Discreteness of subgroups of $\text{PU}(2,1)$ with regular elliptic elements

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Abstract

In [Y.-P. Jiang, S. Kamiya, J.R. Parker, Jørgensen’s inequality for complex hyperbolic space, Geometriae Dedicate 97 (2003) 55–80], Jiang et al. provided Jørgensen’s inequality for non-elementary group of isometries of complex hyperbolic space generated by two elements, one of which is loxodromic or boundary elliptic. In this paper, we give analogues of Jørgensen’s inequality for the subgroup generated by two elements, but one of which is regular elliptic.

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1. Introduction

Jørgensen’s inequality [6] gives a necessary condition for two elements of in $\text{PSL}(2, \mathbb{C})$ to generate a non-elementary discrete group. As a quantitative version of Margulis’ lemma, this inequality has been generalized very widely. For example, Martin [9] and Wielenberg [14] generalized this inequality to non-elementary discrete group of Möbius transformation of any dimension $n > 2$ by using $\text{SO}(n, 1)$. A different approach to the same group is due to Waterman [13] who uses Clifford algebras. Moreover, Friedland and Hersonsky [2] shown some further generalizations who work with normed algebra.

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Since one can view the hyperbolic plane as complex hyperbolic 1-space $\mathbb{H}^1$, it is natural to seek analogues of Jørgensen’s inequality for subgroups of $\text{PU}(n, 1)$ acting on higher dimensional complex hyperbolic space.

Kamiya [7,8] and Parker [10,11] gave generalizations of Jørgensen’s inequality to the two-generator subgroup of $\text{PU}(n, 1)$ when one generator is Heisenberg translation. By using the stable basin theorem, Basmajian and Miner [1] generalized the Jørgensen’s inequality to the two-generator subgroup of $\text{PU}(2, 1)$ when the generators are loxodromic or boundary elliptic, and several other inequalities are due to Jiang, Kamiya and Parker [4] by using matrix method other than the purely geometric method. Also, it was done in [5] for the case when one generator is Heisenberg screw motion.

In [4,7,8,10,11], the authors used a method similar to Jørgensen. That is, constructing a particular sequence of distinct elements of the matrix group and finding conditions on entries of these matrices that force the sequence tending to the identity, thus violating discreteness. In this paper, we extend this method to subgroups of $\text{SU}(2, 1)$ where one generator is a regular elliptic element.

2. Preliminaries

In this section, we give some necessary background materials of complex hyperbolic geometry. More details can be found in [3].

Let $\mathbb{C}^{2,1}$ be the complex vector space of (complex) dimension 3 equipped with a non-degenerate indefinite Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(2, 1)$. There are two standard matrices $J$ which give different Hermitian forms on $\mathbb{C}^{2,1}$. Let $Z$, $W$ be the column vectors $(z_1, z_2, z_3)^t$ and $(w_1, w_2, w_3)^t$ respectively, the first Hermitian form is defined by
\[
\langle Z, W \rangle_1 = Z J_1 W^t = z_1 w_1 + z_2 w_2 - z_3 w_3,
\]
where
\[
J_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix},
\]
and second Hermitian form is defined by
\[
\langle Z, W \rangle_2 = Z J_2 W^t = z_1 w_3 + z_2 w_2 + z_3 w_1,
\]
where
\[
J_2 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

The above two Hermitian forms are equivalent. In fact, the following Cayley transform $C$ satisfying $C J_1 C^{-1} = J_2$ interchanges the first and second Hermitian forms, where
\[
C = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 0 & 1 \\
0 & \sqrt{2} & 0 \\
1 & 0 & -1
\end{bmatrix}.
\]

Let
\[
V_- = \{ Z \in \mathbb{C}^{2,1} : \langle Z, Z \rangle < 0 \}, \quad V_0 = \{ Z \in \mathbb{C}^{2,1} : \langle Z, Z \rangle = 0 \}
\]
and
\[
P : C^3 \setminus \{0\} \rightarrow CP^2
\]
be the canonical projection onto the complex projection space, then the complex hyperbolic space is $H^2_{\mathbb{C}} = PV_-$, and the boundary of $H^2_{\mathbb{C}}$ is $\tilde{H}^2_{\mathbb{C}} = PV_0$. Let the standard lift of $z = (z_1, z_2) \in \mathbb{C}P^2$ is $z = (z_1, z_2, 1)^t$. The first Hermitian form provided
\[ |z_1|^2 + |z_2|^2 < 1. \]
Thus $z = (z_1, z_2)$ is in unit ball in $\mathbb{C}^2$. The second Hermitian form provided
\[ 2Re(z_1) + |z_2| < 0. \]
This model is called the Siegel domain.

