A ground state alternative for singular Schrödinger operators

Yehuda Pinchover\textsuperscript{a,}\textsuperscript{*}, Kyril Tintarev\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Technion, Israel Institute of Technology, Haifa 32000, Israel
\textsuperscript{b}Department of Mathematics, Uppsala University, SE-751 06 Uppsala, Sweden

Received 30 November 2004; accepted 31 May 2005
Communicated by Paul Malliavin
Available online 30 August 2005

Abstract

Let $a$ be a quadratic form associated with a Schrödinger operator $L = -\nabla \cdot (A\nabla) + V$ on a domain $\Omega \subset \mathbb{R}^d$. If $a$ is nonnegative on $C_0^\infty(\Omega)$, then either there is $W > 0$ such that $\int W|u|^2\,dx \leq a[u]$ for all $C_0^\infty(\Omega; \mathbb{R})$, or there is a sequence $\varphi_k \in C_0^\infty(\Omega)$ and a function $\varphi > 0$ satisfying $L\varphi = 0$ such that $a[\varphi_k] \to 0$, $\varphi_k \to \varphi$ locally uniformly in $\Omega \setminus \{x_0\}$. This dichotomy is equivalent to the dichotomy between $L$ being subcritical resp. critical in $\Omega$. In the latter case, one has an inequality of Poincaré type: there exists $W > 0$ such that for every $\psi \in C_0^\infty(\Omega; \mathbb{R})$ satisfying $\int \psi \varphi\,dx \neq 0$ there exists a constant $C > 0$ such that $C^{-1}\int W|u|^2\,dx \leq a[u] + C\int |u\psi|\,dx$ for all $u \in C_0^\infty(\Omega; \mathbb{R})$.

© 2005 Elsevier Inc. All rights reserved.

MSC: primary 35J10; secondary 35J20; 35J70; 49R50

Keywords: Dirichlet form; Ground state; Quadratic form

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a domain. We denote $K \Subset \Omega$, if $K$ is relatively compact in $\Omega$. Let $A : \Omega \to \mathbb{R}^{d^2}$ be a measurable matrix valued function such that for every $K \Subset \Omega$ there

\textsuperscript{*}Corresponding author.

E-mail addresses: pincho@techunix.technion.ac.il (Y. Pinchover), kyril.tintarev@math.uu.se

(K. Tintarev).

0022-1236/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2005.05.015
exists $\lambda_K > 1$ such that
\[
\lambda_K^{-1} I_d \leq A(x) \leq \lambda_K I_d \quad \forall x \in K. \tag{1.1}
\]

Let $V \in L^p_\text{loc}(\Omega; \mathbb{R})$, where $p > \frac{d}{2}$. Throughout the paper, we assume that the bilinear form
\[
a(u, v) := \int_\Omega (A \nabla u \cdot \nabla v + V u v) \, dx \quad u, v \in C_0^\infty(\Omega), \tag{1.2}
\]
associated with the Schrödinger operator
\[
L := -\nabla \cdot (A \nabla) + V \tag{1.3}
\]
is nonnegative on $C_0^\infty(\Omega)$, that is
\[
a[u] := a(u, u) = \int_\Omega \left( A \nabla u \cdot \nabla u + V |u|^2 \right) \, dx \geq 0 \quad \forall u \in C_0^\infty(\Omega; \mathbb{R}). \tag{1.4}
\]

Under this assumption it is known [1] that the Dirichlet problem is uniquely solvable in any bounded subdomain. Therefore, $a[u] = 0$, for $u \in C_0^\infty(\Omega)$, if and only if $u = 0$. Consequently, the bilinear form $a$ defines a scalar product on $C_0^\infty(\Omega)$. Let $H_a(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $\sqrt{a[u]}$.

Generally, $H_a(\Omega)$ cannot be identified as a space of measurable functions or even as a space of distributions. More precisely, it can happen that there is no continuous imbedding of $H_a(\Omega)$ into $D'(\Omega)$. For example, if $\Omega = \mathbb{R}^d$, $d = 1, 2$, and $a[u] = \int |\nabla u|^2 \, dx$, then $H_a(\mathbb{R}^d)$ is not a space of distributions (see e.g. [10, Section 11.3]), and the zero element of $H_a(\mathbb{R}^d)$ contains Cauchy sequences in $C_0^\infty(\mathbb{R}^d)$ that converge in $L^2_\text{loc}(\mathbb{R}^d)$ to the constant function. For $d > 2$ the Sobolev inequality
\[
\left( \int |u|^{\frac{2d}{d-2}} \, dx \right)^{\frac{d-2}{d}} \leq C \int |\nabla u|^2 \, dx \text{ implies that } H_a(\mathbb{R}^d) \text{ is continuously imbedded into } L^{\frac{2d}{d-2}}(\mathbb{R}^d). \]
In the case $\Omega = \mathbb{R}^d \setminus \{0\}$, $d > 2$, and $a[u] = \int_\Omega [|\nabla u|^2 - (d-2)^2 |\frac{u}{|x|^2}|^2 ] \, dx$, the class of the zero element of $H_a(\Omega)$ contains Cauchy sequences in $C_0^\infty(\Omega)$ that converge in $L^2_\text{loc}(\Omega)$ to $|x|^{-\frac{d-2}{2}}$, with $C \in \mathbb{R}$.

