On mixed Ramsey numbers

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Abstract

For positive integers \(m\) and \(n\) the classical Ramsey number \(r(m, n)\) is the least positive integer \(p\) such that if \(G\) is any graph of order \(p\) then either \(G\) contains a subgraph isomorphic to \(K_m\) or the complement \(\overline{G}\) of \(G\) contains a subgraph isomorphic to \(K_n\). Some authors have considered the concept of mixed Ramsey numbers. Given a graph theoretic parameter \(f\), an integer \(m\) and a graph \(H\), the mixed Ramsey number \(v(f; m; H)\) is defined as the least positive integer \(p\) such that if \(G\) is any graph of order \(p\), then either \(f(G) \geq m\) or \(\overline{G}\) contains a subgraph isomorphic to \(H\). In this paper we consider the problem of determining the mixed Ramsey numbers for vertex linear arboricity and some other generalizations of chromatic number. We discuss the above problem for various structures \(H\) such as the complete graph, the claw, the path and the tree. Further, we study the generalized mixed Ramsey number \(v(f; m_1, m_2, ..., m_t; H_1, H_2, ..., H_k)\), where the edge set of the complete graph is partitioned into \(k\) sets.

1. Introduction

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part, our notation and terminology follow that of Bondy and Murty [2]. Thus \(G\) is a graph with vertex set \(V(G)\), edge set \(E(G)\), \(v(G)\) vertices, \(e(G)\) edges, chromatic number \(\chi(G)\), minimum degree \(\delta(G)\) and maximum degree \(\Delta(G)\). However, we denote the complement of the graph \(G\) by \(\overline{G}\). For \(X \subseteq V(G)\), \(G[X]\) denotes the subgraph of \(G\) induced by the vertices of \(X\). Similarly, for \(A \subseteq E(G)\), \(G[A]\) denotes the subgraph of \(G\) induced by the edge set \(A\). We denote the degree of a vertex \(x\) in graph \(G\) by \(d_G(x)\).

\(K_n\) denotes the complete graph of order \(n\) and \(K_{m,n}\) denotes the complete bipartite graph with bipartitioning sets of order \(m\) and \(n\). Further, \(C_n\) and \(P_n\) denote the cycle and path of order \(n\), respectively. For disjoint graphs \(H\) and \(K\) the join \(H \vee K\) is the graph formed from \(H \cup K\) by joining every vertex of \(H\) to every vertex of \(K\). We write \(\bigvee_{i=1}^t F_i\) for the join of the graphs \(F_1, F_2, ..., F_t\).

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For positive integers $m$ and $n$ the \textit{ramsey number} $r(m, n)$ is the least positive integer $p$ such that for each graph $G$ on $p$ vertices either $G$ contains $K_m$ as a subgraph (written $K_m \subseteq G$) or $G$ contains $K_n$ as a subgraph ($K_n \subseteq \overline{G}$). A $k$-factorization of $G$ is a partition of the edges of $G$ into $k$ factors $F_1, F_2, \ldots, F_k$ such that $V(F_i) = V(G)$ for $1 \leq i \leq k$ and $E(G) = \bigcup_{i=1}^{k} E(F_i)$. We represent a $k$-factorization of $G$ by $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$. For graphs $H_1, H_2, \ldots, H_k$, we define the \textit{generalized ramsey number} $r(H_1, H_2, \ldots, H_k)$ to be the least positive integer $p$ such that for each factorization of the complete graph $K_p = F_1 \oplus F_2 \oplus \cdots \oplus F_k$, $H_i \subseteq F_i$ for some $i$, $1 \leq i \leq k$.

For a specified graph parameter $f$, positive integers $m_1, m_2, \ldots, m_l$ and graphs $H_{l+1}, H_{l+2}, \ldots, H_k$, we define the \textit{mixed ramsey number} $v(f; m_1, m_2, \ldots, m_l; H_{l+1}, H_{l+2}, \ldots, H_k)$ to be the least positive integer $p$ such that for each factorization $K_p = F_1 \oplus F_2 \oplus \cdots \oplus F_k$, either $f(F_i) > m_i$, for some $i$, $1 \leq i \leq l$; or $H_i \subseteq F_i$, for some $i$, $l + 1 \leq i \leq k$.

