The Witt kernels of purely inseparable quartic extensions

Hamza Ahmad

Department of Mathematical Sciences, Saginaw Valley State University, University Center, MI 48710, United States

Received 27 April 2004; accepted 16 August 2004
Submitted by R.A. Brualdi

Abstract

For a purely inseparable quartic field extension \( L/k \), we determine the Witt kernel \( W(L/k) \) of quadratic \( k \)-forms that split hyperbolically over \( L \). In particular, we show that \( W(L/k) \) is generated (as a \( W(k) \)-module) by quadratic Pfister forms of dimension four.

© 2004 Elsevier Inc. All rights reserved.

AMS classification: 11E04; 12F05

Keywords: Quadratic form; Witt kernel; Quartic extensions; Purely inseparable extensions; Characteristic 2

1. Introduction

Throughout, we let \( k \) be a field of characteristic 2. \( W(k) \) denotes the Witt ring of symmetric bilinear forms over \( k \), and \( W_f(k) \) denotes the Witt group of \( k \)-quadratic spaces. For a field extension \( L/k \), the inclusion \( i:k \hookrightarrow L \) induces a
homomorphism $i^*: \mathcal{W}(k) \to \mathcal{W}(L)$ between the Witt groups (of isometry classes) of quadratic forms. The kernel of $i^*$, denoted by $W(L/k)$ and called the Witt kernel of $L/k$, consists of the classes represented by anisotropic non-singular quadratic $k$-forms that become hyperbolic over $L$. In this note we compute $W(L/k)$ when $L$ is a purely inseparable quartic extension. In particular, we show that this kernel can be generated by Pfister forms of dimension four. Previously known results determine the Witt kernels of quadratic extension and biquadratic extensions in any characteristic (see [1,2,5,6,8,10]). For fields of characteristic different from 2, Lam et al. [9-Theorem 3.13] determined the Witt kernels of the quartic 2-extensions (i.e. quartic extensions that contain an intermediate quadratic subextensions) and showed it can be generated by Pfister forms.

Terminology and notation

Throughout, the field $k$ will always be of characteristic 2. We follow the notation of [4]. In particular, a quadratic $k$-form (or simply a form) $q$ is a map from a finite dimensional $k$-vector space $V$ to $k$ satisfying: (i) For every $a \in k$ and $x \in V$, $q(ax) = a^2q(x)$, and (ii) $B_q(x,y) := q(x+y) - q(x) - q(y)$ is a bilinear map. (When no ambiguity may arise, the bilinear form $B_q$ will be denoted by $B$.) The quadratic space $(V,q)$ is called anisotropic if $(q(x) = 0$ implies $x = 0)$; and is called non-singular if the subspace $V^\perp := \{x \in V \mid B_q(x,y) = 0 \text{ for all } y \in V\} = \{0\}$. The isometry and the orthogonal sum of forms are denoted by $\cong$ and $\perp$, respectively. The (non-singular) two dimensional quadratic space $(V,q)$ with a basis $\{x,y\}$ such that $q(x) = a$, $q(y) = b$ and $B_q(x,y) = 1$ will be denoted by $[a,b]$.

The form $[0,0]$, denoted by $\mathbb{H}$, is referred to as the hyperbolic plane. The orthogonal sum $m \cdot \mathbb{H} := \mathbb{H} \perp \cdots \perp \mathbb{H}$ ($m$-summands) is called a hyperbolic form. It is well-known (see [3-Satz 2]) that any non-singular $k$-form $q$ decomposes into

$$q \cong m \cdot \mathbb{H} \perp [a_1,b_1] \perp \cdots \perp [a_r,b_r],$$

with $a_i, b_i \in k$ such that the form $q' = [a_1, b_1] \perp \cdots \perp [a_r, b_r]$ is anisotropic. The form $q'$, $m$, and the class of $a_1b_1 + \cdots + a_rb_r$ modulo the additive group $\varphi(k) := \{a^2 + a \mid a \in k\}$ are uniquely determined by the isometry class of $q$ and are, respectively, called the anisotropic part, the Witt index, and the Arf invariant of $q$. Two forms are Witt equivalent if their anisotropic parts are isometric. The set of Witt equivalence classes of non-singular quadratic $k$-forms with the operation $\perp$ defines a group $W_q(k)$, the Witt group of $k$. $W_q(k)$ is a $W(k)$-module with the scalar multiplication induced by $(a_1, \ldots, a_r) \otimes q := a_1q \perp \cdots \perp a_rq$ for $a_i \in k$ and $q \in W_q(k)$.

