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Extremal problems for ordered (hyper)graphs: applications of Davenport–Schinzel sequences

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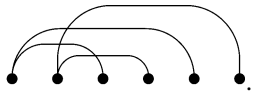
Abstract

We introduce a containment relation of hypergraphs which respects linear orderings of vertices, and we investigate associated extremal functions. We extend, using a more generally applicable theorem, the $n \log n$ upper bound on sizes of $(\{1,3\},\{1,5\},\{2,3\},\{2,4\})$-free ordered graphs with n vertices, due to Füredi, to the $n(\log n)^2(\log\log n)^3$ upper bound in the hypergraph case. We apply Davenport–Schinzel sequences and obtain almost linear upper bounds in terms of the inverse Ackermann function $\alpha(n)$. For example, we obtain such bounds in the case of extremal functions of forests consisting of stars all of whose centres precede all leaves.

1. Introduction and motivation

In this paper we shall investigate extremal problems on graphs and hypergraphs of the following type. Let $G = ([n], E)$ be a simple graph which has the vertex set $[n] = \{1, 2, \ldots, n\}$ and contains no six vertices $1 \le v_1 < v_2 < \cdots < v_6 \le n$ such that $\{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_4\}$, and $\{v_2, v_6\}$ are edges of G, that is, G has no *ordered* subgraph of the form

$G_0 =$  (1)

Determine the maximum possible number $g(n) = |E|$ of edges in G.

What makes this task hard is the linear ordering of $V = [n]$ and the fact that G_0 must not appear in G only as an ordered subgraph. If we ignore the ordering for a while, then

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the problem asks to determine the maximum number of edges in a simple graph G with n vertices and no $2K_{1,2}$ subgraph, and we can easily solve it. Clearly, if G has two vertices of degrees ≥ 3 and ≥ 5, respectively, or if G has ≥ 6 vertices of degrees 4 each, then a $2K_{1,2}$ subgraph must appear. Suppose G has no $2K_{1,2}$ subgraph. If G has a vertex of degree ≥ 5, it has at most $(2(n-1)+n-1)/2 = 3n/2 - 1.5$ edges. If all degrees are ≤ 4, the number of edges is at most $(3(n-5)+4\cdot 5)/2 = 3n/2 + 2.5$. On the other hand, the graph on $[n]$, in which $\deg(n) = n-1$ and $[n-1]$ induces a matching with $\lfloor (n-1)/2 \rfloor$ edges, has $n + \lfloor (n-1)/2 \rfloor - 1$ edges and no $2K_{1,2}$ subgraph. We conclude that in the unordered version of the problem the maximum number of edges equals $3n/2 + O(1)$.

The ordered version is considerably more difficult. Later in this section we prove that the maximum number of edges $g(n)$ satisfies

$$n \cdot \alpha(n) \ll g(n) \ll n \cdot 2^{(1+o(1))\alpha(n)^2} \tag{2}$$

where $\alpha(n)$ is the inverse Ackermann function. Recall that $\alpha(n) = \min\{m : A(m) \geq n\}$ where $A(n) = F_n(n)$, the Ackermann function, is defined as follows. We start with $F_1(n) = 2n$ and for $i \geq 1$ define $F_{i+1}(n) = F_i(F_i(\ldots F_i(1)\ldots))$ with n iterations of F_i. The function $\alpha(n)$ grows to infinity but its growth is extremely slow. We obtain (2) and some generalizations by reductions to *Davenport–Schinzel sequences* (DS sequences). Now we continue with a brief review of the results on DS sequences used in the rest of the paper; the summary of our results is given at the end of this section. The reader interested in more information on DS sequences and their applications in computational and combinatorial geometry may consult Agarwal and Sharir [1], Klazar [17], Sharir and Agarwal [19], and/or Valtr [22].

If $u = a_1a_2\ldots a_r$ and $v = b_1b_2\ldots b_s$ are two finite sequences (words) over a fixed infinite alphabet A, where A contains $\mathbf{N} = \{1, 2, \ldots\}$ and also some symbols $a, b, c, d, \ldots$, we say that v *contains* u and write $v \succ u$ if v has a subsequence $b_{i_1}b_{i_2}\ldots b_{i_r}$ such that for every p and q we have $a_p = a_q$ if and only if $b_{i_p} = b_{i_q}$. In other words, v has a subsequence that differs from u only by an injective renaming of the symbols. For example, $v = ccaaccbaa \succ 22\,244 = u$ because v has the subsequence $cccaa$. On the other hand, $ccaaccbaa \not\succ 12\,121$. A sequence $u = a_1a_2\ldots a_r$ is called *k-sparse*, where $k \in \mathbf{N}$, if $a_i = a_j, i < j$, implies $j - i \geq k$; this means that every interval in u of length at most k consists of distinct terms. The length r of u is denoted $|u|$. For two integers $a \leq b$ we write $[a, b]$ for the interval $\{a, a+1, \ldots, b\}$. For two functions $f, g : \mathbf{N} \to \mathbf{R}$ the notation $f \ll g$ is synonymous to the $f = O(g)$ notation; it means that there exists a constant $c > 0$ such that $|f(n)| < c|g(n)|$ for all $n \in \mathbf{N}$ with $g(n) \neq 0$.

The classical theory of DS sequences investigates, for fixed $s \in \mathbf{N}$, the function $\lambda_s(n)$ defined as the maximum length of 2-sparse sequences v over n symbols which do not contain the $(s+2)$-term alternating sequence $ababa\ldots$ $(a \neq b)$. The notation $\lambda_s(n)$ and the shift $+2$ are due to historical reasons. The term *DS sequences* refers to the sequences v not containing a fixed alternating sequence. The theory of *generalized DS sequences* investigates, for fixed sequence u using exactly k symbols, the function $\mathrm{ex}(u, n)$ defined as the maximum length of k-sparse sequences v which are over n symbols and $v \not\succ u$. Note that $\mathrm{ex}(u, n)$ extends $\lambda_s(n)$ since $\lambda_s(n) = \mathrm{ex}(ababa\ldots, n)$ where $ababa\ldots$ has length $s+2$. In the definition of $\mathrm{ex}(u, n)$ one has to require that v is k-sparse because

no condition, or even only $(k-1)$-sparseness, would allow an infinite v with $v \not\succ u$; for example, $v = 12121212\ldots \not\succ abca = u$ and v is 2-sparse (but not 3-sparse). An easy pigeonhole argument shows that $\mathrm{ex}(u,n) < \infty$ for every v.