The graph theory literature contains a considerable number of papers concerning ramsey numbers and generalized ramsey numbers (see for example, the book by Graham et al. [8]). The objective of this paper is to consider mixed ramsey numbers.

The concept of a mixed ramsey number was introduced by Benedict et al. [1]. In their paper, they studied the relationship between the mixed ramsey number and generalized ramsey number when $f$ is the chromatic number. In Section 2, we generalize this work by establishing the relationship between the mixed ramsey number and the generalized ramsey number for a general graph colouring parameter.

Several authors have studied mixed ramsey numbers for the special case of $k = 2$ and $f$ a colouring parameter. For example, Lesniak et al. [10] investigated the function $v(\chi'; m; H)$, where $\chi'$ is the edge-chromatic number and $H$ a specified graph. Fink [6] and Cleves and Jacobson [4] investigated the total chromatic number.

In the final two sections of this paper we will consider the function $v(f; m; H)$ for graph parameters: vertex linear arboricity and point partition number.

2. Relationship between generalized and mixed Ramsey numbers

We begin our discussion by considering a specific colouring parameter. For an integer $n \geq 2$, the $n$-path chromatic number $\chi_n(G)$ of a graph $G$ is defined as the least number of colours needed to colour the vertices of $G$ such that no path $P_n$ of order $n$ is monocoloured. Note that $\chi_2(G)$ is the usual chromatic number of $G$. The following result is for the case $k = 2$.

\textbf{Theorem 2.1.} Let $m$ and $n$ be integers $\geq 2$ and $H$ a specified connected graph. Then
\[ v(\chi_n; m; H) = (r(P_n, H) - 1)(m - 1) + 1. \]

\textbf{Proof.} Let $G$ be a graph of order $(r(P_n, H) - 1)(m - 1) + 1$ such that $\chi_n(G) \leq m - 1$. We will prove that $\overline{G}$ contains $H$ and thus establish
\[ v(\chi_n; m; H) \leq (r(P_n, H) - 1)(m - 1) + 1. \]
Let $V_1, V_2, \ldots, V_{m-1}$ be a partition of $V(G)$ induced by a colouring of the vertices with $(m - 1)$ colours such that no colour class contains a $P_r$. Obviously there exists a colour class, say $V_1$, such that $|V_1| \geq r(P_r, H)$. Now from the definition of $r(P_r, H)$ and the fact that the subgraph $G[V_1]$ induced by $V_1$ has no $P_r$ we conclude that $G[V_1] \cong H$. Thus $G \cong H$ and so (2.1) is established.

Next we construct a graph $G^*$ of order $(r(P_r, H) - 1)(m - 1)$ such that $\chi(G^*) < (r(P_r, H) - 1)(m - 1) + 1$. This will establish that

$$\nu(\chi_n; m; H) \geq (r(P_n, H) - 1)(m - 1) + 1.$$  \hfill (2.2)

From the definition of $r(P_n, H)$ it follows that there exists a graph $F$ of order $r(P_n, H) - 1$ such that neither $F$ contains $P_n$ nor $\overline{F}$ contains $H$.

Now define $G^* = \bigvee_{i=1}^{m-1} F_i$, where $F_i \cong F$, $1 \leq i \leq m - 1$. Clearly $\chi_n(G^*) < m - 1$ and $G^* \not\cong H$. This proves (2.2). Now combining (2.1) and (2.2) the theorem is proved. 

Remarks

1. Chvátal [3] established that $r(T_n, K_t) = (n - 1)(t - 1) + 1$, where $T_n$ is any tree of order $n$. Now invoking Theorem 2.1 we have

$$\nu(\chi_n; m; K_t) = (n - 1)(t - 1)(m - 1) + 1.$$  \hfill (2.2)

2. Gerencsér and Gyárfás [7] proved that

$$r(P_n, P_t) = \max\{n + \lfloor t/2 \rfloor - 1, t + \lfloor n/2 \rfloor - 1\}.$$  \hfill (2.3)

Thus $\nu(\chi_n; m; P_t)$ can be determined.

3. $\nu(\chi_n; m; C_t)$ can be established using $r(P_n, C_t)$ derived in Faudree et al. [5].