**Definition 1.1.** A sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ satisfying $a[\varphi_k] \to 0$ is called a null sequence for the form $a$. We say that a positive function $\varphi$ is a null state for the form $a$, if there exists a null sequence $\{\varphi_k\}$ for the form $a$ such that $\varphi_k \to \varphi$ in $L^2_\text{loc}(\Omega)$.

**Definition 1.2.** We say that the nonnegative quadratic form $a$ has a weighted spectral gap if there is a function $W > 0$ continuous in $\Omega$ such that
\[
\int_\Omega W |u|^2 \, dx \leq a[u] \quad \forall u \in C_0^\infty(\Omega; \mathbb{R}). \tag{1.5}
\]
Remark 1.3. It is easy to see that a null state $\varphi$ is a distributional solution of the equation $Lu = 0$ in $\Omega$. Indeed, suppose that $L\varphi \neq 0$, then there exists $\psi \in C_0^\infty(\Omega)$ such that $\int \varphi L\psi \, dx < -\varepsilon < 0$. Let $\varphi_n \in C_0^\infty(\Omega)$ be a null sequence that converges to $\varphi$ in $L^2_{\text{loc}}(\Omega)$. Then $a(\varphi_n, \psi) = \int \varphi_n L\psi \, dx < -\varepsilon/2$ for all $n$ sufficiently large. Consequently, $a[\varphi_n + t\psi] \leq a[\varphi_n] + t^2 a[\psi] - \varepsilon t$, which is negative for $t > 0$ sufficiently small and $n$ sufficiently large. This contradicts the assumption that $a \geq 0$. An alternative proof of this statement can be deduced from the proof of Lemma 2.5 which in turn shows that a null state is actually a weak solution.

In the present paper, we establish the dichotomy between the existence of a weighted spectral gap and the existence of a null state for the form $a$.

Theorem 1.4. Let $\Omega \subset \mathbb{R}^d$ be a domain, and assume that the form $a$ is nonnegative. Then $a$ has either a weighted spectral gap or a null state. If $a$ has a null state $\varphi$, then there exists a positive $W \in C(\Omega)$, such that for every $\psi \in L^2(\Omega; \mathbb{R})$ satisfying $\int \psi \varphi \, dx \neq 0$ there exists a constant $C > 0$ such that the following inequality holds:

$$C^{-1} \int_{\Omega} W|u|^2 \, dx \leq a[u] + C \left| \int_{\Omega} \psi u \, dx \right|^2$$

$\forall u \in C_0^\infty(\Omega; \mathbb{R}).$ (1.6)

Moreover, all norms that are induced by the right-hand side of (1.6) with such functions $\psi$ are equivalent.

An inequality similar to (1.6) is discussed by Weidl [18, Section 5.1] in a somewhat different setting, where the counterpart of $|\int_{\Omega} \psi u \, dx|^2$ is an abstract positive homogenous functional satisfying a regularization condition relative to $a$.

Theorem 1.4 answers a natural question that arises in the context of improving inequalities of the form

$$\int_{\Omega} V|u|^2 \, dx \leq C \int_{\Omega} A\nabla u \cdot \nabla \tilde{u} \, dx,$$  

where $V \geq 0$: What is the difference between a potential $V$ that can be refined, that is, can be replaced in the inequality (1.7) by some $V' \geq V$, and a potential that cannot? In other words, what can be said about a nonnegative Schrödinger operator without any weighted spectral gap? The answer given in Theorem 1.4 is of general relevance to the work on refining the spectral gap inequalities (see for example, [2–5,7,9] and the references therein). As it is noted in [7], attainment in the standard Sobolev space of the best constant in the Hardy inequality, or in the Hardy inequality with a further corrected potential cannot serve as a criterion for the nonexistence of a weighted spectral gap. The present paper shows that the relevant energy space for the Schrödinger equation is the completion of $C_0^\infty(\Omega)$ with respect to the quadratic form of the operator appended possibly with a one-dimensional correction. The Poincaré type inequality (1.6) shows that such a completion is continuously imbedded into some weighted $L^2$-spaces. We
show that the existence of a weighted spectral gap is equivalent to the existence of 
continuous imbedding of $\mathcal{H}_a(\Omega)$ into the space of distributions, which in turn implies a continuous imbedding into $L^p_{\text{loc}}(\Omega)$.