DS sequences were introduced by Davenport and Schinzel [7] and strongest bounds on $\lambda_s(n)$ for general s were obtained by Agarwal et al. [2]. We shall need their bound

$$\lambda_6(n) \ll n \cdot 2^{(1+o(1))\alpha(n)^2} \tag{3}$$

(recall that $\lambda_6(n) = \mathrm{ex}(ababababab, n)$). Hart and Sharir [13] proved that

$$n\alpha(n) \ll \lambda_3(n) \ll n\alpha(n). \tag{4}$$

In Klazar [14] we proved that if u is a sequence using $k \geq 2$ symbols and $|u| = l \geq 5$, then for every $n \in \mathbf{N}$

$$\mathrm{ex}(u,n) \leq n \cdot k2^{l-3} \cdot (10k)^{2\alpha(n)^{l-4}+8\alpha(n)^{l-5}}; \tag{5}$$

it is easy to show that for $k = 1$ or $l \leq 4$ we have $\mathrm{ex}(u,n) \ll n$. In particular, for the sequence

$$u(k,l) = 12\ldots k12\ldots k\ldots 12\ldots k \tag{6}$$

with l segments $12\ldots k$ we have, for every fixed $k \geq 2$ and $l \geq 3$,

$$\mathrm{ex}(u(k,l),n) \leq n \cdot k2^{kl-3} \cdot (10k)^{2\alpha(n)^{kl-4}+8\alpha(n)^{kl-5}}. \tag{7}$$

We denote the factor multiplying n in (7) as $\beta(k,l,n)$. Thus

$$\beta(k,l,n) = k2^{kl-3}(10k)^{2\alpha(n)^{kl-4}+8\alpha(n)^{kl-5}}. \tag{8}$$

Let us see now how (3) and the lower bound in (4) imply (2). Let $G = ([n], E)$ be any simple graph not containing G_0 (given in (1)) as an ordered subgraph. Consider the sequence

$$v = I_1 I_2 \ldots I_n$$

over $[n]$ where I_i is the decreasing ordering of the list $\{j : \{j,i\} \in E \ \&\ j < i\}$. Note that $I_1 = \emptyset$ and $|v| = |E|$.

Lemma 1.1. *If $v \succ ababababab$ then G_0 is an ordered subgraph of G.*

Proof. Let v have an 8-term alternating subsequence

$$\ldots a_1 \ldots b_1 \ldots a_2 \ldots b_2 \ldots a_3 \ldots b_3 \ldots a_4 \ldots b_4 \ldots$$

where the appearances of two numbers $a \neq b$ are indexed for further discussion. We distinguish two cases. If $a < b$ then a_2, b_2, a_4, and b_4 lie, respectively, in four distinct intervals I_p, I_q, I_r, and I_s, $p < q < r < s$, (since every I_i is decreasing) and $b < p$ (since b_1 precedes a_2). Hence G_0 is an ordered subgraph of G. If $b < a$ then b_1, a_2, b_3, and a_4 lie, respectively, in four distinct intervals I_p, I_q, I_r, and I_s, $p < q < r < s$, and $a < p$. Again, G_0 is an ordered subgraph of G. $\square$

Thus v has no 8-term alternating subsequence. In v immediate repetitions may appear only on the transitions $I_i I_{i+1}$. Deleting at most $n-1$ (actually $n-2$ because $I_1 = \emptyset$) terms from v we obtain a 2-sparse subsequence w on which we can apply (3). We have

$$|E| = |v| \le |w| + n - 1 \le \lambda_6(n) + n - 1 \ll n \cdot 2^{(1+o(1))\alpha(n)^2}.$$

On the other hand, let $n \in \mathbf{N}$ and v be the longest 2-sparse sequence over $[n]$ such that $v \not\succ ababa$. It uses all n symbols and, by the lower bound in (4), $|v| > cn\alpha(n)$ for an absolute constant $c > 0$. Notice that every $i \in [n]$ appears in v at least twice. We rename the symbols in v so that for every i and j, $1 \le i < j \le n$, the first appearance of j in v precedes that of i; this affects neither the property $v \not\succ ababa$ nor the 2-sparseness. By an *extremal term of* v we mean the first or the last appearance of a symbol in v. The sequence v has exactly $2n$ extremal terms. We decompose v uniquely into intervals $v = I_1 I_2 \dots I_{2n}$ so that every I_i ends with an extremal term and contains no other extremal term. Every I_i consists of distinct terms because a repetition $\dots b \dots b \dots$ in I_i would force a 5-term alternating subsequence $\dots a \dots b \dots a \dots b \dots a \dots$ in v. We define a simple (bipartite) graph $G^* = ([3n], E)$ by

$$\{i, j\} \in E \Longleftrightarrow i \in [n] \ \&\ j \in [n+1, 3n] \text{ and } i \text{ appears in } I_{j-n}.$$

G^* has $3n$ vertices and $|E| = |v| > cn\alpha(n)$ edges. Suppose that G^* contains the forbidden ordered subgraph G_0 on the vertices $1 \le a_1 < a_2 < \dots < a_6 \le 3n$. By the definition of G^*, $a_1 a_2 a_1 a_2$ appears in v as a subsequence z and the four terms of z appear in $I_{a_3-n}, \dots, I_{a_6-n}$, respectively. Since $a_2 > a_1$, number a_2 must appear in v before z starts and therefore v contains a 5-term alternating subsequence, which is forbidden. So G^* does not contain G_0 and shows that

$$g(n) \gg n\alpha(n).$$

This concludes the proof of (2).

Open Problem 1.2. Narrow the gap $\lambda_3(n) \ll g(n) \ll \lambda_6(n)$ in (2). What is the precise asymptotics of $g(n)$?

Our example shows that the ordered version of a simple graph extremal problem may differ dramatically from the unordered one. Classical extremal theory of graphs and hypergraphs, which deals with unordered vertex sets, produced many results of great variety—see, for example, Bollobás [3, 4], Frankl [9], Füredi [11], and Tuza [20, 21]. However, only little attention has been paid to ordered extremal problems. The only systematic studies devoted to this topic known to us are Füredi and Hajnal [12] (bipartite graphs with ordered parts) and Brass et al. [6] (cyclically ordered graphs). We think that ordered extremal problems should be studied and investigated more intensively. First, for their intrinsic combinatorial beauty. Second, since they present to us new orders of growth of extremal functions which are not encountered in the classical theory: nearly linear extremal functions, like $n\alpha(n)$ or $n \log n$, seem characteristic for ordered extremal problems. Third, estimates coming from ordered extremal problems were successfully applied in combinatorial geometry (here often the right key to a problem turns out to

be some linear or partial ordering) and to obtain further applications it is desirable to understand more thoroughly combinatorial cores of these arguments.