Any $k$-form $q$ can be viewed as an $L$-form for any extension $L/k$. This extension of $q$ will be denoted by $q_L$. The bilinear form $\sum_{i=1}^r a_ix_iy_i$ is denoted by $\langle a_1, \ldots, a_r \rangle$. Finally, we recall that (see [4]) $[1,a]$, $a \in k$, is a two-dimensional Pfister form, and, inductively, a Pfister form of dimension $2^n$ is defined as $P \perp aP$, where $P$ is a Pfister form of dimension $2^{n-1}$.  


2. Known results

In characteristic 2, the Witt kernels of algebraic extensions are known in the following situations. For quadratic extensions, we have (see [4,5])

**Theorem 1.** Let $L = k(\beta)$ be a quadratic extension.

1. If $\beta^2 + \beta = a \in k$, then
   $$W(L/k) = \left\{ b_1[1, a] \perp \cdots \perp b_r[1, a] \mid b_i \in k^* \right\} = W(k) \otimes [1, a].$$

2. If $\beta^2 = a \in k$, then
   $$W(L/k) = \left\{ q \perp aq \mid q \in W_q(k) \right\} = \langle 1, a \rangle \otimes W_q(k).$$

For part (1) see [4–Theorem 4.2, p. 121]. For a characterization of part (2) up to Witt equivalence see [5–Lemma 4.3, p. 182], and for the stronger characterization up to isometry see [1–Corollary 2.6, p. 27].

For biquadratic extensions we have (see for example [2])

**Theorem 2.** Let $L = k(\beta_1, \beta_2)$ be a biquadratic extension.

1. If $\beta_i^2 + \beta_i = a_i \in k (i = 1, 2)$, then
   $$W(L/k) = W(k) \otimes [1, a_1] \perp W(k) \otimes [1, a_2].$$

2. If $\beta_1^2 = a_1 \in k$ and $\beta_2^2 + \beta_2 = a_2 \in k$, then
   $$W(L/k) = \langle 1, a_1 \rangle \otimes W_q(k) \perp W(k) \otimes [1, a_2].$$

3. If $\beta_i^2 = a_i \in k (i = 1, 2)$, then
   $$W(L/k) = \langle 1, a_1 \rangle \otimes W_q(k) \perp \langle 1, a_2 \rangle \otimes W_q(k).$$

Part (1) is contained in [4–Corollary 4.16, p. 128]. Part (3) appears in [10–Theorem 2, p. 316].

To the knowledge of the author, the previous theorem is the extent of the known results in literature regarding the Witt kernels of quartic extensions in characteristic 2. In the characteristic different from 2 case, the Witt kernels of field extensions of degree 4 that contain a quadratic subextensions were determined in [9–Theorem 13].

3. Witt kernels of inseparable quartic extensions

An inseparable quartic extension $L/k$ is either a biquadratic extension or a purely inseparable extension. Theorem 2 above describes $W(L/k)$ in the biquadratic case;
hence we will concentrate on the purely inseparable case. For the rest of this note we fix the notation as follows:

\[ M = k(\beta) \] where \( \beta^2 = d \in k - k^2 \),

\[ L = M(\alpha) \] where \( \alpha^4 = \beta^2 = d \).

The main theorem of this note describes the Witt kernel of \( L/k \) as follows.

**Theorem 3.** Let \( L = k(\sqrt[d]{d}) \) be a quartic extension and \( M = k(\sqrt{d}) \). Then the Witt kernel \( W(L/k) \) is generated as \( W(k) \)-module by forms of dimension four of the type

1. \((1, d) \otimes [1, a]\), with \( a \in k - \{0\} \), or
2. \((1, h) \otimes [1, dh^2g^2]\), where \( h, g \in k - \{0\} \) and \( g \in M^2 \).

