On the connectedness of a centralizer

Barbara A. Shipman

Department of Mathematics, The University of Texas at Arlington, Arlington, TX 76019-0408, USA

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Abstract

This letter finds the connected components of certain subgroups of a complex torus that arise in completing the Hamiltonian flows of the full complex Toda lattice.

The group studied in this letter arises when one looks at the completed flows of the full Kostant–Toda lattice, a Hamiltonian system whose equations, written in Lax form, are

\[ \dot{X}(t) = [X(t), L(t)], \] (1)

where \( X(t) \) is a complex \( n \times n \) matrix with 1’s on the superdiagonal and zeros above and \( L(t) \) is the strictly lower triangular part of \( X(t) \). The flow is isospectral so that the functions \( \frac{1}{k} \text{tr}(X^k) \) are constants of motion. For \( k = 2, \ldots, n \), the Hamiltonian flows generated by these functions are not in general complete. However, when the flows are embedded into the flag manifold \( SL(n, \mathbb{C})/B \) (where \( B \) is the upper triangular subgroup), they generate a group action and are thus completed. When the eigenvalues of the system are distinct, this group is the complex torus \( (\mathbb{C}^*)^{n-1} \), whose action on the flag manifold is extremely well studied (see [1] for this part of the picture).

We are interested in the subgroup \( A \) of \( SL(n, \mathbb{C}) \) generated by the flows when there are \( r < n \) different eigenvalues (so that at least one eigenvalue has multiplicity greater than one). It is shown in [2] that \( A \) is composed of \( r \) blocks along the diagonal, where the \( i \)th block is a \( d_i \times d_i \) matrix of the form

\[ A_i = \begin{pmatrix} a_i & b_{i1} & \cdots & b_{i, d_i-1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & b_{i1} & 1 \\ a_i & & & \end{pmatrix} : a_i \in \mathbb{C}^*, \ b_{i1}, \ldots, b_{i, d_i-1} \in \mathbb{C} \]

with \( \prod_{i=1}^{r} a_i^{d_i} = 1 \). A acts on \( SL(n, \mathbb{C})/B \) by multiplication on the right. Let \( K \) be the maximal diagonal subgroup of \( A \) and let \( U \) be the unipotent subgroup obtained by setting all the diagonal entries equal to 1. \( K \) and \( U \) commute, and

E-mail address: bshipman@uta.edu.

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A = U × K. The subgroup of A that fixes all points in the flag manifold is the discrete group D consisting of constant multiples of the identity where the constants are the nth roots of unity. Since D ⊆ K, the group A/D = U × K/D acts effectively on SL(n, C)/B.

Our program in understanding the action of A on the flag manifold has been to first look at the unipotent and diagonal parts separately. For example, [3] describes the fixed point set of K, the connected component of the identity in K (one can check that the fixed point sets of K and K_0 are the same), and [4,5] study the action of U. As part of this program, we will show that K has gcd(d_1, . . . , d_r) connected components and that K/D is connected. Since K is locally path connected, its connected components coincide with its path connected components. So it suffices to show that K has gcd(d_1, . . . , d_r) path connected components. (From a purely algebraic view, K may be obtained as the centralizer of a unipotent element of SL(d_1, C) × · · · × SL(d_r, C), such as the matrix where each d_1 × d_1 block has 1’s on and above the diagonal and zeros below.)

An arbitrary element of K will be denoted as [a_1]^{d_1} · · · [a_r]^{d_r}, where a_i is the element on the diagonal of the ith block of K. (The superscripts d_i will sometimes be omitted when it is clear what they are.) Consider the elements [ω^{q}_{d_1}]^{d_1}[1]^{d_2} · · · [1]^{d_r}, where ω_{d_1} is the primitive d_1-root of unity with the smallest positive angle (ω_{d_1} = exp 2πi/d_1) and q is an integer with 1 ≤ q ≤ d_1. We will first show that every element of K is path connected to one of these.

Given [a_1]^{d_1} · · · [a_r]^{d_r}, choose paths a_i(t), i = 2, . . . , r, such that a_i(0) = a_i and a_i(1) = 1. Since a_1^{d_1} = (∏_{i=2}^{r}a_i^{d_i})^{-1}, a_1 is a d_1-root of (∏_{i=2}^{r}a_i^{d_i})^{-1}. Define a_1(t) by

a_1(0) = a_1 and a_1(t)^{d_1} = ∏_{i=2}^{r}(a_i(t)^{d_i})^{-1}

This uniquely determines a_1(t). (Given a path p(t), the branch of d_1√p(t) chosen at t = 0 determines the branch for all t, since we require that d_1√p(t) be continuous.) Since a_i(1) = 1 for i = 2, . . . , r, a_i(1)^{d_1} = 1. So a_1(1) is a d_1-root of 1. Thus, [a_1]^{d_1} · · · [a_r]^{d_r} is path connected to [ω^{q}_{d_1}]^{d_1}[1]^{d_2} · · · [1]^{d_r}, for some q with 1 ≤ q ≤ d_1.