Before summarizing our results, we return to DS sequences and show that the sequential containment $\prec$ can be naturally interpreted in terms of particular hypergraphs, (set) partitions. A sequence $u = a_1a_2 \dots a_r$ over the alphabet A may be viewed as a partition P of $[r]$ such that i and j are in the same block of P if and only if $a_i = a_j$. Thus blocks of P correspond to the positions of symbols in u. For example, $u = abaccba$ is the partition $\{\{1, 3, 7\}, \{2, 6\}, \{4, 5\}\}$. If $u = ([r], \sim_u)$ and $v = ([s], \sim_v)$ are two sequences given as partitions by equivalence relations, then $u \prec v$ if and only if there is an *increasing* injection $f : [r] \to [s]$ such that $x \sim_u y \Leftrightarrow f(x) \sim_v f(y)$ for every $x, y \in [r]$.

In this paper we investigate hypergraph containment generalizing both the ordered subgraph relation and the sequential containment. The containment and its associated extremal functions $\mathrm{ex}_e(F, n)$ and $\mathrm{ex}_i(F, n)$ are introduced in Definitions 2.1 and 2.2. The function $\mathrm{ex}_e(F, n)$ counts edges in extremal simple hypergraphs H not containing a fixed hypergraph F and the function $\mathrm{ex}_i(F, n)$ counts sums of edge cardinalities. In Theorem 2.3 we show that for many F one has $\mathrm{ex}_i(F, n) \ll \mathrm{ex}_e(F, n)$. Theorem 3.1 shows that if F is a simple graph, then in some cases good bounds on $\mathrm{ex}_e(F, n)$ can be obtained from bounds on the ordered graph extremal function $\mathrm{gex}(F, n)$. We apply Theorem 3.1 to prove in Theorem 3.3 that for $G_1 = (\{1, 3\}, \{1, 5\}, \{2, 3\}, \{2, 4\})$ one has $\mathrm{ex}_e(G_1, n) \ll n \cdot (\log n)^2 \cdot (\log \log n)^3$ and the same bound for $\mathrm{ex}_i(G_1, n)$; this generalizes the bound $\mathrm{gex}(G_1, n) \ll n \cdot \log n$ of Füredi. In another application, Theorem 3.5, we prove that for every forest F the *unordered* hypergraph extremal function $\mathrm{ex}_e^u(F, n)$ is $\ll n$. In Theorem 4.1 we generalize the bound (5) to hypergraphs. In Theorem 4.3 we prove that if F is a star forest, then $\mathrm{ex}_e(F, n)$ has an almost linear upper bound in terms of $\alpha(n)$; this generalizes the upper bound in (2). In the concluding section we introduce the notion of orderly bipartite forests and pose some problems.

This paper is a revised version of about one half of the material in the technical report [16]. We present the other half in [18].

2. Definitions and bounding weight by size

By a *hypergraph* $H = (E_i : i \in I)$ we shall understand a finite list of finite nonempty subsets E_i of $\mathbf{N} = \{1, 2, \dots\}$, called *edges*. H is *simple* if $E_i \neq E_j$ for every $i, j \in I$, $i \neq j$. H is a *graph* if $|E_i| = 2$ for every $i \in I$. H is a *partition* if $E_i \cap E_j = \emptyset$ for every $i, j \in I$, $i \neq j$. The elements of $\bigcup H = \bigcup_{i \in I} E_i \subset \mathbf{N}$ are called *vertices*. Note that our hypergraphs have no isolated vertices. The *simplification* of H is the simple hypergraph obtained from H by keeping from each family of mutually equal edges just one edge. The standard linear order on $\mathbf{N}$ induces a linear ordering on every vertex set $\bigcup H$ and this ordering is crucial for our extremal theory. It would be more precise to speak of *ordered hypergraphs* and *ordered graphs* but hopefully the shorter terms will cause no confusion.

Definition 2.1. Let $H = (E_i : i \in I)$ and $H' = (E'_i : i \in I')$ be two hypergraphs. H *contains* H', written $H \succ H'$, if there exist an *increasing* injection $F : \bigcup H' \to \bigcup H$ and

an injection $f : I' \to I$ such that

$$F(E'_i) \subset E_{f(i)}$$

for every index $i \in I'$. Else we say that H is H'*-free* and write $H \not\succ H'$.

The hypergraph containment $\prec$ extends the sequential containment and the ordered subgraph relation. $H = (E_i : i \in I)$ and $H' = (E'_i : i \in I')$ are *isomorphic* (as ordered hypergraphs) if there are an *increasing* bijection $F : \bigcup H' \to \bigcup H$ and a bijection $f : I' \to I$ such that $F(E'_i) = E_{f(i)}$ for every $i \in I'$. H' is a *reduction* of H if $I' \subset I$ and $E'_i \subset E_i$ for every $i \in I'$. Hence the containment $H' \prec H$ means that H' is isomorphic to a reduction of H. We call that reduction of H an H'*-copy* in H. For example, if $H' = (\{1\}, \{1\})$ (H' is a singleton edge repeated twice) then $H \not\succ H'$ if and only if H is a partition. Another example: if $H' = (\{1, 3\}, \{2, 4\})$ then H is H'-free if and only if H has no four vertices $a < b < c < d$ such that a and c lie in one edge of H while b and d lie in another edge.

The *order* $v(H)$ of $H = (E_i : i \in I)$ is the number of vertices $v(H) = |\bigcup H|$, the *size* $e(H)$ is the number of edges $e(H) = |H| = |I|$, and the *weight* $i(H)$ is the number of incidences between the vertices and the edges $i(H) = \sum_{i \in I} |E_i|$. Trivially, $v(H) \le i(H)$ and $e(H) \le i(H)$ for every H.

Definition 2.2. Let F be any hypergraph. We associate with F the extremal functions $\mathrm{ex}_e(F), \mathrm{ex}_i(F) : \mathbf{N} \to \mathbf{N}$, defined by

$$\mathrm{ex}_e(F, n) = \max\{e(H) : H \not\succ F \ \&\ H \text{ is simple } \&\ v(H) \le n\}$$
$$\mathrm{ex}_i(F, n) = \max\{i(H) : H \not\succ F \ \&\ H \text{ is simple } \&\ v(H) \le n\}.$$

We considered $\mathrm{ex}_e(F, n)$ and $\mathrm{ex}_i(F, n)$ implicitly already in Klazar [15]. Except for this paper, to our knowledge, this extremal setting is new and was not investigated before. Obviously, for every $n \in \mathbf{N}$ and F, $\mathrm{ex}_e(F, n) \le 2^n - 1$ and $\mathrm{ex}_i(F, n) \le n2^{n-1}$ but much better bounds can be usually given. The *reversal* of a hypergraph $H = (E_i : i \in I)$ with $N = \max(\bigcup H)$ is the hypergraph $\overline{H} = (\overline{E_i} : i \in I)$ where $\overline{E_i} = \{N - x + 1 : x \in E_i\}$. Thus reversals are obtained by reverting the linear ordering of vertices. It is clear that $\mathrm{ex}_e(F, n) = \mathrm{ex}_e(\overline{F}, n)$ and $\mathrm{ex}_i(F, n) = \mathrm{ex}_i(\overline{F}, n)$ for every F and n.