The facts in Remark 4 and Proposition 5 below are well-known and will be used freely in this note.

**Remark 4**

1. For any form \( \varphi \) and \( c \in k - \{0\} \), \( \varphi \cong c^2\varphi \).
2. If \( \varphi \) is a Pfister form that represents \( a \in k - \{0\} \), then \( \varphi \cong a\varphi \).
3. Any two quadratic forms of dimension two are isometric if and only if they have the same Arf invariant and represent a common element. In particular, the following isometries hold:

\[ [1, a] \cong [1, a^2], \quad [a, b] \cong a[1, ab], \quad \text{and} \quad a[1, b] \cong [a, a^{-1}b]. \]

**Proposition 5.** Let \( K/k \) be a purely inseparable extension.

1. If \( a \in k \) and \( a + \wp(K) = \wp(K) \), then \( a + \wp(k) = \wp(k) \). In particular, if a k-form has a trivial Arf invariant over \( K \), then it has a trivial Arf invariant over \( k \).
2. For any \( a \in K \), there exists \( b \in k \) such that \( a = b \mod \wp(K) \). In particular, the Arf invariant of any \( K \)-form is defined over \( k \).

**Lemma 6**

1. In \( W_q(k) \), \( e[1, b] \perp a[1, c] = (e, a) \otimes [1, b] \perp a[1, b + c] \).
2. \( \left[ 1, \frac{(a + de)e}{a^2f^2} \right] \perp a \left[ e, \frac{(a + de)}{a^2f^2} \right] \cong \left[ 1, \frac{de^2}{a^2f^2} \right] \perp \frac{e}{a} \left[ 1, \frac{de^2}{a^2f^2} \right] \).

**Proof.** (1) In \( W_q(k) \), we have

\[ e[1, b] \perp a[1, c] = e[1, b] \perp a[1, b] \perp a[1, b] \perp a[1, c] \]
\[ = e[1, b] \perp a[1, b] \perp a[1, b + c] \]
\[ = (e, a) \otimes [1, b] \perp a[1, b + c]. \]
(2) In $W_q(k)$, we have

$$
\left[1, \frac{(a + de)e}{a^2 f^2}\right] \perp a \left[1, \frac{(a + de)}{a^2 f^2}\right]
$$

$$
= \left[1, \frac{de^2}{a^2 f^2}\right] \perp \left[1, \frac{ae}{a^2 f^2}\right] \perp ae \left[1, \frac{ae + de^2}{a^2 f^2}\right]
$$

$$
= \left[1, \frac{de^2}{a^2 f^2}\right] \perp \left[1, \frac{ae}{a^2 f^2}\right] \perp ae \left[1, \frac{ae}{a^2 f^2}\right] \perp ae \left[1, \frac{de^2}{a^2 f^2}\right].
$$

Since $ae\left[1, \frac{ae}{a^2 f^2}\right]$ represents $\frac{ae^2}{a^2 f^2}$, this form is isometric to $\left[1, \frac{ae}{a^2 f^2}\right]$. Therefore $\left[1, \frac{ae}{a^2 f^2}\right] \perp ae\left[1, \frac{ae}{a^2 f^2}\right]$ is hyperbolic, and the claim follows since $ae\left[1, \frac{de^2}{a^2 f^2}\right] \cong \frac{ae}{a}\left[1, \frac{de^2}{a^2 f^2}\right]$. □

The next theorem shows that the forms in our main Theorem 3 are the only four-dimensional forms in $W(L/k)$.

**Theorem 7.** A four-dimensional anisotropic $k$-form $\psi$ is hyperbolic over $L$ if and only if $\psi$ is isometric to a scalar multiple of a form of the type

1. $(1, d) \otimes [1, a]$ with $a \in k - \{0\}$, or
2. $(1, h) \otimes [1, dh^2g^2]$ where $h, g \in k - \{0\}$ and $g \in M^2$.