It suffices then to determine which [ω^{q}_{d_1}]^{d_1}[1]^{d_2} · · · [1]^{d_r} are path connected to each other. We can restrict the paths between them to those where every entry has unit modulus. For, suppose there is a path [a_1(t)]^{d_1} · · · [a_r(t)]^{d_r} from g = [ω^{q}_{d_1}]^{d_1}[1]^{d_2} · · · [1]^{d_r} to g̃ = [ω^{q}_{d_1}]^{d_1}[1]^{d_2} · · · [1]^{d_r}. Then [ω^{q}_{d_1}]^{d_1}[1]^{d_2} · · · [1]^{d_r} is also a path from g to g̃. Notice that it has the form

=e^{(d_2 + 1+ ω(1))2πi/d_1}e^{ω(2)2πi/d_2} . . . e^{ω(r)2πi/d_r},

where a_i(0) ∈ Z for all i, a_i(1) ∈ Z for i = 2, . . . , r, and

d_1a_1(t) + · · · + d_rα_r(t) = n ∈ Z,

where n is the same for all t.

Lemma. Suppose gcd(d_1, d_k) = p_k. Then e^{ρ_1 2πi/d_1}g_1[1]^{d_2} · · · [1]^{d_r} is path connected to the identity.

Proof. Since gcd(d_1, d_k) = p_k, there are integers n_1 and n_k such that n_1d_1 + n_kd_k = p_k. So

=e^{ρ_1 2πi/(d_1)}g_1[d_1] · · · [1]e^{n_1 2πi/d_1}d_1[1] · · · [1]

is a path in K from the identity (at t = 0) to e^{ρ_1 2πi/d_1}g_1[1] · · · [1] (at t = 1).

Proposition. K has exactly gcd(d_1, . . . , d_r) connected components.

Proof. Let p = gcd(d_1, . . . , d_r). Then there are positive integers g_k such that gcd(d_1, d_k) = g_kp and gcd(g_2, . . . , g_r) = 1. By the lemma, e^{ρ_1 2πi/d_1}g_1[1] · · · [1] is path connected to the identity, via a path s_k(t). Let n_2, . . . , n_r be integers such that n_2g_2 + · · · + n_rg_r = 1. Then s(t) = s_2(t)^n_2 · · · s_r(t)^n_r is a path in K from e^{ρ_1 2πi/d_1}g_1[1] · · · [1] to the identity. Furthermore, for any integers a and m, s(t) = s^m(t) · (e^{ρ_1 2πi/d_1}g_1[1] · · · [1]) is a path from e^{ρ_1 2πi/d_1}g_1[1] · · · [1] to e^{ρ_1 2πi/d_1}g_1[1] · · · [1]. Since we have already seen that any element of K is path connected to an element of the form e^{ρ_1 2πi/d_1}g_1[1] · · · [1], this shows that K has at most p path connected components.
To see that $K$ has at least $p$ path connected components, let $h(t)$ be a path in $K$ from $[e^{\frac{2\pi i}{n}}]d_1[1] \cdots [1]$ to an element of the form $[z]^d[1] \cdots [1]$

$$h(t) = [e^{(\frac{\phi_1(t)+\beta_1(t)}{n}+\beta_2(t)+\cdots+\beta_r(t))2\pi i}]d_1 \cdots [e^{\beta_r(t)2\pi i}]d_r,$$

where

$$d_1\beta_1(t) + d_2\beta_2(t) + \cdots + d_r\beta_r(t) = n \in \mathbb{Z},$$

$$\beta_i(0) \in \mathbb{Z} \quad \forall \quad i, \quad \text{and} \quad \beta_i(1) \in \mathbb{Z}, \quad i = 2, \ldots, r.$$

Notice that $p|n$ since $p = \gcd(d_1, \ldots, d_r)$. So

$$h(1) = [e^{\frac{a+n-(d_2\beta_2(1)+\cdots+d_r\beta_r(1))2\pi i}{d_1}}]d_1[1] \cdots [1]$$

$$= \left[e^{\frac{a+mp}{d_1}2\pi i}\right]d_1[1] \cdots [1],$$

where $m$ is an integer. Thus, the elements $[e^{\frac{q}{d_1}2\pi i}]d_1[1] \cdots [1]$ with $1 \leq q \leq p$ lie in distinct connected components of $K$. ■

Finally, to see that $K/D$ is connected, observe that the element $[e^{\frac{2\pi ik}{n}}] \cdots [e^{\frac{2\pi ik}{n}}]$ of $D$ is connected to the element $[e^{\frac{2\pi ik}{n}}]d_1[1] \cdots [1]$ of $K$ via the path

$$[e^{\frac{2\pi ik}{n}d_1(i+d_2\beta_2(1)+\cdots+d_r\beta_r(1))}] [e^{\frac{2\pi ik}{n}(1-t)}] \cdots [e^{\frac{2\pi ik}{n}(1-t)}].$$

Since $D$ contains an element in each connected component of $K$, $K/D$ is connected.

References