We give a few comments on Definitions 2.1 and 2.2. Note that our containment $\prec$ is not an induced one. For graphs, if $H_2 \succ H_1$ and H_2 is simple then H_1 is simple as well. But a simple hypergraph may contain nonsimple hypergraphs. In Definition 2.2 H must be simple because allowing all H would usually produce the value $+\infty$ (the simplicity of H may be dropped only for $F = (\{1\}, \{1\}, \ldots, \{1\})$). On the other hand, for the forbidden F we allow any hypergraph: F need not be simple and may have singleton edges. Another perhaps unusual feature of our extremal theory is that in H and F edges of all cardinalities are allowed; in extremal theories with forbidden substructures it is more common to have edges of just one cardinality. This led naturally to the function $\mathrm{ex}_i(F, n)$ which accounts for edges of all sizes. Trivially, $\mathrm{ex}_i(F, n) \ge \mathrm{ex}_e(F, n)$ for every hypergraph F and $n \in \mathbf{N}$. On the other hand, Theorem 2.3 shows that for many F one has $\mathrm{ex}_i(F, n) \ll \mathrm{ex}_e(F, n)$. In Definition 2.2 we take all H with $v(H) \le n$ that the extremal functions be automatically nondecreasing. Replacing $v(H) \le n$ with $v(H) = n$ would give more information on the

extremal functions but would also bring the complication that then extremal functions are not always nondecreasing. It happens for $F = (\{1\}, \{2\}, \ldots, \{k\})$ and we analyse this phenomenon in [16, 18].

Theorem 2.3. *Suppose that no edge of the hypergraph F precedes completely (in the linear ordering of $\bigcup F$) another edge. Let $p = v(F)$ and $q = e(F) > 1$. Then for every $n \in \mathbf{N}$,*

$$\mathrm{ex}_i(F, n) \le (2p-1)(q-1) \cdot \mathrm{ex}_e(F, n).$$

Proof. Let H be a simple hypergraph satisfying $\bigcup H = [m]$, $m \le n$, $H \not\succ F$ and $i(H) = \mathrm{ex}_i(F, n)$. We transform H in a new hypergraph H' by keeping all edges with less than p vertices and replacing every edge $E = \{v_1, v_2, \ldots, v_s\}$, $v_1 < v_2 < \cdots < v_s$, with at least p vertices by $t = \lfloor |E|/p \rfloor$ new p-element edges $\{v_1, \ldots, v_p\}$, $\{v_{p+1}, \ldots, v_{2p}\}, \ldots$, $\{v_{(t-1)p+1}, \ldots, v_{tp}\}$. H' may not be simple and we let H'' be the simplification of H'. Two observations: (i) no edge of H' repeats q or more times and (ii) H'' is F-free. If (i) were false, there would be q distinct edges $E_1, \ldots, E_q$ in H such that $|\bigcap_{i=1}^q E_i| \ge p$. But this implies the contradiction $F \prec H$. As for (ii), any F-copy in H'' may use from every $E \in H$ at most one new edge $E'' \subset E$ (each two new edges born from E are separated in the manner excluded in F) and so it is an F-copy in H as well. The observations and the definitions of H' and H'' give

$$\begin{aligned}\mathrm{ex}_i(F, n) = i(H) &\le \frac{(2p-1) \cdot i(H')}{p} \le \frac{(2p-1)(q-1) \cdot i(H'')}{p} \\ &\le (2p-1)(q-1) \cdot e(H'') \\ &\le (2p-1)(q-1) \cdot \mathrm{ex}_e(F, n).\end{aligned}$$

The last inequality follows from the fact that $\mathrm{ex}_e(F, n)$ is nondecreasing by definition. □

However, $\mathrm{ex}_i(F, n) \ll \mathrm{ex}_e(F, n)$ does not hold for $F = F_k = (\{1\}, \{2\}, \ldots, \{k\})$, $k \ge 2$, because $\mathrm{ex}_e(F_k, n) = 2^{k-1} - 1$ for $n \ge k-1$ and $\mathrm{ex}_i(F_k, n) = (k-1)n - (k-2)$ for $n > \max(k, 2^{k-2})$ ([16, 18]). Note that for $F = (\{1\})$ both extremal functions are undefined and that, since F_k is highly symmetric, the ordering of vertices is irrelevant for the containment $H \succ F_k$.

Open Problem 2.4. Prove that if F is not isomorphic to $(\{1\}, \{2\}, \ldots, \{k\})$ then $\mathrm{ex}_i(F, n) \ll \mathrm{ex}_e(F, n)$.

3. Bounding hypergraphs by means of graphs

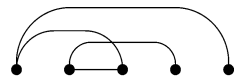
For a family of simple graphs R and $n \in \mathbf{N}$ we define

$$\begin{aligned}\mathrm{gex}(R, n) = \max\{e(G) : G \not\succ G' \text{ for all } G' \in R \ \&\ G \text{ is a simple graph} \\ \&\ v(G) \le n\}\end{aligned}$$

and for one simple graph G we write $\mathrm{gex}(G, n)$ instead of $\mathrm{gex}(\{G\}, n)$. Füredi proved

in [10], see also [12], that for

$$G_1 = (\{1,3\},\{1,5\},\{2,3\},\{2,4\}) = \tag{9}$$



one has

$$n \log n \ll \mathrm{gex}(G_1, n) \ll n \log n. \tag{10}$$

(In [10] and [12] the investigated objects are 0-1 matrices, which can be viewed as bipartite graphs with ordered parts, but in the case of G_1 the bounds are easily extended to all ordered graphs.) Attempts to generalize the upper bound in Eq. (10) to hypergraphs motivated the next theorem.

For $k \in \mathbf{N}$ we say that a simple graph G' is a *k-blow-up* of a simple graph G if for every edge colouring $\chi : G' \to \mathbf{N}$ using every colour at most k times there is a G-copy in G' with totally different colours, that is, χ is injective on the G-copy. For $k \in \mathbf{N}$ and a simple graph G we write $B(k, G)$ to denote the set of all k-blow-ups of G.

