S_i. By Lemma 2, each jump of the robber is favorable to some cop. By the pigeonhole principle, if the robber takes $n\sum_{i=0}^{n-1} d_i$ jumps, then at least $\sum_{i=0}^{n-1} d_i$ of them would be favorable to some specific cop. By Lemma 3, this implies that a terminating configuration has been reached. Thus, the set of cop strategies $\{S_i \mid 0 \le i < n\}$ guarantees capture of the robber. □

Since, for each of $n\sum_{i=0}^{n-1} d_i$ jumps of the robber, the n cops take simultaneous jumps in $O(1)$ time, the worst case run-time of the set of cop strategies of Algorithm 1 is $O(n\sum_{i=0}^{n-1} d_i)$.

The set of strategies outlined in Algorithm 1 is optimal in the sense that they guarantee the attainment of a terminating configuration using the minimum number of cops.

3.3. Some experimental results for two dimensions

The robber cannot ensure evasion in two dimensions if two cops are chasing her. But she may want to delay her capture. We empirically observe how many jumps are taken by the robber before she is caught. We consider three different robber strategies (assuming that exactly two cops are present) described below. The cops move in accordance with the winning set of strategies presented in this section.

Robber strategy 1: For each adjacent position (x, y), she evaluates the expression $\{(x - C_{0,0})^2 + (y - C_{0,1})^2\} + \{(x - C_{1,0})^2 + (y - C_{1,1})^2\}$ and moves to the adjacent position for which the expression is maximized.

Robber strategy 2: For each adjacent position (x, y), she evaluates the expression $\{|x - C_{0,0}| + |y - C_{0,1}|\} + \{|x - C_{1,0}| + |y - C_{1,1}|\}$ and moves to the adjacent position for which the expression is maximized.

Robber strategy 3: For each adjacent position (x, y), she evaluates the expression $\sqrt{(x - C_{0,0})^2 + (y - C_{0,1})^2} + \sqrt{(x - C_{1,0})^2 + (y - C_{1,1})^2}$ and moves to the adjacent position for which the expression is maximized.

Table 1 shows the average number of jumps for the three different robber strategies when the game is repeated 1000,000 times, each time with a random initial configuration.

As the table shows, Strategy 3 seems to be most effective for the robber.

4. Additional theoretical results for two dimensions

Consider a single cop chasing the robber in two dimensions. According to Theorem 1, the robber can always evade capture for certain *bad* initial configurations, that is, when $D_0^{0(0)}$ is odd. If the starting positions of the cop and the robber are chosen uniformly at random, then the probability that a bad initial configuration will be encountered is 0.5. In all other situations, we show that the cop strategy S presented in Algorithm 2 guarantees capture of the robber. This scenario was analyzed earlier in [14]. However, here we present an independent analysis and the proof.

Definition 5 takes the following form in a two-dimensional grid: $D_0^{0(t)} = D_{0,0}^{0(t)} + D_{0,1}^{0(t)} = |C_{0,0}^{(t)} - R_0^{(t)}| + |C_{0,1}^{(t)} - R_1^{(t)}|$, $D_0^{1(t)} = D_{0,0}^{1(t)} + D_{0,1}^{1(t)} = |C_{0,0}^{(t)} - R_0^{(t+1)}| + |C_{0,1}^{(t)} - R_1^{(t+1)}|$.

Algorithm 2: Cop strategy S in 2D for the $(t+1)$-th jump	
1.	$C_0^{(t+1)} \leftarrow C_0^{(t)}$;
2.	If $\mid C_{0,0}^{(t)} - R_0^{(t+1)} \mid > \mid C_{0,1}^{(t)} - R_1^{(t+1)} \mid$ then
2.1	If $C_{0,0}^{(t)} > R_0^{(t+1)}$ then $C_{0,0}^{(t+1)} \leftarrow C_{0,0}^{(t)} - 1$;
2.2	Else $C_{0,0}^{(t+1)} \leftarrow C_{0,0}^{(t)} + 1$;
3.	Else if $\mid C_{0,0}^{(t)} - R_0^{(t+1)} \mid < \mid C_{0,1}^{(t)} - R_1^{(t+1)} \mid$ then
3.1	If $C_{0,1}^{(t)} > R_1^{(t+1)}$ then $C_{0,1}^{(t+1)} \leftarrow C_{0,1}^{(t)} - 1$;
3.2	Else $C_{0,1}^{(t+1)} \leftarrow C_{0,1}^{(t)} + 1$;
4.	Else jump to any adjacent node;