We now generalize the colouring parameter to prohibit any specified graph in a colour class. Let $G_1$ be a specified graph. We define the $G_1$-chromatic number $\chi_{G_1}(G)$ of a graph $G$ as the least number of colours needed to colour the vertices of $G$ such that no colour class of $G$ contains $G_1$ as a subgraph.

Lemma 2.2. Let $m_1, \ldots, m_l (l \geq 1)$ be positive integers $\geq 2$ and $H_1, \ldots, H_k (l < k)$ be connected graphs. Then for a specified connected graph $G_1$,

$$\nu(\chi_{G_1}; m_1, m_2, \ldots, m_l; H_{l+1}, \ldots, H_k) \geq mr + 1,$$  \hfill (2.3)

where

$$m = \prod_{i=1}^{l} (m_i - 1); \quad r = r(H_1, \ldots, H_{l+1}, \ldots, H_k) - 1;$$

$$H_i \cong G_1, \quad 1 \leq i \leq l.$$
Proof. To prove (2.3), we construct a $k$-factorization $F_1^* \oplus \cdots \oplus F_k^*$ of $K_{m_r}$ such that $\chi_{G_1}(F_i^*) < m_i$, $1 \leq i \leq l$ and $F_i^* \not\supseteq H_i$, $l+1 \leq i \leq k$. We first construct an $l$-factorization $F_i^* \oplus \cdots \oplus F_i^*$ of $K_m$.

Let

$$P = \left\{ (p_1, \ldots, p_l) : \text{each } p_i \text{ is a positive integer with}\right. \left. \text{value at most } m_i - 1. \right\}$$

Note that $|P| = \prod_{i=1}^l (m_i - 1) = m$. We use the $m$ elements of $P$ as the vertex labels of the graph $K_m$. Define an $l$-factorization $F_1^* \oplus \cdots \oplus F_l^*$ of $K_m$ as follows: For $i = 1, \ldots, l$, let

$$F_i^* = \left\{ e : e \text{ is an edge joining } (p_1, \ldots, p_l) \text{ and } (p_1', \ldots, p_l') \text{ such that}\right. \left. p_j = p_j', 1 \leq j \leq i - 1 \text{ and } p_i \neq p_i'. \right\}$$

It is easy to check that $F_i^* \oplus \cdots \oplus F_l^*$ is a valid $l$-factorization of $K_m$.

Now for integers $i$ and $c$, $1 \leq c \leq m_i - 1$ and $1 \leq i \leq l$, we define

$$V_{i,c} = \{(p_1, \ldots, p_{i-1}, c, p_{i+1}, \ldots, p_l) \in P\}.$$ 

Note that $V_{i,c}$ is an independent set of vertices in $F_i$, for $1 \leq c \leq m_i - 1$. Thus $\chi(F_i^*) \leq m_i - 1$ and hence $\chi_{G_1}(F_i^*) \leq m_i - 1$ for each $i$, $1 \leq i \leq l$.

Next, from the definitions of generalized ramsey number and $r$ we know that there exists a $k$-factorization $F_1^* \oplus F_2^* \oplus \cdots \oplus F_k^*$ of $K_r$ such that $F_i^* \not\supseteq G_1$, $1 \leq i \leq l$ and $F_i^* \not\supseteq H_i$, $l+1 \leq i \leq k$.

Now using $K_m = F_1^* \oplus \cdots \oplus F_l^*$ and $K_r = F_1^* \oplus \cdots \oplus F_k^*$, construct a $K_{m_r}$ and its $k$-factorization $F_1^* \oplus \cdots \oplus F_k^*$ as follows:

1. First replace each vertex $v$ of $K_m$ by $\bar{v}$, and denote it by $(\bar{v})_v$.

2. Next construct $F_i^*$, $1 \leq i \leq l$ as follows:
   (a) Consider $F_i^*$. For each vertex $v$ of $F_i^*$, introduce the edges of $F_i^*$ in $(\bar{v})_v$, and denote the resulting subgraph by $(\bar{F}_i^*)_v$.
   (b) Join each vertex of $(\bar{F}_i^*)_v$ to each vertex of $(\bar{F}_j^*)_u$ if and only if $u$ and $v$ are adjacent in $F_i^*$.