**Proof.** Since $d \in M^2 \subset L^2$, the forms of type (1) are in $W(M/k) \subseteq W(L/k)$. If $\psi$ is of type (2), setting $g = c^2, c \in M$, we have $[1, dh^2g^2] \cong [1, \beta hc^2] \cong \beta h[1, \beta hc^2]$ by Remark 4. Hence

$$
\psi_M \cong [1, \beta hc^2] \perp h[1, \beta hc^2] \cong [1, \beta hc^2] \perp \beta[1, \beta hc^2] \in W(L/M).
$$

Hence $\psi \in W(L/k)$.

Conversely, let $\psi$ be a four-dimensional anisotropic $k$-form that becomes hyperbolic over $L$. Then $\psi_L$ has trivial Arf invariant, and by Proposition 5, $\psi$ has a trivial Arf invariant over $k$. In particular $\psi$ is isometric to a multiple of a Pfister form, and we can write $\psi \cong \varphi \perp \alpha \varphi$ for some $a \in k - \{0\}$. Scaling $\psi$ by any element of $k$ that is represented by $\varphi$, we may assume that $\varphi$ and $\psi$ are both Pfister forms. Let $V$ be the underlying two-dimensional $k$-vector space of $\varphi$. Since $(\varphi \perp \alpha \varphi)_L$ is hyperbolic, $a = \varphi(v)$ for some $v \in V \otimes L$. Writing $v = v_1 + \alpha v_2$ with $v_1 \in V \otimes M$, we have

$$
a = \varphi(v_1) + \beta \varphi(v_2) \quad \text{and} \quad B_\varphi(v_1, v_2) = 0. \tag{1}
$$

If $v_2 = 0$, then $\psi \in W(M/k)$, hence $\psi \cong (1, d) \otimes [1, b], b \in k$, by part (2) of Theorem 1. If $v_1 = 0$, then $a\beta = \varphi(w)$, where $w = \beta v_2 \in V \otimes M$. Writing $w = w_1 + \beta w_2, w_1, w_2 \in V$, it follows that

$$
\varphi(w_1) = d\varphi(w_2) \quad \text{and} \quad B_\varphi(w_1, w_2) = a \neq 0.
$$
In particular, \( w_1, w_2 \) are independent. Let \( c = \psi(w_2) \). Then relative to the basis \( \{ w_2, (1/a)w_1 \} \), \( \psi \equiv [ c, \frac{dc}{a} ] \equiv c[1, \frac{dc}{a^2}] \) by Remark 4(3). Since \( \psi \) is a Pfister form, we also have \( \psi \equiv [1, \frac{dc^2}{a^2}] \). Hence

\[
\psi \equiv \left[ 1, \frac{dc^2}{a^2} \right] \perp ac \left[ 1, \frac{dc^2}{a^2} \right] \equiv \left[ 1, \frac{dc^2}{a^2} \right] \perp \left[ \frac{c}{a}, \frac{dc^2}{a^2} \right]
\]

\[
\equiv [1, dh^2] \perp b[1, dh^2],
\]

where \( h = c/a \). So, in this case, \( \psi \) is of type (2), and we are done.

Finally suppose that \( v_1 \neq 0 \neq u_2 \). Since \( B_\psi(v_1, v_2) = 0 \) and since \( (V, \psi) \) is non-singular of dimension two, it follows that \( v_2 = bv_1 \) for some \( b \in M - \{0\} \). Using Eq. (1), we have \( a(1 + \beta b^2) = (1 + \beta b^2)\psi(v_1) + \beta \psi(bv_1) = (1 + \beta b^2)\psi(v_1) = \psi((1 + \beta b^2)v_1) \). Let \( u_1, u_2 \in V \otimes k \) such that \( u_1 + \beta u_2 = (1 + \beta b^2)v_1 \in V \otimes M \). Then \( \psi(u_1) + d\psi(u_2) + \beta B_\psi(u_1, u_2) = \psi((1 + \beta b^2)v_1) = a(1 + \beta b^2) \). Since \( b^2 \in k \), we have

\[
a = \psi(u_1) + d\psi(u_2) \quad \text{and} \quad B_\psi(u_1, u_2) = ab^2 \neq 0.
\]