Another equivalent formulation of Theorem 1.4 is the dichotomy between critical and subcritical elliptic operators. Let $P$ be a linear second-order elliptic operator with real coefficients which is defined on a domain $\Omega$ (if $P$ is a symmetric operator of the form (1.3), then we denote the operator by $L$). Let $C_P(\Omega)$ be the cone of all positive solutions of the equation $Pu = 0$ in $\Omega$. Let $W \in L^p_{\text{loc}}(\Omega; \mathbb{R})$, $p > d/2$. The \textit{generalized principal eigenvalue} is defined by

$$\lambda_0(P, W, \Omega) := \sup\{\lambda \in \mathbb{R} \mid C_{P - \lambda W}(\Omega) \neq \emptyset\}.$$

Let $K \Subset \Omega$. Recall [1,16] that $u \in C_P(\Omega \setminus K)$ is said to be a \textit{positive solution of the operator $P$ of minimal growth in a neighborhood of infinity in $\Omega$}, if for any $K_1 \Subset \Omega$ and any $v \in C(\Omega \setminus K_1) \cap C_P(\Omega \setminus K_1)$, the inequality $u \leq v$ on $\partial K_1$ implies that $u \leq v$ in $\Omega \setminus K_1$. A positive solution $u \in C_P(\Omega)$ which has minimal growth in a neighborhood of infinity in $\Omega$ is called a \textit{ground state} of $P$ in $\Omega$.

The operator $P$ is said to be \textit{critical} in $\Omega$, if $P$ admits a ground state in $\Omega$. The operator $P$ is called \textit{subcritical} in $\Omega$, if $C_P(\Omega) \neq \emptyset$, but $P$ is not critical in $\Omega$. If $C_P(\Omega) = \emptyset$, then $P$ is \textit{supercritical} in $\Omega$.

Assume that $P$ is critical in $\Omega '\subseteq \Omega$, then $P$ is subcritical in any domain $\Omega$ such that $\Omega_1 \subseteq \Omega'$, and supercritical in any domain $\Omega_2$ such that $\Omega' \subseteq \Omega_2 \subseteq \Omega$. Furthermore, for any nonzero nonnegative function $W$ the operator $P + W$ is subcritical and $P - W$ is supercritical in $\Omega'$. Moreover, if $P$ is critical in $\Omega'$, then $\dim C_P(\Omega') = 1$ (see e.g. [16]).

If $P$ is subcritical in $\Omega$, then $P$ admits a positive minimal Green function $G^\Omega_P(x, y)$ in $\Omega$. For each $y \in \Omega$, the function $G^\Omega_P(\cdot, y)$ is a positive solution of the equation $Pu = 0$ in $\Omega \setminus \{y\}$ that has minimal growth in a neighborhood of infinity in $\Omega$.

**Theorem 1.5.** Let $a$ be a nonnegative quadratic form associated with a Schrödinger operator $L = -\nabla \cdot (A \nabla) + V$ defined on $\Omega$. Then the following conditions are equivalent:

(i) The form $a$ has a weighted spectral gap.
(ii) The space $\mathcal{H}_a(\Omega)$ is continuously imbedded into $L^2_{\text{loc}}(\Omega)$.
(iii) The space $\mathcal{H}_a(\Omega)$ is continuously imbedded into $\mathcal{D}'(\Omega)$.
(iv) The operator $L$ is subcritical in $\Omega$.

Since $a$ is nonnegative, the operator $L$ is either critical or subcritical in $\Omega$ [1]. Thus, Theorems 1.4 and 1.5 imply:

**Corollary 1.6.** The form $a$ has a null state if and only if $L$ is critical in $\Omega$. In other words, $\phi$ is a null state for the form $a$ if and only if $\phi$ is a ground state of the operator $L$. 
2. Preliminary results

Given a domain $\Omega$, we fix an exhaustion $\{\Omega_N\}_{N=1}^\infty$ of $\Omega$, i.e. a sequence of smooth, relatively compact domains such that $x_0 \in \Omega_1 \neq \emptyset$, $\overline{\Omega_N} \subset \Omega_{N+1}$ and $\cup_{N=1}^\infty \Omega_N = \Omega$. Denote by $B(x_0, \delta)$ the ball of radius $\delta$ centered at $x_0$.

Recall that if $P$ is subcritical in $\Omega$, and $W$ has a compact support in $\Omega$, then there exists $\varepsilon_0 > 0$ such that $P + \varepsilon W$ is subcritical in $\Omega$ for all $|\varepsilon| < \varepsilon_0$ (see [13,16]). We need the following stronger assertion.

**Lemma 2.1.** Suppose that $P$ is a subcritical operator in $\Omega$. Then there exists a strictly positive function $W$ such that $\lambda_0(P, W, \Omega) > 0$.