   Denote the resulting subgraph by $F_i^*$.

3. Finally, for $i$, $l+1 \leq i \leq k$, define $F_i^*$ to be $m$ disjoint copies of $F_i^*$.

It is easy to observe that $F_1^* \oplus \cdots \oplus F_k^*$ is a valid factorization of $K_{m_r}$. For $i$, $1 \leq i \leq l$ we have $F_i^* \not\supseteq G_1$ and using this we note that $\chi_{G_1}(F_i^* \oplus \cdots \oplus F_k^*) = \chi_{G_1}(F_i^*) \leq m_i - 1$. Further, from the fact that $F_i^* \not\supseteq H_i$, for $i$, $l+1 \leq i \leq k$ we have $F_i^* \not\supseteq H_i$. This establishes (2.3) and proves the lemma. □

Theorem 2.3. Let $m_1, m_2, \ldots, m_l$ ($l \geq 1$) be positive integers $\geq 2$ and $H_{i+1}, \ldots, H_k$ ($l < k$) be connected graphs. Then for a specified connected graph $G_1$,

$$v(\chi_{G_1}; m_1, \ldots, m_l; H_{i+1}, \ldots, H_k) = [r(H_1, \ldots, H_l, H_{l+1}, \ldots, H_k) - 1] \prod_{i=1}^l (m_i - 1) + 1,$$

where $H_i \cong G_1$, $1 \leq i \leq l$. 

\[ \]
Proof. Let \( r = r(H_1, \ldots, H_l, H_{l+1}, \ldots, H_k) - 1 \) and \( m = \prod_{i=1}^{l} (m_i - 1) \). Now by Lemma 2.2 we have \( v(\chi_{G_1}; m_1, \ldots, m_l; H_{l+1}, \ldots, H_k) \geq mr + 1 \). To prove equality, we consider any factorization \( F_1 \oplus \cdots \oplus F_k \) of \( K_{mr+1} \) and establish the following claim:

either \( (\chi_{G_1}(F_i) \geq m_i, \text{ for some } i, 1 \leq i \leq l) \) or \( (F_i \supseteq H_i, \text{ for some } i, l+1 \leq i \leq k) \).

Define \( F_i^{(1)} = F_i, 1 \leq i \leq k \). Note that

\[ v(F_i^{(1)}) = r \prod_{i=1}^{l} (m_i - 1) + 1. \]

Start with \( j = 1 \) and perform the following steps (a)–(d) until termination:

(a) If \( j = l + 1 \) stop. Otherwise go to step (b).

(b) If \( \chi_{G_1}(F_i^{(j)}) \geq m_j \) stop. Otherwise, go to step (c).

(c) Consider a \( G_1 \)-colouring of the graph \( F_i^{(j)} \) with \( m_j - 1 \) colours and let the corresponding partition of the vertices of \( F_i^{(j)} \) be \( U_1^{(j)}, U_2^{(j)}, \ldots, U_{m_j-1}^{(j)} \). Note that \( F_i^{(j)}[U_i^{(j)}] \) does not contain \( G_1 \) as a subgraph for \( 1 \leq i \leq m_j - 1 \). Since \( v(F_i^{(j)}) \geq r \prod_{i=1}^{l} (m_i - 1) + 1 \), then by the pigeon hole principle, we have, without loss of generality,

\[ |U_j^{(i)}| \geq r \prod_{i=1}^{l} (m_i - 1) + 1. \]

Go to step (d).

(d) Define \( F_i^{(j+1)} = F_i^{(j)}[U_i^{(j)}], 1 \leq i \leq k \). Using the facts that \( V(F_1^{(1)}) \supseteq U_1^{(1)} \supseteq U_1^{(2)} \supseteq \cdots \supseteq U_1^{(j)} \) and \( F_i^{(j)}[U_i^{(j)}] \) is \( G_1 \)-free for \( 1 \leq i \leq j \), it follows that \( \chi_{G_1}(F_i^{(j+1)}) = 1, 1 \leq i \leq j \). Further, note that

\[ K_{|U_j^{(i)}|} = F_1^{(j+1)} \oplus F_2^{(j+1)} \oplus \cdots \oplus F_k^{(j+1)}. \]

Increase \( j \) by 1 and go to step (a).