In particular \( u_1, u_2 \) are independent. With \( e = \psi(u_2) \), \( \psi \) can be written relative to the basis \( \{ u_2, (1/ab^2)u_1 \} \) as \( \psi \equiv [ e, \frac{e + de}{a^2b^4} ] \equiv e[1, \frac{e + de}{a^2b^4}] \) by Remark 4(3). Since \( \psi \) is a Pfister form, we also have \( \psi \equiv [1, \frac{e + de}{a^2b^4}] \). Therefore

\[
\psi \equiv \psi \perp a\phi \equiv \left[ 1, \frac{e(a + de)}{a^2b^4} \right] \perp a \left[ \frac{e}{a^2}, \frac{a + de}{a^2b^4} \right]
\]

\[
\equiv \left[ 1, \frac{de^2}{a^2b^4} \right] \perp \frac{e}{a} \left[ 1, \frac{de^2}{a^2b^4} \right],
\]

where the last isometry follows by part (2) of Lemma 6. Setting \( h = e/a \) and \( g = 1/b^2 \) we see that \( \psi \) is of type (2). This completes our proof. \( \Box \)

**Lemma 8.** Let \( (V, \psi) \) be a non-singular anisotropic quadratic \( k \)-space. If \( \psi \) is hyperbolic over \( L \), there exist vectors \( u, v, z, w \in V \) such that

- The vector \( x := u + \beta v + \alpha z + \alpha \beta w \in V \otimes L \) is isotropic,
- \( u \notin \text{Span}[v, z, w] \), and
- The vectors \( v, z, w \) are pairwise orthogonal.

**Proof.** Set \( \psi \equiv [a_1, b_1] \perp \cdots \perp [a_7, b_7] \), and choose a basis \( \{ x_i, y_j \} \) of the subspace \( [a, b] \) of \( V \) such that \( \psi(x_i) = a_i, \psi(y_j) = b_i, \) and \( B_\psi(x_i, y_j) = 1 \). We will show (possibly after re-indexing) that there exists an isotropic vector \( \xi = ax_1 + \sum_{i=1}^7 c_i y_i \in V \otimes L \) of \( \psi_L \) with \( ax_1 + c_1 y_1 \neq 0 \). Assuming this was done, and by interchanging \( x_1 \) and \( y_1 \) and scaling if necessary, we may assume without loss of generality that \( a = 1 \). So, \( \xi = x_1 + \sum_{i=1}^7 c_i y_i \). Write the coefficients \( c_1 = c_{11} + \beta c_{12} + \alpha c_{13} + \alpha \beta c_{14} \), where \( c_{11}, \ldots, c_{14} \in k \). Set \( u = x_1 + \sum_{i=1}^7 c_{1i} y_i, \ v = \sum_{i=1}^7 c_{2i} y_i, \)
\[ z = \sum_{i=1}^{r} c_i y_i \text{ and } w = \sum_{i=1}^{r} c_i y_i. \] It can be easily seen that \( x := u + \beta v + \alpha z + \alpha \beta w \) has the desired properties.

To prove the existence of the vector \( \zeta \), let \( \gamma := [b_1] \perp \cdots \perp [b_r] \) be the subform of \( \psi \) with underlying subspace generated by \( \{y_1, \ldots, y_r\} \). If \( \gamma_L \) is isotropic, we take \( \zeta = c_1 y_1 + \cdots + c_r y_r \) to be any non-trivial isotropic vector of \( \gamma_L \). By re-indexing the basis \( \{x_i, y_i\}_{i=1}^{r} \) of \( V \), we may assume \( c_1 \neq 0 \), and we are done. Now assume that \( \gamma_L \) has no non-trivial zeros. Since the space \( (V \otimes L, \psi_L) \) is hyperbolic, the form \( \psi \) has a trivial Arf invariant by Proposition 5, and therefore \( \dim(\psi) = 2r \geq 4 \) as \( \psi \) is anisotropic. Since \( \psi_L \cong r \times H \), we get

\[ (\psi \perp [b_2] \perp \cdots \perp [b_r])_L \cong (r \times H \perp [b_2] \perp \cdots \perp [b_r])_L. \]