Theorem 3.1. *Let F be a simple graph with $p = v(F)$ and $q = e(F) > 1$ and let $B \subset B\left(\binom{p}{2}, F\right)$. If $f : \mathbf{N} \to \mathbf{N}$ is a nondecreasing function such that*

$$\mathrm{gex}(B, n) < n \cdot f(n)$$

for every $n \in \mathbf{N}$, then

$$\mathrm{ex}_e(F, n) < q \cdot \mathrm{gex}(F, n) \cdot \mathrm{ex}_e(F, 2f(n) + 1) \tag{11}$$

for every $n \in \mathbf{N}$, $n \geq 3$.

Proof. Let H be a simple hypergraph satisfying $\bigcup H = [m]$, $m \leq n$, $H \not\succ F$ and $e(H) = \mathrm{ex}_e(F, n)$. We put in H' every edge of H with more than 1 and less than p vertices, and for every $E \in H$ with $|E| \geq p$ we put in H' an arbitrary subset $E' \subset E$, $|E'| = p$. So $2 \leq |E| \leq p$ for every $E \in H'$ and no edge of H' repeats more than $q - 1$ times, for else we would have $H \succ F$. Let H'' be the simplification of H'. H'' is F-free. We have

$$e(H) \leq n + (q-1)e(H'').$$

Let G be the simple graph consisting of all edges E^* such that $E^* \subset E$ for some $E \in H''$. Observe that if $F' \in B$ and $F' \prec G$, then $F \prec H''$ and thus $F \prec H$. (For the edges $E^* \in G$ forming an F'-copy consider the colouring $\chi(E^*) = E$ where $E \in H''$ is such that $E^* \subset E$. Every colour is used at most $\binom{p}{2}$ times and therefore, since F' is a $\binom{p}{2}$-blow-up of F, we have an F-copy in G for which the correspondence $E^* \mapsto E$ is injective.) Hence $F' \prec G$ for no $F' \in B$. Let $v(G) = n'$; $n' \leq n$. We have

$$e(G) \leq \mathrm{gex}(B, n') < n' \cdot f(n').$$

There exists a vertex $v_0 \in \bigcup G$ such that

$$d = \deg_G(v_0) < 2f(n') \leq 2f(n).$$

Fix an arbitrary edge E_0^*, $v_0 \in E_0^* \in G$. Let $X \subset [n]$ be the union of all edges $E \in H''$ satisfying $E_0^* \subset E$ and m be the number of such edges in H''. We have the inequalities

$$m \le \mathrm{ex}_e(F, |X|) \qquad \text{and} \qquad |X| \le d + 1.$$

Thus

$$m \le \mathrm{ex}_e(F, |X|) \le \mathrm{ex}_e(F, d+1) \le \mathrm{ex}_e(F, 2f(n)+1).$$

We see that the two-element set E_0^* is contained in at least one and at most $\mathrm{ex}_e(F, 2f(n)+1)$ edges of H''. More generally, for every subhypergraph H_1'' of H'' there is a two-element set that is contained in at least one and at most $\mathrm{ex}_e(F, 2f(n)+1)$ edges of H_1''. It follows that there is a mapping M from H'' to two-element subsets of $[n]$ such that $M(E) \subset E$ for every $E \in H''$ and $|M^{-1}(E^*)| \le \mathrm{ex}_e(F, 2f(n)+1)$ for every $E^* \subset [n]$, $|E^*| = 2$. Let G' be the simple graph consisting of the image of M and $v(G') = n'$; $n' \le n$. Clearly, $e(H'') \le \mathrm{ex}_e(F, 2f(n)+1) \cdot e(G')$. The containment $F \prec G'$ implies, by the definition of G', that $F \prec H''$ and hence $F \prec H$, which is not allowed. Thus

$$e(G') \le \mathrm{gex}(F, n') \le \mathrm{gex}(F, n).$$

Putting it all together, we obtain (since $\mathrm{gex}(F, n) \ge n - 1$ if $q > 1$)

$$\begin{aligned} \mathrm{ex}_e(F, n) = e(H) &\le n + (q-1) \cdot e(H'') \\ &\le n + (q-1) \cdot \mathrm{ex}_e(F, 2f(n)+1) \cdot e(G') \\ &< q \cdot \mathrm{ex}_e(F, 2f(n)+1) \cdot \mathrm{gex}(F, n) \end{aligned}$$

for every $n \ge 3$. $\square$

We give three applications of this theorem. The first one is the promised generalization of the upper bound in (10). We need a technical lemma.

For fixed $k \in \mathbf{N}$ consider all simple graphs G having this structure: $\bigcup G = A \cup \{v\} \cup B \cup C$ with $A < v < B < C$, $|A| = k$, the vertex v has degree k and is connected to every vertex in A, every vertex in A has degree $2k+1$ and is besides v connected to k vertices in B and to k vertices in C, and G has no other edges. The edges incident with v are called *backward edges* and the edges incident with vertices in $B \cup C$ are called *forward edges*. We denote the set of all such graphs by $M(k)$.

Lemma 3.2. *Let G_1 be as defined in* (9) *and $M(k)$ be the above sets of graphs.*

1. *For every k, $M(3k+1) \subset B(k, G_1)$. In particular, $M(31) \subset B\left(\binom{5}{2}, G_1\right)$.*
2. *For every k, $\mathrm{gex}(M(k), n) \ll n \log n$.*

Proof. 1. Let $G \in M(3k+1)$ and $\chi : G \to \mathbf{N}$ be an edge colouring using each colour at most k times. We select in G two backward edges $E_1 = \{i, v\}$ and $E_2 = \{j, v\}$, $i < j < v$, with different colours. It follows that we can select in G two forward edges $E_3 = \{i, l\}$ and $E_4 = \{j, l'\}$ such that $v < l' < l$ and the four colours $\chi(E_i)$, $i = 1, \ldots, 4$ are distinct. Edges $E_1, \ldots, E_4$ form a G_1-copy on which χ is injective. Thus $G \in B(k, G_1)$.