The above procedure terminates in one of the following two cases:

Case (i): \( j \leq l \) and \( \chi_{G_1}(F_j^{(j)}) \geq m_j \).

In this case we have obviously

\[ \chi_{G_1}(F_j) \geq \chi_{G_1}(F_j^{(j)}) \geq m_j \]

and this proves the claim.

Case (ii): \( j = l + 1 \).

Note that we have

\[ K_{|U_j^{(i)}|} = F_1^{(l+1)} \oplus \cdots \oplus F_k^{(l+1)} \]

and

\[ |U_j^{(i)}| \geq r + 1 = r(H_1, \ldots, H_k). \]

Therefore \( F_i^{(l+1)} \supseteq H_i \) for some \( i, 1 \leq i \leq k \). Since \( F_i^{(l+1)} \not\supseteq G_1 \) for all \( i, 1 \leq i \leq l \), we have \( F_i^{(l+1)} \supseteq H_i \) for some \( i, l+1 \leq i \leq k \). This proves the claim for any \( K_{mr+1} = F_1 \oplus \cdots \oplus F_k \) and completes the proof of Theorem 2.3. \( \square \)
3. Vertex linear arboricity

The colouring parameter $\chi_{G_1}$ forces the colouring of vertices to be such that a colour class is free of $G_1$. Thus the results of the last section deal with mixed ramsey numbers of colouring parameters defined by forbidding a specified graph structure in a colour class. In this section we will present results on mixed ramsey numbers of colouring parameters defined through the insistence of a specified graph structure.

A linear forest is a forest where every component is a path. The vertex linear arboricity $\rho(G)$ of a graph $G$ is the minimum number of colours required to colour the vertices of $G$ such that the subgraph induced on each colour class is a linear forest.

The vertex arboricity $a(G)$ of a graph $G$ is the minimum number of colours required to colour the vertices of $G$ such that the subgraph induced on each colour class is acyclic.

Lesniak-Foster [9] has proved that

$$v(a; m; K_t) = 1 + (2t - 2)(m - 1) \quad (3.1)$$

and

$$v(a; m; T_t) = 1 + t(m - 1), \quad (3.2)$$

where $T_t$ is a tree of order $t \geq 2$.

Noting that $\rho(G) \geq a(G)$ for any graph $G$, we have the following inequality:

$$v(\rho; m; H) \leq v(a; m; H), \quad (3.3)$$

where $H$ is a specified graph. We now determine $v(\rho; m; H)$ for some special cases of $H$.

**Theorem 3.1.** Let $m$ and $t$ be positive integers $\geq 2$. Then

$$v(\rho; m; K_t) = (2t - 2)(m - 1) + 1$$

and

$$v(\rho; m; K_{1,t}) = (t + 1)(m - 1) + 1.$$  

**Proof.** From the inequalities (3.1) and (3.3) we have

$$v(\rho; m; K_t) \leq (2t - 2)(m - 1) + 1.$$  

To establish equality we construct a graph $G^*$ of order $(2t - 2)(m - 1)$ such that $\rho(G^*) \leq m - 1$ and $G^*$ does not contain $K_t$ as a subgraph. Define

$$G^* \cong \bigvee_{i=1}^{m-1} F_i,$$

where $F_i \cong P_{2t-2}$, $1 \leq i \leq m - 1$.

Clearly $\rho(G^*) \leq m - 1$ and $G^* \nneq K_t$. 


From inequalities (3.2) and (3.3) it follows that
\[ v(\rho; m; K_{1,t}) \leq 1 + (t + 1)(m - 1). \]
To establish equality we define \( G^* \) to be the join of \( m - 1 \) copies of \( P_{t-1} \). Clearly \( \rho(G^*) \leq m - 1 \) and \( G^* \not\cong K_{1,t} \).

This completes the proof of Theorem 3.1. \( \square \)

We next determine \( v(\rho; m; T_t) \), where \( T_t \) is a tree of order \( t \) different from \( K_{1,t-1} \). We first prove a necessary lemma.

**Lemma 3.2.** Let \( T \) and \( F \) be forests of order \( k \) (\( \geq 4 \)) different from \( K_{1,k-1} \). Then \( F \) contains \( T \) as a subgraph.