Using the isometry \([a_1, b_1] \perp [b_1] \cong H \perp [b_1] \), we get

\[ ((r-1) \times H \perp [a_1, b_1] \perp [b_1] \perp \cdots \perp [b_r])_L \cong (r \times H \perp [b_2] \perp \cdots \perp [b_r])_L. \]

By the Witt cancellation for non-singular quadratic forms [7–Proposition 1.2, p. 282], we get that the Witt index of \( (\psi \perp [b_2] \perp \cdots \perp [b_r])_L \) is at least 1. Hence there exists \( \zeta \in V \otimes L \) which is an isotropic vector for the form \( (\psi \perp [b_2] \perp \cdots \perp [b_r])_L \). Write \( \zeta = ax_1 + \sum_{i=1}^{r} c_i y_i \), where \( a, c_j \in L \). Note that \( a \neq 0 \) because \( \gamma_L \) is anisotropic. This concludes the proof of the lemma. \( \square \)

We are now ready to prove our main theorem.

**Proof of Theorem 3.** Let \( I \) denote the \( W(k) \)-submodule of \( W_q(k) \) generated by all four-dimensional forms of the shape

\[ \langle 1, d \rangle \otimes [1, a] \quad \text{or} \quad \langle 1, h \rangle \otimes [1, dh^2g^2] \]

with \( a, h, g \in k - \{0\} \) and \( g \in M^2 \).

Let \( (V, \psi) \) be a non-singular anisotropic \( k \)-space such that \( \psi_L \) is hyperbolic. By Proposition 5, \( \psi \) has a trivial Arf invariant, and therefore \( \dim(\psi) = 4 \) as \( \psi \) is anisotropic. Since the anisotropic part of \( \psi_M \) is defined over \( k \), we let \( (\psi_M)_{an} \cong \phi_M \) where \( \phi \) is an anisotropic \( k \)-form, the form \( \psi \perp -\phi \in W(M/k) = \langle 1, d \rangle \otimes W_q(k) \subset I \) (by (2) of Theorem 1). In particular, \( \psi \in I \) if and only if \( \phi \in I \). So, to prove our theorem we may assume, without loss of generality, that \( \psi_M \) is anisotropic.

Let \( u, v, z, w \in V \) be as predicted by Lemma 8. Since \( \psi_M \) is anisotropic and \( 0 = \psi(u + \beta v + \alpha z + \alpha \beta w) \), it follows that \( z \) and \( w \) are not both 0 and

\[ 0 = \psi(u) + d\psi(v), \quad (2) \]
\[ 0 = B_\psi(u, v) + \psi(z) + d\psi(w), \quad (3) \]
\[ 0 = B_\psi(u, z) = B_\psi(u, w). \quad (4) \]
Since $\psi_M$ is anisotropic, $\mathbf{0} \neq \psi(z + \beta w) = \psi(z) + d\psi(w)$. Thus, Eq. (3) implies that $B_{\psi}(u, v) \neq 0$. In particular $u$ and $v$ are independent and span a non-singular quadratic subspace $V_0$ which is orthogonal to both $w, z$. Set $a = \psi(v)$ and $b = B_{\psi}(u, v)$. Then relative to the ordered basis $\{v, \frac{1}{B(u,v)}u\}$, $\psi|_{V_0}$ is isometric to the form $[a, \frac{da}{b^2}] \cong a[1, \frac{da^2}{b^2}]$ (by Remark 4(3)). Thus, by [4–Chap 1, Proposition 3.2, p. 10]. $(V, \psi)$ has an orthogonal decomposition

$$V = V_0 \oplus V_0^\perp \quad \text{and} \quad \psi \cong a \left[ 1, \frac{da^2}{b^2} \right] \perp \psi_1$$

with $\dim(\psi_1) = \dim(\psi) - 2$. Also $z, w \in V_0^\perp$, are orthogonal to each other, and are not both zero.