**Proof.** Let $u \in C_P(\Omega)$. It follows from the proof of Lemma 3.3 in [15], that it is sufficient to find a positive function $W$ and a constant $C > 0$ such that

$$\int_{\Omega} G_P^\Omega(x, z)W(z)u(z)\,dz \leq C u(x) \quad \forall x \in \Omega. \quad (2.1)$$

Let $\{\chi_N\}$ be a locally finite partition of unity on $\Omega$ subordinated to the exhaustion $\{\Omega_N\}$. Since $\chi_N$ has a compact support it follows from Theorem 2.10 in [13] and Remark 3.5 in [15] that there exists $C_N > 0$ such that

$$\int_{\Omega} G_P^\Omega(x, z)\chi_N(z)u(z)\,dz \leq C_N u(x) \quad (2.2)$$

for all $x \in \Omega$. Let $\varepsilon_N := 2^{-N}/C_N$, and define

$$W(x) := \sum_{N=1}^\infty \varepsilon_n \chi_N(x).$$

Then

$$\int_{\Omega} G_P^\Omega(x, z)W(z)u(z)\,dz \leq u(x) \quad (2.3)$$

for all $x \in \Omega$. Hence, $\lambda_0(P, W, \Omega) > 0$. □

**Remark 2.2.** For further necessary and sufficient conditions for $\lambda_0 > 0$, see [11,15] and the references therein.

**Corollary 2.3.** Suppose that the operator $L$ is of the form (1.3), and let $a$ be the (nonnegative) quadratic form (1.4) associated with $L$. Then
(i) The operator $L$ is subcritical in $\Omega$ if and only if $a$ has a weighted spectral gap, i.e. there exists a strictly positive function $W$ such that

$$\int_{\Omega} W |u|^2 \, dx \leq a[u] \quad \forall u \in C_0^\infty(\Omega; \mathbb{R}).$$

(2.4)

(ii) If $L$ is critical in $\Omega$, then for every nonempty open set $B \subset \Omega$ there is a strictly positive continuous function $W$ such that

$$\int_{\Omega} W |u|^2 \, dx \leq a[u] + \int_B |u|^2 \, dx \quad \forall u \in C_0^\infty(\Omega; \mathbb{R}).$$

(2.5)

(iii) The operator $L$ is critical in $\Omega$ if and only if there is an open set $B_0 \subset \Omega$, such that $\int_{B_0} |u|^2 \, dx$ is not bounded by $a[u]$ on $C_0^\infty(\Omega; \mathbb{R})$.

(iv) If the quadratic form $a$ has a null state, then $L$ is critical in $\Omega$.

Proof. Recall that $P$ is critical in $\Omega$ if and only if for any nonzero nonnegative function $Q$, the operator $P + Q$ is subcritical in $\Omega$, and $P - Q$ is supercritical in $\Omega$. It is well known that if $L$ is symmetric, then $C_P(\Omega) \neq \emptyset$ if and only if the quadratic form $a$ is nonnegative (see [1] and the references therein). Therefore (i)–(iii) follow from Lemma 2.1. Part (iii) clearly implies (iv). □

The following key lemma is well known in the case of Schrödinger operators (see, e.g. [6]).

Lemma 2.4. Let $\psi$ be a solution of the equation $Lu = 0$ in a bounded domain $D \subset C^1$, and let $v \in H^1(D; \mathbb{R})$ such that $v\psi = 0$ on $\partial D$. Then

$$\int_D \left[ A \nabla (v\psi) \cdot \nabla (v\psi) + V|v\psi|^2 \right] \, dx = \int_D (A \nabla v \cdot \nabla v)|\psi|^2 \, dx.$$  

(2.6)

Proof. Apply the Gauss divergence theorem, and calculate. □

Lemma 2.5. Suppose that $L$ is critical in $\Omega$, and let $\varphi$ be its ground state. Let $\{u_N\} \subset C_0^\infty(\Omega)$ be a null sequence, and assume that $\{u_N\}$ is locally bounded in $L^2$. Then $\{u_N\}$ has a converging subsequence in $L^2_{\text{loc}}(\Omega)$ that converges to $c\varphi$ with some $c \in \mathbb{R}$. Moreover, any converging subsequence of $\{u_N\}$ in $L^2_{\text{loc}}(\Omega)$ converges to $c\varphi$ for some $c \in \mathbb{R}$.

If further, for some $B \subset \Omega$ the sequence $\{u_N\}$ is normalized such that $\|u_N\|_{L^2(B)} = 1$, then $c \neq 0$. Such a normalized null sequence does exist. In particular, $\varphi$ is a null state.

Proof. For any $K \subset \Omega$ there exists $C_K > 0$ such that

$$(C_K)^{-1} \leq \varphi(x) \leq C_K \quad \forall x \in K.$$  

(2.7)
Invoking Lemma 2.4 with \( v = u_N / \phi \), and \( \psi = \phi \), and using (1.1) and (2.7), we infer that \( \nabla (u_N / \phi) \) tends to zero in \( L^2_{\text{loc}}(\Omega) \). By our assumption \( u_N / \phi \) is locally bounded in \( L^2(\Omega) \). In light of the Sobolev compact embedding in smooth bounded domains, it follows that (up to a subsequence) \( u_N / \phi \) converges in \( L^2_{\text{loc}}(\Omega) \). Therefore, \( u_N / \phi \) converges in \( H^1_{\text{loc}}(\Omega) \) to a function \( u \) with a zero gradient, consequently, \( u \) is locally constant in \( \Omega \). Since \( \Omega \) is connected, \( u = \text{constant} \) in \( \Omega \). Hence, (2.7) implies that \( \{u_N\} \) converges in \( L^2_{\text{loc}}(\Omega) \) to \( c\phi \).