**Proof.** It is enough to prove the lemma when both \( T \) and \( F \) are trees. We proceed by induction on \( k \). The lemma can easily be verified for \( k = 4 \) and 5. Assume that it is true for trees of order \( k - 2 \) (\( \geq 4 \)). We prove the result for trees of order \( k \).

Let \( T \) and \( F \) be trees different from \( K_{1,k-1} \) of order \( k \). It is easy to see that there exist vertices \( u \) and \( v \) in \( T \) such that

(i) \( d_T(u) = d_T(v) = 1 \),

(ii) the neighbour of \( u \) is different from the neighbour of \( v \), and

(iii) \( T - \{u, v\} \) is not isomorphic to \( K_{1,k-3} \).

Let \( u' \) and \( v' \) be the neighbours of \( u \) and \( v \) respectively in \( T \). Similarly, let \( x \) and \( y \) be vertices of \( F \) such that conditions (i)–(iii) are satisfied. Further, let \( x' \) and \( y' \) be the neighbours in \( F \) of \( x \) and \( y \), respectively. From the induction hypothesis we have
\[ T - \{u, v\} \cong F - \{x, y\}. \]

Thus we have a 1–1 mapping \( g \) from \( V(T - \{u, v\}) \) to \( V(F - \{x, y\}) \) such that if \( (x, y) \in E(T - \{u, v\}) \) then \( (g(x), g(y)) \in E(F - \{x, y\}) \).

Now we extend the mapping \( g \) to \( V(T) \) as follows: Define

(i) \( g(u) = x \) and \( g(v) = y \) if \( g(u') \neq x' \) and \( g(v') \neq y' \) and

(ii) \( g(u) = y \) and \( g(v) = x \), otherwise.

It is easy to check that \( g \) is a one–to–one mapping from \( V(T) \) to \( V(F) \) such that if \( (x, y) \in E(T) \) then \( (g(x), g(y)) \in E(F) \). This establishes \( T \cong F \) and completes the proof of Lemma 3.2. \( \square \)

**Theorem 3.3.** For integers \( m \geq 2 \) and \( t \geq 4 \), we have
\[ v(\rho; m; T_t) = 1 + (t - 1)(m - 1), \]
where \( T_t \) is a tree of order \( t \), different from \( K_{1,t-1} \).

**Proof.** Let \( G^* \) be a graph isomorphic to \( \bigvee_{i=1}^{m-1} F_i \), where \( F_i \cong P_{t-1} \), for \( 1 \leq i \leq m - 1 \). Clearly \( \rho(G^*) = m - 1 \) and \( \bar{G}^* \) does not contain \( T_i \). Therefore
\[ v(\rho; m; T_t) \geq 1 + (t - 1)(m - 1). \]
To prove the reverse inequality let $G$ be a graph of order $1 + (t - 1)(m - 1)$ such that $\rho(G) \leq m - 1$. Consider a colouring of $V(G)$ with $m - 1$ colours such that each colour class is a linear forest. Let $V_i$ be the set of vertices of $G$ assigned to colour $i$, for $1 \leq i \leq m - 1$. Without loss of generality let $|V_1| = \max |V_i|$. Then $|V_1| \geq t$. Note that $G[V_1]$ is a linear forest different from $K_{1,t-1}$. Let $U \subseteq V_1$ be such that $|U| = t$. Then by Lemma 3.2, $G[U] \supseteq T_t$, and hence $G \supseteq T_t$. This establishes the inequality

$$v(\rho; m; T_t) \leq 1 + (t - 1)(m - 1)$$

and completes the proof of Theorem 3.3. $\square$

4. Point partition numbers

A graph $G$ is said to be $k$-degenerate if and only if for every induced subgraph $F$ of $G$, $\delta(F) \leq k$. Note that a graph is 0-degenerate if and only if it is totally disconnected. Also a graph is 1-degenerate if and only if it is a forest. We first state the following two lemmas needed for proving our results.