Proposition 2. *If $|C_{0,0} - R_0|$ becomes equal to $|C_{0,1} - R_1|$ at a stage when the cop and the robber have taken same number of jumps, then from that point onwards the sign of $C_{0,0} - R_0$ (as well as that of $C_{0,1} - R_1$) does not change, provided the cop moves in accordance with strategy S.*

Proof. Let for some t, $|C_{0,0}^{(t)} - R_0^{(t)}| = |C_{0,1}^{(t)} - R_1^{(t)}|$. Without any loss of generality, let $C_{0,0}^{(t)} < R_0^{(t)}$ and $C_{0,1}^{(t)} > R_1^{(t)}$. The robber may now either increment (decrement) R_0, or she may increment (decrement) R_1. Irrespective of the choice she makes, the cop following strategy S will move in such a fashion as to maintain the equality $|C_{0,0}^{(t+1)} - R_0^{(t+1)}| = |C_{0,1}^{(t+1)} - R_1^{(t+1)}|$. Further, we shall have $C_{0,0}^{(t+1)} \le R_0^{(t+1)}$ and $C_{0,1}^{(t+1)} \ge R_1^{(t+1)}$. In other words, either $C_{0,0}^{(t+1)} = R_0^{(t+1)}$, $C_{0,1}^{(t+1)} = R_1^{(t+1)}$ and the game terminates; or we have $C_{0,0}^{(t+1)} < R_0^{(t+1)}$ and $C_{0,1}^{(t+1)} > R_1^{(t+1)}$. The same way of reasoning may be repeated an arbitrary number of times. □

Lemma 4. *Consider an initial configuration such that $D_0^{0(0)}$ is of even parity. After the cop, who follows strategy S, moves along coordinate i for the first time, the sign of the expression $C_{0,i} - R_i$ is never going to change, for $i = 0, 1$.*

Proof. We validate the above statement only for coordinate 0. The case for coordinate 1 can be proved in a similar way. Let the cop move along coordinate 0 for the first time in her (t_1+1)th jump. Since $D_0^{0(0)}$ is even, $|C_{0,0}^{(t_1)} - R_0^{(t_1+1)}| - |C_{0,1}^{(t_1)} - R_1^{(t_1+1)}|$ must be odd and hence nonzero. According to strategy S, $|C_{0,0}^{(t_1)} - R_0^{(t_1+1)}| - |C_{0,1}^{(t_1)} - R_1^{(t_1+1)}| > 0$. Without any loss of generality, we assume that $C_{0,0}^{(t_1)} < R_0^{(t_1+1)}$ and the cop increments $C_{0,0}$ in her (t_1+1)th jump.

If possible, let the sign of $C_{0,0} - R_0$ become positive at some point of time after the cop has taken her (t_1+1)th jump. But prior to that, $C_{0,0} - R_0$ must touch the value 0, for $C_{0,0} - R_0$ changes by at most 1 during each step. When $C_{0,0} - R_0 = 0$, $|C_{0,0} - R_0| - |C_{0,1} - R_1| \le 0$. The value of $|C_{0,0} - R_0| - |C_{0,1} - R_1|$ also changes by at most 1 during each jump of the cop or robber. Consequently the game must have gone through a stage where $|C_{0,0} - R_0| - |C_{0,1} - R_1| = 0$ and $C_{0,0} \le R_0$. Moreover, this particular stage must have been attained after the (t_1+1)th jump by the cop and prior to the moment, when $C_{0,0} - R_0$ becomes positive for the first time. Since $D_0^{0(0)}$ was even, the cop and the robber must have taken the same number (say t_2) of jumps before reaching the above mentioned stage. If $C_{0,0}^{(t_2)} = R_0^{(t_2)}$, then the game terminates immediately, ruling out the option for $C_{0,0} - R_0$ to become positive. Otherwise, if $C_{0,0}^{(t_2)} < R_0^{(t_2)}$, we apply Proposition 2 to show that $C_{0,0}^{(t_2)} - R_0^{(t_2)}$ will never become positive as the game proceeds. This leads to a contradiction. □