From part (iii) of Corollary 2.3 it follows that there exist \( B \Subset \Omega \) and a null sequence \( \{u_N\} \) such that \( \|u_N\|_{L^2(B)} = 1 \). Thus, \( \phi \) is a null state. \( \square \)

Remark 2.6. Clearly, without the assumption of locally boundedness in \( L^2 \) it is not true that \( \{u_N\} \) converges. Take for instance a bounded smooth domain and \( u_N = N\phi \).

Remark 2.7. In the subcritical case, any null sequence converges to zero in \( L^2_{\text{loc}}(\Omega) \).

Lemma 2.8. If \( a \) admits a null state \( \phi \), then for any \( \psi \in C^\infty_0(\Omega) \) such that \( \int \psi \phi \, dx \neq 0 \), the mapping \( u \mapsto \int u \psi \, dx \) is not continuous in \( \mathcal{H}_a(\Omega) \). Consequently, \( \mathcal{H}_a(\Omega) \) is not continuously imbedded into \( \mathcal{D}'(\Omega) \).

Proof. Part (iv) of Corollary 2.3 implies that \( L \) is critical in \( \Omega \). It follows from Lemma 2.5 that there exists a null sequence \( \phi_k \to c\phi \) in \( L^2_{\text{loc}}(\Omega) \) with \( \int_B |\phi_k|^2 \, dx = 1 \), and \( c > 0 \). Thus, \( \int \phi_k \psi \, dx \to \int c\phi \psi \, dx \neq 0 \), although \( \phi_k \to 0 \) in \( \mathcal{H}_a(\Omega) \). \( \square \)

3. Poincaré inequality and the space \( \mathcal{D}^{1,2}_a(\Omega) \)

Proof of Theorem 1.5. The equivalence of (i) and (iv) follows from Corollary 2.3. From (i) follows immediately (ii) which implies (iii). If (iv) does not hold, then Lemmas 2.5 and 2.8 imply that condition (iii) is false. \( \square \)

Proof of Theorem 1.4. If the form \( a \) has a weighted spectral gap, then every null sequence \( w_k \) converges to 0 in \( L^2_{\text{loc}}(\Omega) \), so \( a \) has no null state. If the form \( a \) has no weighted spectral gap, then by Theorem 1.5, \( L \) is critical in \( \Omega \), and by Lemma 2.5, the form \( a \) admits a null state.

Let us prove now (1.6). Due to (2.5) it suffices to verify that for some nonempty open \( B \Subset \Omega \),

\[
\int_B |u|^2 \, dx \leq C \left( a[u] + \int_\Omega u \psi \, dx \right)^2. \tag{3.1}
\]

Assume that this is false. Then there is a sequence \( \{u_k\} \) such that \( a[u_k] \to 0 \), \( \int_\Omega u_k \psi \, dx \to 0 \), and \( \int_B |u_k|^2 \, dx = 1 \). By (2.5), \( \{u_k\} \) is bounded in \( L^2_{\text{loc}}(\Omega) \), so by Lemma 2.5, \( u_k \to \lambda \phi \neq 0 \) in \( L^2_{\text{loc}}(\Omega) \). Then \( \int_\Omega u_k \psi \, dx \to \lambda \int_\Omega \phi \psi \, dx \neq 0 \), and we arrive at a contradiction.
Let $\psi_1, \psi_2 \in C_0^\infty(\Omega)$ satisfy $\int_\Omega \psi_i \varphi \, dx \neq 0$, $i = 1, 2$. Then the equivalence of norms follows from
\[ \left| \int_\Omega u \psi_1 \, dx \right|^2 \leq C \left( a[u] + \left| \int_\Omega u \psi_2 \, dx \right|^2 \right), \tag{3.2} \]
which in turn follows from the Cauchy-Schwartz inequality and (1.6). □

Recall the standard notation $D^{1,2}(\mathbb{R}^d)$ for $H^1(\mathbb{R}^d)$, where $d > 2$, and $a[u] = \int_{\mathbb{R}^d} |\nabla u|^2 \, dx$. Therefore, it is natural to use the notation $D^{1,2}_a(\Omega)$ for the space $H^1_a(\Omega)$ in case of a weighted spectral gap, and for the closure of $C_0^\infty(\Omega)$ in the norm induced by the right-hand side of (1.6) in the case of the existence of null state.

By analogy with the compactness of local imbeddings in $D^{1,2}(\mathbb{R}^d)$, we have the following statement.

**Proposition 3.1.** The space $D^{1,2}_a(\Omega)$ is continuously imbedded into $H^1_{\text{loc}}(\Omega)$ (and therefore, it is compactly imbedded into $L^2_{\text{loc}}(\Omega)$).