2. Let $n \ge 2$ and G be any simple graph such that $\bigcup G = [n]$ and $G \not\succ F$ for every $F \in M(k)$. For each $i \in [n]$ we denote $J_i = \{E \in G : \min E = i\}$. In every J_i we mark the 1st, $(k+1)$th, $(2k+1)$th, …, $(pk+1)$th edge where $p = \lfloor |J_i|/k \rfloor - 1$ (the edges

in J_i are ordered by their endpoints). Then each two marked edges are separated by $k-1$ unmarked edges, the last marked edge is followed by at least $k-1$ unmarked edges, and we have marked $\lfloor |J_i|/k \rfloor > |J_i|/k - 1$ edges. The graph G' formed by all marked edges satisfies

$$e(G') > e(G)/k - n.$$

Also, for every edge $\{i, j\} \in G'$, $i < j$, there are at least $k-1$ edges $\{i, l\} \in G$ with $l > j$, and for every two edges $\{i, j\}, \{i, j'\} \in G'$, $i < j < j'$, there are at least $k-1$ edges $\{i, l\} \in G$ with $j < l < j'$. Now we proceed as in Füredi [10]. We say that $\{i, j\} \in G'$, $i < j$, has *type* (j, m), where $m \geq 0$ is an integer satisfying $2^m < n$, if there are two edges $\{i, l\}$ and $\{i, l'\}$ in G' such that $j < l < l'$ and $l - j \leq 2^m < l' - j$. Consider the partition

$$G' = G^* \cup G^{**}$$

where G^* is formed by edges with at least one type and G^{**} by edges without type. It follows from the definition of type and of G' that if k edges of G^* have the same type, then $F \prec G$ for some $F \in M(k)$ which is forbidden. Thus any type is shared by at most $k-1$ edges. Since the number of types is less than $n(1 + \log_2 n)$, we have

$$e(G^*) < (k-1)n + (k-1)n \log_2 n.$$

To bound $e(G^{**})$, we fix a vertex $i \in [n]$ and consider the endpoints $i < j_0 < j_1 < \cdots < j_{t-1} \leq n$ of all t edges $E \in G'$ which have no type and $\min E = i$. Let $d_r = j_r - j_{r-1}$ for $1 \leq r \leq t-1$ and $D = d_1 + \cdots + d_{t-1} = j_{t-1} - j_0$. If $d_1 \leq D/2$, then $d_1 \leq 2^m < D$ for some integer $m \geq 0$ and the edge $\{i, j_0\}$ would have type (j_0, m) because of the edges $\{i, j_1\}$ and $\{i, j_{t-1}\}$. Thus $d_1 > D/2$ and $D - d_1 < D/2$. By the same argument applied to $\{i, j_1\}$, $d_2 > (D - d_1)/2$ and thus $D - d_1 - d_2 < D/4$. In general, $1 \leq D - d_1 - \cdots - d_r < D/2^r$ for $1 \leq r \leq t-2$. Thus $t \leq \log_2 D + 2 < 2 + \log_2 n$. Summing these inequalities for all $i \in [n]$, we have

$$e(G^{**}) < 2n + n \log_2 n.$$

Altogether we have

$$e(G) < kn + k(e(G^*) + e(G^{**})) < (k^2 + 2k)n + k^2 n \log_2 n.$$

We conclude that $\mathrm{gex}(M(k), n) \ll n \log n$ and the constant in $\ll$ depends quadratically on k. $\square$

Theorem 3.3. *Let G_1 be the simple graph given in* (9). *We have the following bounds.*

1. $n \cdot \log n \ll \mathrm{ex}_e(G_1, n) \ll n \cdot (\log n)^2 \cdot (\log \log n)^3$.
2. $n \cdot \log n \ll \mathrm{ex}_i(G_1, n) \ll n \cdot (\log n)^2 \cdot (\log \log n)^3$.

Proof. 1. The lower bound follows from the lower bound in (10). To prove the upper bound, we use Theorem 3.1. By 2 of Lemma 3.2, we have $\mathrm{gex}(M(31), n) \ll n \log n$. Also, $\mathrm{gex}(G_1, n) \ll n \log n$ (by the upper bound in (10) or by $\mathrm{gex}(G_1, n) \leq \mathrm{gex}(M(k), n)$). By 1 of Lemma 3.2, we can apply Theorem 3.1 with $B = M(31)$. Starting with the trivial

bound $\mathrm{ex}_e(G_1, n) < 2^n$, (11) with $f(n) \ll \log n$ gives

$$\mathrm{ex}_e(G_1, n) \ll n^c$$

where $c > 0$ is a constant. Feeding this bound back to (11), we get

$$\mathrm{ex}_e(G_1, n) \ll n \cdot (\log n)^{c+1}.$$

Two more iterations of (11) give

$$\mathrm{ex}_e(G_1, n) \ll n \cdot (\log n)^2 \cdot (\log \log n)^{c+1}$$

and

$$\mathrm{ex}_e(G_1, n) \ll n \cdot (\log n)^2 \cdot (\log \log n)^2 \cdot (\log \log \log n)^{c+1}$$

which is slightly better than the stated bound.

2. The lower bound follows from $\mathrm{ex}_i(G_1, n) \geq \mathrm{ex}_e(G_1, n)$. The upper bound follows by Theorem 2.3 from the upper bound in 1. □

Open Problem 3.4. What is the exact asymptotics of $\mathrm{ex}_e(G_1, n)$?

The second application of Theorem 3.1 concerns unordered extremal functions $\mathrm{ex}_e^u(F, n)$ and $\mathrm{gex}^u(G, n)$. They are defined as $\mathrm{ex}_e(F, n)$ and $\mathrm{gex}(G, n)$ except that in the containment the injection between vertex sets need not be increasing. So $\mathrm{gex}^u(G, n)$ is the classical graph extremal function. It is well known, see for example Bollobás [5, Exercise 24 in IV.7], that for every forest F one has $\mathrm{gex}^u(F, n) \leq (e(F) - 1) \cdot n$. We extend this linear bound to unordered hypergraphs. Theorem 3.1 holds also in the unordered case because the proof is independent of ordering. Ordering is crucial only for obtaining linear or almost linear bounds on $\mathrm{gex}(F, n)$ and $\mathrm{gex}(B, n)$ because the inequality (11) is useless if $f(n)$ is not almost constant. The proof of Theorem 3.1 also shows that if F is a forest and all members of B are forests (which is not the case for $B = M(k)$) then $\binom{p}{2}$ can be replaced by $p - 1$ (because if $|E| = p$ then every p two-element sets contained in E force cycle but no $F' \in B$ has a cycle).

Theorem 3.5. *Let F be a forest. Its unordered hypergraph extremal function satisfies*

$$\mathrm{ex}_e^u(F, n) \ll n.$$

Proof. Let $v(F) = p$ and $e(F) = q > 1$ (case $q = 1$ is trivial). Adapting the construction of graphs in the sets $M(k)$, it is not hard to construct a forest F' with Q edges (one can take $Q \leq (pq^2)^{q+1}$) that is a $(p - 1)$-blow-up of F. We set $B = \{F'\}$ and use (11) with the bounds $\mathrm{gex}^u(F, n) \leq (q - 1)n$, $f(n) = Q - 1$ (since $\mathrm{gex}^u(B, n) = \mathrm{gex}^u(F', n) \leq (Q - 1)n$), and $\mathrm{ex}_e^u(F, n) < 2^n$ (trivial):

$$\mathrm{ex}_e^u(F, n) < q \cdot (q - 1)n \cdot 2^{2Q-1} = \binom{q}{2} 4^Q \cdot n. \quad \square$$

One can prove the bound $\mathrm{ex}_e^u(F, n) \ll n$ also directly, without Theorem 3.1, by adapting the proof of $\mathrm{gex}^u(F, n) \ll n$ to hypergraphs. The third application of Theorem 3.1 follows in the next section.