Theorem 3. *If the initial configuration is such that $D_0^{0(0)}$ is even, then the strategy presented in Algorithm* 2 *ensures a win for the cop.*

Proof. Let $D_0^{0(0)}$ be even and assume that the cop fails to nab the robber. Lemma 4 implies that the cop will never backtrack along any of its coordinates. Moreover, the cop has to take an infinite number of jumps. Since we only consider finite grids, this leads to a contradiction. □

Note that the evenness of $D_0^{0(0)}$ does not necessarily ensure a win for the cop if the same strategy is extended in higher dimensions. For example, consider a three-dimensional grid where the cop is distance 2 away from the robber. This means that the cop and robber have the same coordinate in at least one dimension, say j. Whatever their positions, the robber can always make a move that takes her away from the cop in dimension j, and their mutual distance now becomes 3. At the next step, whatever the action taken by the cop, the distance will be at least 2. Thus, the robber can always evade capture.

Theorem 4. *The cop strategy in Algorithm* 2 *succeeds in capturing the robber on average half the times the game is repeated, given that the initial positions of the cop and the robber are decided uniformly at random.*

Proof. By definition, $D_0^{0(0)} = D_{0,0}^{0(0)} + D_{0,1}^{0(0)} = |C_{0,0}^{(0)} - R_0^{(0)}| + |C_{0,1}^{(0)} - R_1^{(0)}|$. $C_{0,0}^{(0)}, R_0^{(0)}, C_{0,1}^{(0)}$, and $R_1^{(0)}$ are each chosen uniformly at random. Hence each of these is expected to be odd (or even) half of the times, and so will be each of $|C_{0,0}^{(0)} - R_0^{(0)}|$ and $|C_{0,1}^{(0)} - R_1^{(0)}|$, and their sum. Now the result follows immediately from Theorem 3. □

If $D_0^{0(0)}$ is even, then the cop always moves in a fixed direction along coordinate 0 (as well as along coordinate 1). The robber will be caught within $O(d_0 + d_1)$ jumps of the cop (recall that the game is being played in a $d_0 \times d_1$ grid). A cop following Algorithm 2 can decide, in constant time, where to jump. Hence the time complexity of strategy S is also $O(d_0 + d_1)$. Here we exclude all initial configurations with an odd value of $D_0^{0(0)}$, as the robber can perpetually evade capture in such cases.

Now consider the situation, where initially the cop and the robber are situated at diagonally opposite corners of the grid and the robber's strategy dictates that she stays as close as possible to her initial position. Obviously the cop will have to take at least $\Omega(d_0 + d_1)$ jumps to catch the robber. We thus have the following result.

Theorem 5. *Unless the initial configuration is such that the robber has the privilege of evading capture indefinitely, the cop strategy in Algorithm* 2 *ensures a win for the cop in asymptotically optimal time.*

5. Conclusion

We have analyzed the cops and robber game in an n-dimensional grid structure and show that n cops are both necessary and sufficient for capturing the robber. We have presented a set of cop strategies which satisfy this sufficiency condition. Moreover, in a two-dimensional grid, we have shown that even a single cop can catch the robber in certain cases. In our future work, we plan to investigate whether such strategies exist in general in n dimensions, i.e., strategies that would guarantee capture of the robber in some special cases with less than n cops.

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