**Proof.** Consider a ball $B \subset D^{1,2}_a(\Omega)$. Fix $\psi \in C_p(\Omega)$. From (2.6) it follows that
\[ \int_\Omega (A \nabla (u/\psi) \cdot \nabla (u/\psi)) |\psi|^2 \, dx = a[u] \quad \forall u \in C_0^\infty(\Omega; \mathbb{R}). \tag{3.3} \]

By density, this implies that the set $B_{\psi} = \{u/\psi : u \in B\}$ is locally bounded with respect to the Dirichlet norm (i.e. it is bounded in $D^{1,2}_{\text{loc}}(\Omega)$). At the same time, from either (1.5) or (1.6), it follows that $B$ is bounded in $L^2_{\text{loc}}(\Omega)$. Consequently, using the Leibniz product rule and the Young inequality, we infer that $B$ is bounded in $D^{1,2}_{\text{loc}}(\Omega)$, and thus also in $H^1_{\text{loc}}(\Omega)$. □

4. Null sequence converging locally uniformly to null state

Let $P$ be a second-order elliptic operator with real coefficients which is not necessarily symmetric and which is defined on a domain $\Omega \subset \mathbb{R}^d$. Assume that $P$ is critical in $\Omega$. Fix $x_0 \in \Omega$, and let $\varphi$ be the ground state, satisfying $\varphi(x_0) = 1$. Let $\{\Omega_N\}_{N=1}^\infty$ be an exhaustion of $\Omega$. Without loss of generality assume that $x_0 = 0$ and $B(0, 1) \subset \Omega_1$.

We begin this section with the following lemma (cf. [14, Theorem 1.2]).

**Lemma 4.1.** Suppose that $P$ is critical in $\Omega \subset \mathbb{R}^d$. Let $x_1 \in \Omega_1$, $0 < |x_1| < 1$. Consider the function
\[ \psi_N(x) := \begin{cases} \frac{G_P^{\Omega_N}(x, 0)}{G_P^{\Omega_N}(x_1, 0)} & x \in \Omega_N, \ x \neq 0, \\ \frac{1}{\varphi(x_1)} & x = 0. \end{cases} \]

Then

\[ \lim_{N \to \infty} \psi_N(x) = \frac{\varphi(x)}{\varphi(x_1)}, \]

locally uniformly in \( \Omega \setminus \{0\} \).

**Proof.** By criticality,

\[ \lim_{N \to \infty} G_P^{\Omega_N}(x_1, 0) = \infty. \]

Therefore,

\[ \lim_{N \to \infty} \left( \frac{G_P^{\Omega_1}(x, 0)}{G_P^{\Omega_N}(x_1, 0)} \right) = 0, \quad (4.1) \]

locally uniformly in \( B(0, |x_1|) \setminus \{0\} \).

Consider the function

\[ \tilde{\psi}_N(x) := \frac{G_P^{\Omega_N}(x, 0) - G_P^{\Omega_1}(x, 0)}{G_P^{\Omega_N}(x_1, 0)}. \quad (4.2) \]

Note that the function \( \tilde{\psi}_N \) has a removable singularity at the origin, therefore it may be considered as a positive solution of the equation \( Pu = 0 \) in \( \Omega_1 \). Moreover by (4.1), \( \tilde{\psi}_N(x_1) = 1 + o(1) \). Therefore, by Harnack’s inequality \( c^{-1} \leq \tilde{\psi}_N(0) \leq c \). By a standard elliptic argument \( \{ \tilde{\psi}_N \} \) has a subsequence \( \{ \tilde{\psi}_{N_k} \} \) which converges uniformly in any compact set \( K \subseteq \Omega_1 \), to a positive solution \( u \) satisfying \( u(x_1) = 1 \). Hence \( \{ \tilde{\psi}_{N_k} \} \) converges locally uniformly in any punctured ball \( B(0, r) \setminus \{0\} \subseteq \Omega_1 \) to \( u \).

On the other hand, \( \psi_{N_k}(x) \) is a positive solution of the equation \( Pu = 0 \) in \( \Omega_{N_k} \setminus \{0\} \), and \( \psi_{N_k}(x_1) = 1 \). Therefore, the sequence \( \{ \psi_{N_k}(x) \} \) has a subsequence which converges locally uniformly in \( \Omega \setminus \{0\} \) to a positive solution \( u_1(x) \) of the equation \( Pu = 0 \) in \( \Omega \setminus \{0\} \) and \( u_1(x_1) = 1 \). It follows that in \( \Omega_1 \setminus \{0\} \) we have \( u_1 = u \), and therefore this subsequence converges to a global positive solution \( u_1 \) of the equation \( Pu = 0 \) in \( \Omega \), and by uniqueness, \( u_1(x) = \frac{\varphi(x)}{\varphi(x_1)} \). Moreover, since this is true for any subsequence, it
follows that
\[
\lim_{N \to \infty} \psi_N(x) = \frac{\varphi(x)}{\varphi(x_1)}
\]
locally uniformly in \( \Omega \setminus \{0\} \). \( \square \)