**Lemma 4.1** (Lick and White [11]). For integers $k \geq 2$ and $t \geq 2$, if $G$ is a $k$-degenerate graph of order $k + t$ for minimal $k$, then

$$e(G) \leq k(t + k) - \binom{k + 1}{2}.$$  

**Lemma 4.2** (Woodall [12]). Let $k$ and $t$ be positive integers $\geq 2$ and $G$ a graph of order $k + t$ which contains no $P_t$. Then

$$e(G) \leq \binom{t - 1}{2} + \binom{k + 1}{2}.$$  

For a non-negative integer $k$, the point partition number $\rho_k(G)$ of a graph $G$ is the least positive integer $l$ such that $V(G)$ can be partitioned into $l$ sets $V_1$, $V_2$, ..., $V_l$ where $G[V_i]$ is $k$-degenerate for all $i$, $1 \leq i \leq l$. Note that $\rho_0(G)$ and $\rho_1(G)$ are the usual chromatic number and the vertex arboricity of $G$, respectively.

We define $g(k, t)$ to be the least positive integer $\lambda$ such that if $G$ is a graph of order $\lambda$, then either $G$ is not $k$-degenerate or $G$ contains $K_t$.

We determine $g(k, t)$ in the following theorem.

**Theorem 4.3.** For integers $k \geq 0$ and $t \geq 2$ we have

$$g(k, t) = (k + 1)(t - 1) + 1.$$  

**Proof.** Let $G^* \cong \bigcup_{i=1}^{t-1} F_i$, where $F_i \cong K_{k+1}$, for all $i$, $1 \leq i \leq t - 1$. Clearly $G^*$ is $k$-degenerate and $G^* \not\supseteq K_t$. Thus we have

$$g(k, t) \geq (k + 1)(t - 1) + 1.$$  


To establish equality, consider a graph $G$ of order $(k + 1)(t - 1) + 1$ such that $G$ is $k$-degenerate. We shall prove that $\bar{G} \cong K_t$.

Start with $l = 1$, $V_0 = V(G)$ and perform the following steps for $l, 1 \leq l \leq t - 1$.

(a) Choose a vertex $x_l$ of minimum degree in $G[V_{l-1}]$.

(b) Define $V_l$ to be the set of vertices of $G[V_{l-1}]$ that are not adjacent to $x_l$ in $G[V_{l-1}]$.

Note that $|V_l| \geq (k + 1)(t - 1) - l(k - l + 1)$, for $1 \leq l \leq t - 1$. Let $x_l \in V_{l-1}$. Note that $x_1, x_2, \ldots, x_t$ are mutually non-adjacent vertices in $G$. Therefore $\bar{G} \cong K_t$. This completes the proof of Theorem 4.3. $\square$

**Theorem 4.4.** For integers $k \geq 0$ and $t \geq 2$, we have

$$v(\rho_k; m; K_t) = (k + 1)(t - 1)(m - 1) + 1.$$ 

**Proof.** Define $G^*$ as the join of $(m - 1)$ copies of $\cup_{i=1}^{t-1} F_i$, where $F_i \cong K_{k+1}$, $1 \leq i \leq t - 1$. It can easily be seen that $\rho_k(G^*) = m - 1$ and $\bar{G}^* \not\cong K_t$, Therefore we have,

$$v(\rho_k; m; K_t) \geq (k + 1)(t - 1)(m - 1) + 1.$$ 

Now equality can be established using Theorem 4.3. $\square$

In the remaining part of this section we attempt to determine $v(\rho_k; m; P_t)$. We define $h(k, t)$ as the least positive integer $\lambda$ such that if $G$ is a graph of order $\lambda$, then either $G$ is not $k$-degenerate or $G$ contains $P_t$ as a subgraph.

We now determine $v(\rho_k; m; P_t)$ in terms of $h(k, t)$.

**Theorem 4.5.** For integers $k$ and $t$, $k \geq 0$ and $t \geq 2$, we have

$$v(\rho_k; m; P_t) = (h(k, t) - 1)(m - 1) + 1.$$ 

where $h(k, t)$ is the least positive integer $\lambda$ such that if $G$ is a graph of order $\lambda$, then either $G$ is not $k$-degenerate or $\bar{G} \cong P_t$.