4. Partitions and star forests

The bound (5) tells us that if F is any fixed partition with k blocks and H is a k-sparse partition with $H \not\succ F$, then $v(H)(= i(H))$ has an almost linear upper bound in terms of $e(H)$. The following theorem bounds $i(H)$ almost linearly in terms of $e(H)$ in the wider class of (not necessarily simple) hypergraphs H. The proof is based on (7).

Theorem 4.1. *Let F be a partition with $p = v(F)$ and $q = e(F) > 1$ and H be a F-free hypergraph, not necessarily simple. Then*

$$i(H) < (q-1) \cdot v(H) + e(H) \cdot \beta(q, 2p, e(H)) \tag{12}$$

where $\beta(k, l, n)$ is the almost constant function defined in (8).

Proof. Let $\bigcup H = [n]$ and the edges of H be $E_1, E_2, \ldots, E_e$ where $e = e(H)$. We set, for $1 \le i \le n$, $S_i = \{j \in [e] : i \in E_j\}$ and consider the sequence

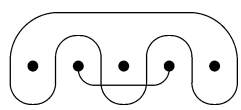
$$v = I_1 I_2 \ldots I_n$$

where I_i is an arbitrary ordering of S_i. Clearly, v is over $[e]$ and $|v| = i(H)$. To bound $|v|$ by means of (7) we need v be sufficiently sparse but this may be violated on the transitions $I_i I_{i+1}$. We fix this by selecting an appropriate subsequence w. It is easy to see that we can delete at most $q-1$ terms from the beginning of each I_i, $i > 1$, so that the resulting subsequence w is q-sparse; then $|w| \ge |v| - (q-1)(n-1)$. It follows that if w (or v) contains $u(q, 2p)$, where $u(k, l)$ is defined in (6), then H contains F but this is forbidden. (Note that the subsequence aab in v forces the first a and the b to appear in two distinct segments I_i and thus it gives incidences of E_a and E_b with two distinct vertices.) Hence $w \not\succ u(q, 2p)$ and we can bound $|w|$ by means of (7):

$$i(H) = |v| < (q-1)n + |w| \le (q-1)n + e \cdot \beta(q, 2p, e). \quad \square$$

We show that for the partition

$$F = H_2 = (\{1, 3, 5\}, \{2, 4\}) =$$



the factor multiplying $e(H)$ in (12) must be $\gg \alpha(e(H))$. We proceed as in the proof of $g(n) = \mathrm{gex}(G_0, n) \gg n\alpha(n)$ in (2) and take a 2-sparse sequence v over $[n]$ such that $v \not\succ 12\,121$, $|v| \gg n\alpha(n)$, and $v = I_1 I_2 \ldots I_{2n}$ where every interval I_i consists of distinct terms. We define the hypergraph

$$H = (E_i : i \in [n]) \qquad \text{with } E_i = \{j \in [2n] : i \text{ appears in } I_j\}.$$

We have $i(H) = |v| \gg n\alpha(n)$, $\bigcup H = [2n]$, $v(H) = 2n$, and $e(H) = n$. It is clear that $H \not\succ H_2$ because $v \not\succ 12\,121$.

Corollary 4.2. *If F is a partition, $p = v(F)$ and $q = e(F) > 1$, then*

$$\mathrm{ex}_i(F, n) < (q-1)n + \mathrm{ex}_e(F, n) \cdot \beta(q, 2p, \mathrm{ex}_e(F, n)).$$

Proof. Take H to be simple, F-free, $\bigcup H = [n]$ and with the maximum weight, and apply Theorem 4.1. $\square$

This bound is slightly weaker than the linear bound in Theorem 2.3 but on the other hand it applies to every partition F, while Theorem 2.3 says nothing about partitions with separated edges, such as $F = (\{1, 3, 5\}, \{2, 4, 8\}, \{6, 7\})$.

Our last theorem generalizes in two ways the upper bound in (2). First, we consider a class of forbidden forests that contains G_0 as a member. Second, we extend the almost linear upper bound to hypergraphs. The class consists of *star forests* which are forests F with this structure: $\bigcup F = A \cup B$ for some sets $A < B$ such that every vertex in B has degree 1 and every edge of F connects A and B. In other words, F is a star forest if every component of F is a star and every central vertex of a star is smaller than every leaf.

Theorem 4.3. *Let F be a star forest with $r > 1$ components, p vertices, and $q = p - r$ edges. Let $t = (p-1)(q-1)+1$ and $\beta(k, l, n)$ be the almost constant function defined in* (8). *We have the following bounds.*

1. $\mathrm{gex}(F, n) < (r-1)n + n \cdot \beta(r, 2q, n)$.
2. $\mathrm{ex}_e(F, n) \ll n \cdot \beta(r, 2tq, n)^3$.
3. $\mathrm{ex}_i(F, n) \ll n \cdot \beta(r, 2tq, n)^3$.

Proof. 1. We give the leaves of the star with the smallest centre label 1, the leaves of the star with the second smallest centre label 2, and so on. All labels form a sequence u over $[r]$ of length $p - r$. Now let G be any simple graph with $\bigcup G = [n]$ and $G \not\succ F$. We consider the sequence

$$v = I_1 I_2 \dots I_n$$

where I_j is any ordering of the set $\{i \in [n] : \{i, j\} \in G, i < j\}$. As in the proof of Theorem 4.1, we select an r-sparse subsequence w of v with length $|w| \geq |v| - (r-1)(n-1)$. Suppose that $w \succ u(r, 2(p-r))$ where $u(k, l)$ is defined in (6). This means that w has a (not necessarily consecutive) subsequence z of the form

$$a_1 a_2 \dots a_r a_1 a_2 \dots a_r \dots a_1 a_2 \dots a_r$$

with $2(p-r)$ segments $a_1 a_2 \dots a_r$. For a permutation $i_1, i_2, \dots, i_r$ of $[r]$, $a_{i_1} < a_{i_2} < \dots < a_{i_r}$. We give every term a_{i_j} in z label j. If we select one term from the 2nd, 4th, ..., $2(p-r)$th segment of z so that the labels on the selected terms form the sequence u, which is clearly possible, then the selected terms lie in $p - r$ distinct intervals $I_{j_1}, \dots, I_{j_{p-r}}$, $j_1 < \dots < j_{p-r}$. Since the selected terms are preceded by one segment $a_1 a_2 \dots a_r$, we have $a_{i_r} < j_1$. The edges connecting $a_1, \dots, a_r$ and $j_1, \dots, j_{p-r}$ corresponding to the selected terms form an F-copy in G, which is a contradiction. Therefore $w \not\succ u(r, 2(p-r))$ and we can apply (7):