We use the following cutoff functions. For \( d \geq 3 \) define
\[
a_N(x) = \begin{cases} 
1 & |x| > \frac{2}{N}, \\
N \left( |x| - \frac{1}{N} \right) & \frac{1}{N} \leq |x| \leq \frac{2}{N}, \\
0 & |x| < \frac{1}{N}.
\end{cases}
\]
For \( d = 2 \), and for \( M < N \), define
\[
a_{N,M}(x) = \begin{cases} 
1 & |x| > \frac{1}{M}, \\
\log \frac{|x|N}{\log \frac{N}{M}} & \frac{1}{M} \leq |x| \leq \frac{1}{M}, \\
0 & |x| < \frac{1}{N},
\end{cases}
\]
and denote \( a_N := a_{N, \sqrt{N}} \). We have

**Theorem 4.2.** Suppose that the operator \( L \) of the form (1.3) is critical in \( \Omega \), and let \( \varphi \) be its ground state. There exists a null sequence \( \{u_N\} \subset H^1(\Omega) \) such that \( \text{supp } u_N \subset \Omega_N \) and \( \{u_N\} \) converges locally uniformly in \( \Omega \setminus \{0\} \) to \( \varphi \).

**Proof.** Set \( C := \max_{|x| \leq 1} \varphi(x) \). By Lemma 4.1, the sequence \( \{\varphi(x_1)\psi_N\} \) converges locally uniformly in \( \Omega \setminus \{0\} \) to \( \varphi \). Hence, there exists an increasing subsequence \( \{M_N\}_{N=1}^{\infty} \subset \mathbb{N} \) such that
\[
\sup_{\frac{1}{N} \leq |x| \leq 1} \varphi(x_1)\psi_{M_N}(x) \leq 2C.
\]
Consider the function \( u_N(x) := a_N(x)\varphi(x_1)\psi_{M_N}(x) \). It follows that \( \{u_N\} \) converges locally uniformly in \( \Omega \setminus \{0\} \) to \( \varphi \).

Note that
\[
\lim_{N \to \infty} \int_{\Omega} |\nabla a_N|^2 \, dx = 0. \tag{4.3}
\]
On the other hand, by the definition of $M_N$, $0 < \varphi(x_1)\psi_{M_N}(x) \leq 2C$ for all $\frac{1}{N} \leq |x| \leq 1$ and $N \geq 1$. Therefore, (4.3) and (1.1) imply that

$$ \lim_{N \to \infty} \int_{\Omega} A \nabla a_N \cdot \nabla a_N \left[ \varphi(x_1)\psi_{M_N}(x) \right] dx = 0. \quad (4.4) $$

Now, use (2.6) with $v = a_N$, and $\psi = \varphi(x_1)\psi_{M_N}(x)$, and (4.4) to verify that $\{u_N\}$ is indeed a null sequence. □

### 5. Finding a critical potential for a subcritical operator

If $P$ is subcritical, then the set $S_+$ (resp., $S$) of all continuous $W$, such that $P + W$ is subcritical (resp., $P + W$ is not supercritical) is convex [14], and as we have seen, contains a positive function. The set $S_0$ of all continuous $W$, such that $P + W$ is critical is contained in the set of all extreme points of $S$. Furthermore, it is known that if $P$ is subcritical and $W$ is a continuous function that takes a positive value at some point, and decays sufficiently fast at infinity (for example $W$ has a compact support), then there is $t_0$ such that $P - tW$ is subcritical for all $0 < t < t_0$, critical for $t = t_0$ and supercritical for $t > t_0$. The precise sufficient decay condition for $W$ to have such a property depends on $P$, $W$ and $\Omega$ via the Green function. An almost optimal general condition is that $W$ is a small or even semismall perturbation (notions that were introduced by Pinchover and Murata, respectively, see, for example [12]).

A critical positive weight $W$ can be found, in particular, via a minimizer of a nonlinear problem.

**Theorem 5.1.** Let $a$ be the quadratic form associated with a symmetric subcritical operator $L$ in $\Omega$. Let $W_0 \in L^\infty_{\text{loc}}(\Omega)$ be a nonnegative nonzero function, and let $p > 2$. If the minimum of the following constrained problem

$$ \kappa = \inf \left\{ a[u] \mid u \in \mathcal{H}_a(\Omega), \int_{\Omega} W_0|u|^p \, dx = 1 \right\} \quad (5.1) $$

is attained at some positive $v \in \mathcal{H}_a(\Omega)$, then the operator $L - W$ with $W = \kappa W_0 v^{p-2}$ is critical in $\Omega$.

Note that the conditions of the theorem fail if $\mathcal{H}_a(\Omega)$ is not continuously imbedded into $L^p(\Omega, W_0(x)dx)$, in particular, if $L$ is critical or $p > \frac{d+2}{d-2}$. Moreover, these conditions imply that $\kappa > 0$.