From now on, we will always work with the first Hermitian form given by $J_1$. The Bergman metric on $H^2_{\mathbb{C}}$ is given by the distance formula
\[ \cosh^2 \rho(z, w) = \frac{\langle z, w \rangle}{\langle z, z \rangle \langle w, w \rangle}. \]

The holomorphic isometry group of $H^2_{\mathbb{C}}$ with respect to the Bergman metric is the projective unitary group $PU(2, 1)$ and acts on $PC^2$ by matrix multiplication. An unitary matrix $A$ preserves the first Hermitian form. This means that $A^{-1}$ is given by $J_1$. The general forms of $A$ and $A^{-1}$ are given by
\[
A = \begin{bmatrix} a & b & c \\
       d & e & f \\
       g & h & j \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \bar{a} & \bar{d} & -\bar{g} \\
        \bar{b} & \bar{e} & -\bar{h} \\
    -\bar{c} & -\bar{f} & \bar{j} \end{bmatrix}.
\]

As the composition of an element of $PU(2, 1)$ with its inverse is the identity, we obtain a list of equations that the matrix entries in (2.1) must satisfy
\[
|a|^2 + |b|^2 - |c|^2 = 1, \quad a\bar{d} + b\bar{e} - c\bar{f} = 0, \quad |a|^2 + |d|^2 - |g|^2 = 1,
\]
\[
|j|^2 - |g|^2 - |h|^2 = 1, \quad g\bar{a} + h\bar{b} - j\bar{c} = 0, \quad |j|^2 - |e|^2 - |f|^2 = 1, \quad |d|^2 + |e|^2 - |f|^2 = 1.
\]

As in real hyperbolic geometry, $A$ holomorphic complex hyperbolic isometry $g$ is said to be:

(i) loxodromic if it fixes exactly two points of $\partial H^2_{\mathbb{C}}$;
(ii) parabolic if it fixes exactly one point of $\partial H^2_{\mathbb{C}}$;
(iii) elliptic if it fixes at least one point of $H^2_{\mathbb{C}}$.

The matrices correspond to loxodromic element and parabolic element can be find in [10]. We will only give some matrices correspond to elliptic element in this paper. For the case of elliptic element, suppose that $A$ has eigenvalue with unit modulus. In particular, we say that $A$ is boundary elliptic when $A$ has two eigenvalues and one of which is repeated. The another case that $A$ is regular elliptic when $A$ has distinct eigenvalues of unit modulus.

**Proposition 2.1**

(1) If $A$ is boundary elliptic element, then $A$ conjugates to
\[
A = \begin{bmatrix} u^{-\frac{1}{2}} & 0 & 0 \\
0 & u^{\frac{1}{2}} & 0 \\
0 & 0 & u^{-\frac{1}{2}} \end{bmatrix},
\]
where $u = e^{i\theta}$. 
(2) If $A$ is a regular elliptic element, then $A$ conjugates to

$$A = \begin{bmatrix}
    u & 0 & 0 \\
    0 & v & 0 \\
    0 & 0 & w
\end{bmatrix},$$

where $|u| = |v| = |w| = 1$, and $uvw = 1$.

3. Main results

We consider the subgroup generated by $A$ and $B$ in $\text{SU}(2, 1)$, where $A$ is a regular elliptic. As we use the first Hermitian form, $A$ and $B$ have the following form:

$$A = \begin{bmatrix}
    u & 0 & 0 \\
    0 & v & 0 \\
    0 & 0 & w
\end{bmatrix}, \quad B = \begin{bmatrix}
    a & b & c \\
    d & e & f \\
    g & h & j
\end{bmatrix}. \quad (3.1)$$

Then we have $A(q_0) = q_0$, where $q_0 = (0, 0, 1)$.

Let $q \in H_2^C$ and $A \in \text{PU}(2, 1)$ be a regular elliptic element, if $A$ and $BAB^{-1}$ fixes $q$, then $B$ also fixes $q$. This implies that if $A$ is regular elliptic, then $w = \overline{uv}$ is invariant under conjugation. In order to prove our main results, we shall require the following three lemmas.

**Lemma 3.1.** If $A \in \text{SU}(2, 1)$ is a regular elliptic element, then

$$\max \{|u - w|^2, |v - w|^2\} \text{ invariant under conjugation.}$$

It is easy to verify Lemma 3.1, since eigenvalues of matrix are invariant under conjugation.

**Lemma 3.2.** Suppose that $A$ is a regular elliptic element fixing $q_0$, let $B$ be any element of form (3.1), then

$$\cosh \frac{\rho(q_0, B(q_0))}{2} = |j|. \quad (3.2)$$

It is easy to get (3.2) by using the distance formula of Bergman metric.

**Lemma 3.3** [12]. Suppose that $A$ is an elliptic element of $\text{SU}(2, 1)$ with real trace, then the eigenvalues of $A$ are $1, e^{i\theta}, e^{-i\theta}$, where $\text{tr}A = 2\cos(\theta) + 1$.

We now give our main theorem.

**Theorem 3.4.** Let $A$ and $B$ be elements of $\text{PU}(2, 1)$ so that $A$ is regular elliptic element with fixed point $q_A$. If

$$\cosh^2 \frac{\rho(q_A, B(q_A))}{2} \max \{|u - w|^2, |v - w|^2\} < 1, \quad (3.3)$$

then the group $\langle A, B \rangle$ is either elementary or not discrete.