$$e(G) = |v| \leq (r-1)n + |w| < (r-1)n + n \cdot \beta(r, 2(p-r), n).$$

2. Suppose that F has the vertex set $[p]$ (so that $[r]$ are the centres of the stars and $[r+1, p]$ are the leaves). For $k \in \mathbf{N}$ we denote $F(k)$ the star forest with the vertex set $[r + (p-r)k]$ in which $[r]$ are again the centres of stars and for $i = 1, 2, \dots, p-r$ the vertices in $[r + (i-1)k + 1, r + ik]$ are joined to the same vertex in $[r]$ as $r + i$

is joined in F. It is easy to see that $F(t) = F((p-1)(q-1)+1)$ is a $(p-1)$-blow-up of F. Also, $e(F(k)) = kq$. We set $B = \{F(t)\}$ and use (11) with the bounds $\mathrm{gex}(F,n) \ll n \cdot \beta(r, 2q, n) = n \cdot \beta'$ (bound 1 for F), $f(n) = c\beta(r, 2tq, n) = c\beta$ for a constant $c > 0$ (bound 1 for $F(t)$), and $\mathrm{ex}_e(F,n) < 2^n$ (trivial):

$$\mathrm{ex}_e(F,n) \ll n \cdot \beta' \cdot 2^{2c\beta+1} < n \cdot 2^{2(c+1)\beta}.$$

The second application of (11) gives

$$\mathrm{ex}_e(F,n) \ll n \cdot \beta' \cdot \beta \cdot 2^{2(c+1)\cdot\beta(r,2tq,2c\beta+1)} \ll n \cdot \beta^3$$

because $\beta' \le \beta$ and

$$\beta(r, 2tq, x) \ll \log\log x$$

(this is true with any number of logarithms).

3. This follows from 2 by Theorem 2.3. □

The lower bound in (2) shows that in general in the bounds 1–3 of Theorem 4.3 the factor multiplying n cannot be replaced with a constant and may be as big as $\gg \alpha(n)$. The bounds of Theorem 4.3 also hold for the reversals of star forests.

5. Concluding remarks

One can call a function $f : \mathbf{N} \to \mathbf{R}$ *nearly linear* if $n^{1-\varepsilon} \ll f(n) \ll n^{1+\varepsilon}$ holds for every $\varepsilon > 0$. We identify a candidate for the class of hypergraphs F with nearly linear $\mathrm{ex}_e(F,n)$. If F is isomorphic to the hypergraph $(\{1\},\{2\},\ldots,\{k\})$, then $\mathrm{ex}_e(F,n)$ is eventually constant ([18]) and thus is not nearly linear. For other hypergraphs we have $\mathrm{ex}_e(F,n) \ge n$ because $F \not\prec (\{1\},\{2\},\ldots,\{n\})$. An *orderly bipartite forest* is a simple graph F such that F has no cycle and $\min E < \max E'$ holds for every two edges of F. In other words, F is a forest and there is a partition $\bigcup F = A \cup B$ such that $A < B$ and every edge of F connects A and B. We denote the class of orderly bipartite forests by OBF. We say that F is an *orderly bipartite forest with singletons* if $F = F_1 \cup F_2$ where $F_1 \in \mathrm{OBF}$ and F_2 is a hypergraph consisting of possibly repeated singleton edges. For example, F may be

$$F = (\{8\},\{6\},\{6\},\{2\},\{1,6\},\{3,6\},\{4,5\},\{4,7\}).$$

The class OBF subsumes star forests and their reversals. G_1 defined in (9) belongs to OBF but is neither a star forest nor a reversed star forest.

Lemma 5.1. *If the hypergraph F is not an orderly bipartite forest with singletons, then there is a constant $\gamma > 1$ such that*

$$\mathrm{ex}_e(F,n) \gg n^{\gamma}$$

and hence $\mathrm{ex}_e(F,n)$ *is not nearly linear.*

Proof. If F is not an orderly bipartite forest with singletons, then F has (i) an edge with more than two elements or (ii) two separated two-element edges or (iii) a two-path isomorphic to $(\{1,2\},\{2,3\})$ or (iv) a repeated two-element edge or (v) an even cycle

of two-element edges (odd cycles are subsumed in (iii)). In the cases (i)–(iv) we have $\mathrm{ex}_e(F, n) \gg n^2$ because the complete bipartite graph with parts $[\lfloor n/2 \rfloor]$ and $[\lfloor n/2 \rfloor + 1, n]$ does not contain F. As for the case (v), an application of the probabilistic method (Erdős [8]) provides an unordered graph that has n vertices, $\gg n^{1+1/k}$ edges, and no even cycle of length k. Thus, in the case (v), $\mathrm{ex}_e(F, n) \gg n^{1+1/k}$ for some $k \in \mathbf{N}$. $\square$

We conjecture that $\mathrm{ex}_e(F, n)$ is nearly linear if and only if F is an orderly bipartite forest with singletons not isomorphic to $(\{1\}, \{2\}, \ldots, \{k\})$. Since every orderly bipartite forest with singletons is contained in some orderly bipartite forest, it suffices to consider only orderly bipartite forests.

Open Problem 5.2. Prove (or disprove) that for every orderly bipartite forest F we have

$$\mathrm{ex}_e(F, n) \ll n(\log n)^c$$

for some constant $c > 0$.

It is not hard to construct, for every $F \in \mathrm{OBF}$ and $k \in \mathbf{N}$, an $F' \in \mathrm{OBF}$ that is a k-blow-up of F. Thus the previous bound would follow by Theorem 3.1 from the graph bound $\mathrm{gex}(F, n) \ll n(\log n)^c$.

It is natural to consider two subclasses $\mathrm{OBF}^l \subset \mathrm{OBF}^\alpha \subset \mathrm{OBF}$ where OBF^l consists of all $F \in \mathrm{OBF}$ with $\mathrm{ex}_e(F, n) \ll n$ and OBF^α consists of all $F \in \mathrm{OBF}$ with $\mathrm{ex}_e(F, n) \ll n \cdot f(\alpha(n))$ for a primitive recursive function $f(n)$. Both inclusions are strict, as witnessed by G_0 and G_1 (defined in (1) and (9)). In this paper the class OBF^l was ignored and we showed that OBF^α contains all star forests (and their reversals). It would be interesting to learn more about OBF^l and OBF^α. Does the latter class consist only of star forests and their reversals?

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