**Proof.** From the definition of $h(k, t)$ it follows that there exists a graph $F$ of order $h(k, t) - 1$ such that $F$ is $k$-degenerate and $\bar{F} \not\cong P_t$. Let $G^*$ be the join of $m - 1$ copies of $F$. Clearly $\rho_k(G^*) = m - 1$ and $\bar{G}^* \not\cong P_t$. Thus we have

$$v(\rho_k; m; P_t) \geq (h(k, t) - 1)(m - 1) + 1.$$ 

To prove the reverse inequality, consider a graph $G$ of order $(h(k, t) - 1)(m - 1) + 1$ such that $\rho_k(G) \leq m - 1$. This implies that there exists a subset $U$ of $V(G)$ such that $|U| \geq h(k, t)$ and $G[U]$ is $k$-degenerate. From the definition of $h(k, t)$ it follows that $\bar{G}[U] \cong P_t$ and hence $\bar{G} \cong P_t$. This establishes the inequality

$$v(\rho_k; m; P_t) \leq (h(k, t) - 1)(m - 1) + 1$$ 

and completes the proof of Theorem 4.5. $\square$
In the following we address the problem of determining $h(k, t)$. It is easy to see that $h(0, t) = t$ and $h(1, t) = t + 1$.

**Theorem 4.6.** For integers $k \geq 2$ and $t \geq 2$ we have

$$h(k, t) \geq t + k.$$  

Further, equality holds whenever $t \geq \frac{1}{2}k(k + 1) + 2$.

**Proof.** Define $G^* = K_k \cup \bar{K}_{t-1}$. Obviously $G^*$ is $k$-degenerate and $\bar{G}^* \not\subseteq P_t$. Thus we have

$$h(k, t) \geq t + k - 1 + 1 = t + k.$$  

Now we assume that $t \geq \frac{1}{2}k(k + 1) + 2$ and establish the equality $h(k, t) = t + k$.

Let $G$ be a graph of order $t + k$ such that $G$ is $k$-degenerate for minimal $k$ and $\bar{G} \not\subseteq P_t$. Now from Lemmas 4.1 and 4.2 we have

$$e(G) \leq k(t + k) - \binom{k + 1}{2}$$  

and $e(\bar{G}) \leq \binom{t - 1}{2} + \binom{k + 1}{2}$.

Adding these two inequalities we get

$$\binom{t + k}{2} = e(G) + e(\bar{G}) \leq \frac{1}{2}(2kt + 2k^2 + t^2 - 3t + 2),$$

and thus $t \leq \frac{1}{2}k(k + 1) + 1$. This contradicts our assumption that $t \geq \frac{1}{2}k(k + 1) + 2$. This contradiction proves the theorem. $\square$

**Theorem 4.7.** Let $k = \alpha t + \beta$, $t \geq 4$, $\alpha \geq 1$ and $0 \leq \beta \leq t - 1$. If $t \leq \beta + \alpha \theta$, where $\theta = \lceil t/2 \rceil - 1$, then $h(k, t) \geq k + t + \lfloor t/2 \rfloor$.

**Proof.** We will construct a graph $G^*$ of order $k + t + \lfloor t/2 \rfloor - 1$ such that $G^*$ is $k$-degenerate and $\bar{G}^* \not\subseteq P_t$. Let $\gamma = t - \theta$. Define

$$G^* = \bigvee_{i=1}^{\gamma + 2} F_i,$$

where $F_i \cong \bar{K}_\gamma \cup K_i$; $F_i \cong \bar{K}_\theta \cup K_i$, $2 \leq i \leq \alpha + 1$ and $F_{\gamma + 2} \cong \bar{K}_\beta$. Clearly $\delta(G^*) = k$ and it is easy to check that $G^*$ is $k$-degenerate and $\bar{G}^* \not\subseteq P_t$.

This proves that $h(k, t) \geq k + t + \lfloor t/2 \rfloor$. $\square$

**Remarks**

(i) From Theorems 4.5 and 4.6 it follows that

$$v(\rho_k; m; P_t) = (t + k - 1)(m - 1) + 1$$

whenever $t \geq \frac{1}{2}k(k + 1) + 2$. 

(ii) It can easily be checked that
\[ v(p_2; m; P_t) = (t + 1)(m - 1) + 1 \quad \text{for } t = 2, 3, 4 \]
\[ v(p_3; m; P_t) = (t + 2)(m - 1) + 1 \quad \text{for } 2 \leq t \leq 7. \]

References