**Proof.** Since (3.3) is invariant under conjugation, we assume that $A$ may be written in the form (3.1). Thus (3.3) can be written as

$$|j|^2 \max \{|u - w|^2, |v - w|^2\} < 1. \quad (3.4)$$
We form the sequence $B_n$ by defining $B_0 = B$ and $B_{n+1} = B_n AB_n^{-1}$, that is,
\[
B_{n+1} = \begin{bmatrix}
    a_{n+1} & b_{n+1} & c_{n+1} \\
    d_{n+1} & e_{n+1} & f_{n+1} \\
    g_{n+1} & h_{n+1} & j_{n+1}
\end{bmatrix} = \begin{bmatrix}
    a_n & b_n & c_n \\
    d_n & e_n & f_n \\
    g_n & h_n & j_n
\end{bmatrix} \begin{bmatrix}
    u & 0 & 0 \\
    0 & v & 0 \\
    0 & 0 & w
\end{bmatrix} \begin{bmatrix}
    \tilde{a}_n & \tilde{d}_n & -\tilde{g}_n \\
    \tilde{b}_n & \tilde{e}_n & -\tilde{h}_n \\
    -\tilde{c}_n & -\tilde{f}_n & \tilde{j}_n
\end{bmatrix}.
\]

After performing the matrix multiplication, we have
\[
j_{n+1} = w|j_n|^2 - u|g_n|^2 - v|h_n|^2. \tag{3.5}
\]

We first consider the case that $|j| \neq 1$. Combining (3.5) with $|j_n|^2 = 1 + |g_n|^2 + |h_n|^2$, we obtain
\[
|j_{n+1}|^2 = |g_n|^4 + |h_n|^4 + |j_n|^4 + |u\bar{v} + v\bar{u}|g_n|^2|h_n|^2 - (u\bar{w} + w\bar{v})|g_n|^2|j_n|^2 - (v\bar{w} + w\bar{v})|h_n|^2j_n|^2 = 1 - |u - v|^2|g_n|^2|h_n|^2 + |u - w|^2|g_n|^2|j_n|^2 + |v - w|^2|h_n|^2|j_n|^2 \\
\leq 1 + (|j_n|^2 - 1)|j_n|^2 \max(|u - w|^2, |v - w|^2) = 1 + (|j_n|^2 - 1)|j_n|^2 \max(|u - w|^2, |v - w|^2).
\]

It follows that
\[
|j_{n+1}|^2 - 1 \leq (|j_n|^2 - 1)|j_n|^2 \max(|u - w|^2, |v - w|^2). \tag{3.6}
\]

Noting that $|j_n|^2 = 1 + |g_n|^2 + |h_n|^2 \geq 1$, from (3.4) and (3.6), we can get
\[
|j_{n+1}|^2 - 1 < (|j|^2 - 1)(|j|^2 - 1) \max(|u - w|^2, |v - w|^2)^n+1
\]
and
\[
|j_{n+1}| < |j_n|.
\]

Thus $B_{n+1}$ are distinct for all $n$ and $|j_n| \rightarrow 1$.

Now from $|j_n|^2 = 1 + |g_n|^2 + |h_n|^2$ and $|j_n|^2 = 1 + |c_n|^2 + |f_n|^2$, we have
\[
e_n \rightarrow 0, \quad f_n \rightarrow 0, \quad g_n \rightarrow 0, \quad h_n \rightarrow 0. \tag{3.7}
\]

Finally, from $|j_n| \rightarrow 1$ and (3.7), we obtain $j_n \rightarrow w$.

Since $|a_n|^2 + |b_n|^2 - |c_n|^2 = 1$ and $|d_n|^2 + |e_n|^2 - |f_n|^2 = 1$, we have
\[
|a_n|^2 + |b_n|^2 \rightarrow 1, \quad |d_n|^2 + |e_n|^2 \rightarrow 1.
\]

By passing to the subsequence, we may assume
\[
a_{n_k} \rightarrow a^*, \quad b_{n_k} \rightarrow b^*, \quad d_{n_k} \rightarrow d^*, \quad e_{n_k} \rightarrow e^*,
\]
so the matrix $B_{n+1}$ converges to $B_\infty$ defined by
\[
B_\infty = \begin{bmatrix}
    a^* & b^* & 0 \\
    d^* & e^* & 0 \\
    0 & 0 & w
\end{bmatrix}.
\]

Obviously, $B_\infty \in SU(2, 1)$.

If $|j| = 1$, then $g = h = c = f = 0$. $A$ and $B$ sharing exactly one fixed point so $\langle A, B \rangle$ is elementary. This completes the proof of Theorem 3.4. \qed
By Lemma 3.3, we can give a Jørgensen’s inequality for subgroups containing regular elliptic with real trace.

**Corollary 3.5.** Let $A$ be a regular elliptic element of $\text{PU}(2, 1)$ with real trace fixing $q_A$ and preserving a Lagrangian plane, and let $B$ be any element of $\text{PU}(2, 1)$. If

$$(3 - \text{tr}A) \cosh^2 \frac{\rho(q_A, B(q_A))}{2} < 1,$$

then the group $\langle A, B \rangle$ is either elementary or not discrete.

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**References**