Case 1: Suppose that $z$ and $w$ are independent. Set $c = \psi(w)$. By Eq. (3), $\psi(z) = cd + b$. Since $z$ and $w$ are also orthogonal, it follows by [5–Lemma 3.1, p. 176] (or [1–Lemma 2.2, p. 25]) that

$$\psi_1 \cong c[1, e] \perp (cd + b)[1, f] \perp \psi_2$$

for some $e, f \in k$, where $\dim(\psi_2) = \dim(\psi) - 6$. Note that, in $W_q(k)$, we have

$$c[1, e] = c(1, d) \otimes [1, e] \perp cd[1, e].$$

(7)

Also, since $cd[1, e] \perp (cd + b)[1, f]$ represents $b$, it follows by [5–Lemma 3.1, p. 176] that there exist $e_1, c_1, f_1 \in k$ so that

$$cd[1, e] \perp (cd + b)[1, f] \cong b[1, e_1] \perp c_1[1, f_1].$$

(8)

By (6)–(8) we get in $W_q(k)$

$$\psi_1 = c(1, d) \otimes [1, e] \perp b[1, e_1] \perp \psi_3,$$

(9)

where $\psi_3 = c_1[1, f_1] \perp \psi_2$ has dimension $\dim(\psi) - 4$. By part (1) of Lemma 6, we have $a[1, \frac{da^2}{b^2}] \perp b[1, e_1] = a[1, \frac{a}{b}] \otimes [1, \frac{da^2}{b^2}] \perp b[1, \frac{da^2}{b^2} + e_1]$. Thus, it follows by (5) and (9) that in $W_q(k)$

$$\psi = c(1, d) \otimes [1, e] \perp a \left[ 1, \frac{a}{b} \right] \otimes \left[ 1, \frac{da^2}{b^2} \right] \perp b \left[ 1, \frac{da^2}{b^2} + e_1 \right] \perp \psi_3.$$  

Note the first two forms in the last expression belong to $I$. Therefore, modulo $I$, $\psi \cong b[1, \frac{da^2}{b^2} + e_1] \perp \psi_3$. The form on the right side has dimension $\dim(\psi) - 4$. Hence, we may proceed in proving the theorem by induction on the dimension.

Case 2: Now assume that $z$ and $w$ are dependent. Since $z$ and $w$ are not both zero, let us assume that $w \neq 0$. (The case $z \neq 0$ and $w = 0$ could be treated in a similar manner.) Then $z = cw$ for some $c \in k$, and by (3), we have $\psi(w) = \frac{b}{d + c^2}$. By [5–Lemma 3.1, p. 176], there exist $e \in k$ and a form $\psi'$ (of dimension $\dim(\psi) - 4$) such that $\psi_1 \cong \frac{b}{d + c^2} \left[ 1, e \right] \perp \psi'$. Hence, from (5), we have
\[ \psi \cong a \left[ 1, \frac{d a^2}{b^2} \right] \perp \frac{b}{d + c^2} [1, e] \perp \psi' \]
\[ = a \left[ 1, \frac{a b}{d + c^2} \right] \otimes \left[ 1, \frac{d a^2}{b^2} \right] \perp \psi'' \] by (1) of Lemma 6
\[ = a(1, h) \otimes [1, d h^2 g^2] \perp \psi'', \]
where \( h = \frac{a b}{d + c^2} \), \( g = \frac{d + c^2}{b^2} \), and \( \psi'' \) is a form of dimension \( \dim(\psi) - 2 \). Since \( a(1, h) \otimes [1, d h^2 g^2] \in I \), we have \( \psi \in I \) if and only if \( \psi'' \in I \). Also, since \( \psi \in W(L/k) \) and \( I \subset W(L/k) \), it follows that \( \psi'' \in W(L/k) \), and we can continue by induction on the dimension. This completes the proof of the main theorem. \( \Box \)

Acknowledgment

The author is grateful to an anonymous referee who reviewed an earlier version of this note, particularly for the improvement in the proofs of Lemma 8 and part (2) of Lemma 6, and also for bringing to the author’s attention the paper [10].

References