**Proof.** If the minimum in (5.1) is attained at $v > 0$, then $Lv = \kappa W_0 v^{p-1}$. In other words, $v \in C_{L-W}(\Omega)$. In particular, the operator $L - W$ is not supercritical. Let us show that $v$ is a null state for the quadratic form $b$ associated with $L - W$. Since $v \in \mathcal{H}_a(\Omega)$, there exists $v_k \in C^\infty_0$ such that $v_k \to v$ in $\mathcal{H}_a(\Omega)$. Theorem 1.5 implies
that \( v_k \to v \) in \( L^2_{\text{loc}}(\Omega) \). By the Fatou lemma \( \liminf \int_{\Omega} W_0|v_k|^{-2} |v_k|^2 \, dx \geq 1 \). Thus,
\[ 0 \leq \limsup b[v_k] = \lim a[v_k] - \liminf \int_{\Omega} W|v_k|^2 \, dx \leq \kappa - \kappa = 0. \]
Hence, \( v \) is a null state. \( \Box \)

The assumption that the minimum of problem (5.1) is attained is not trivial, and its verification typically requires a concentration-compactness argument. In a recent work \[17\], it is shown that the conditions of Theorem 5.1 are satisfied when \( \Omega = \mathbb{R}^d \setminus \mathbb{R}^{d-m} \), \( d > 3 \), \( 1 \leq m \leq d-1 \), \( p = \frac{2d}{d-2} \), \( W_0 = 1 \), and \( L = -\Delta - \frac{(m-2)^2}{2} \rho(x)^{-2} \), where \( \rho \) is the distance function to \( \partial \Omega \). Consequently, the minimum point \( v \) for

\[ \kappa = \inf_{\Omega} \frac{\int_{\Omega} |\nabla u|^2 - \left( \frac{m-2}{2} \right)^2 \frac{|u|^2}{\rho(x)^2} \, dx}{\int_{\Omega} |u|^2 \, dx = 1} \]  

(5.2)
is attained, and the operator \(-\Delta - \left( \frac{m-2}{2} \right)^2 \rho(x)^{-2} - \kappa v^{\frac{4}{d-2}}\) is critical.

Note added in proofs: The following converse of Theorem 1.4 can be easily proved.

**Claim.** If for any \( \psi \in L^2(\Omega) \) there exists \( C > 0 \) such that the corrected form \( a[u] + C \left| \int_{\Omega} \psi u \, dx \right|^2 \) is nonnegative on \( C_0^\infty(\Omega) \), then \( a \geq 0 \). More precisely, if there exists a function \( \psi \in L^2(\Omega) \) and \( C \in \mathbb{R} \) such that

\[ 0 \leq a[u] + C \left| \int_{\Omega} \psi u \, dx \right|^2 \quad \forall u \in C_0^\infty(\Omega; \mathbb{R}), \]  

(5.3)
then the negative \( L^2 \)-spectrum of \( L \) is either empty or consists of a single eigenvalue (cf. [8, Problem 4.5]). Moreover, if \( \psi \) lies in the positive spectral space of \( L \), then \( a \geq 0 \).

Note that under the Claim’s conditions, \( L \) is bounded below on \( L^2(\Omega) \), and therefore the spectral theorem applies.

To prove the first assertion note that the quadratic form in (5.3) is convex, therefore it is weakly lower semicontinuous in \( L^2(\Omega) \), and therefore \( a \) is weakly lower semicontinuous. The restriction of \( a \) to \( P_{(-\infty, 0]} L^2(\Omega) \) (the negative spectral space of \( L \)) is concave, and therefore also a weakly upper semicontinuous function, and thus weakly continuous. We conclude that the negative spectrum of \( a \) is discrete. If the assertion is false, then there exist two independent eigenvectors \( \phi_1 \) and \( \phi_2 \) corresponding to two negative eigenvalues \( \lambda_1 \leq \lambda_2 \). Then there is a linear combination \( \phi \) of \( \phi_1 \) and \( \phi_2 \) which is orthogonal to \( \psi \). Substituting \( u = \phi \) into (5.3) we arrive at a contradiction. To prove the second assertion, assume that \( a \) has a negative eigenvalue corresponding to an eigenfunction \( \phi \) and note that (5.3) fails for \( u = \phi \).
Acknowledgments

The authors wish to thank S. Agmon, S. Filippas, V. Maz’ya, L. Peres-Hari, G. Rozenblum, and A. Tertikas for valuable discussions. This research was partly done at the Technion as K.T. was a Lady Davis Visiting Professor, and continued at the University of Queensland, where K.T. was supported by the Ethel Raybould Visiting Fellowship and a grant from the Swedish Research Council. The work of Y.P. was partially supported by the RTN network “Nonlinear Partial Differential Equations Describing Front Propagation and Other Singular Phenomena”, HPRN-CT-2002-00274, and the Fund for the Promotion of Research at the Technion.

References