A Quillen Model Structure for Chu Spaces

Jeffrey M. Egger\textsuperscript{1,2}

Department of Mathematics and Statistics
University of Ottawa
Ottawa, Ontario, Canada

Abstract

Barr introduced Chu categories as a general construction for generating $*$-autonomous categories, the basic framework for the semantics of Girard’s linear logic. Barr singles out two classes of objects in a Chu category for special consideration, the separated and extensional objects. It is shown in [2] that, under certain circumstances, one can induce a $*$-autonomous structure on the full subcategory of these objects. The manner in which this is done, and the nature of the hypotheses involved, suggest the existence of a homotopy-theoretic interpretation of these ideas. In this paper, we show that this is indeed the case. In particular, we show that it is possible to put a Quillen model structure on certain Chu categories in such a way that a Chu space is separated if and only if it is fibrant, and extensional if and only if it is cofibrant.

Keywords: Linear logic, Abstract homotopy theory, Chu spaces.

1 Introduction

Girard’s linear logic has become one of the most important logical frameworks in the field of categorical logic, and the study of its semantics has led to a number of breakthroughs in the field. The basic unit in the semantics of linear logic is the theory of monoidal categories. The study of linear logic has led to new approaches to producing coherence theorems for various types of symmetric monoidal closed categories. It has led to numerous new examples of monoidal closed categories such as Girard’s coherence spaces and

\textsuperscript{1} Partially funded by NSERC.
\textsuperscript{2} Email: jeffegger@yahoo.ca
Ehrhard’s Köthe spaces, and has led to several new constructions for producing ∗-autonomous categories, i.e. symmetric monoidal closed categories with a strong notion of duality. The most well-known such construction is the Chu construction, due to Barr.

In [2], Barr singles out a special class of objects within any Chu category. These are the separated, extensional objects. While these do not necessarily form a ∗-autonomous category, there are many special cases in which this is indeed the case. In that case, the smaller category is typically more tractable. As a key example, if one begins with the category of discrete vector spaces, then the corresponding separated, extensional category is indeed ∗-autonomous and has a simple interpretation as a category of topological linear spaces, as introduced by Lefschetz [9]. The separated, extensional subcategory is also of interest when one begins with the category of Banach spaces. There one obtains a category of mixed topological spaces, see [1].

The main goal of this paper and of the author’s thesis research is to apply ideas of homotopy theory to analyse the semantics of linear logic; more specifically we consider Quillen model categories. A Quillen model structure [7] is a structure on an abstract category which allows one to ‘do homotopy’, i.e. to mimic the algebraic and topological manipulations inherent in homotopy theory. The impact of Quillen’s ideas has been enormous. Homotopy-theoretic ideas have been applied to a wide variety of structures in many branches of mathematics. Consideration of the many examples in [7] show how widespread these ideas are.

Our main result is that many Chu categories can be equipped with a natural Quillen model structure in such a way that the resultant homotopy category is equivalent to the smaller, and in many ways more interesting, category of separated and extensional Chu spaces.

2 Background

First, we recall some of the basic notions which we will be using throughout this paper.

We draw particular attention to the concept of Quillen model category, one of the principal tools of abstract homotopy theory. Concrete homotopy theory is, of course, one of the pillars of algebraic topology. But Quillen observed that the concrete notions of cofibration, weak homotopy equivalence, and Serre fibration (see, for example, [13], where Serre fibrations are also called weak fibrations) which arise in algebraic topology could be axiomatised—not individually, but in terms of the categorical properties they share in relation to one another. Moreover, all of the remaining concepts of (concrete) homotopy
theory can be phrased in terms of these classes of continuous maps. Thus, one
is able to import topological intuition to categories other than those consisting
of topological spaces and continuous maps, see for example [8].

2.1 Monoidal model categories

A Quillen model category is a finitely complete and cocomplete category \( \mathcal{A} \)
with three distinguished classes of morphisms: \( \mathcal{C} \), whose elements are called
cofibrations; \( \mathcal{W} \), whose elements are called weak equivalences; and \( \mathcal{F} \), whose
elements are called fibrations. \( \mathcal{C} \), \( \mathcal{W} \), and \( \mathcal{F} \), which we collectively refer to as
the Quillen model structure of \( \mathcal{A} \), are required to satisfy a list of axioms which
can be summarised as follows: both \((\mathcal{C} \cap \mathcal{W}, \mathcal{F})\) and \((\mathcal{C}, \mathcal{W} \cap \mathcal{F})\) should form
weak factorisation systems [12]; and, \( \mathcal{W} \) should satisfy the 2-out-of-3 rule—
i.e., if any two of \( \omega, \psi, \psi \circ \omega \) are in \( \mathcal{W} \) then so is the third. These axioms are
self-dual in the sense that if \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) is a Quillen model structure for \( \mathcal{A} \), then
\((\mathcal{F}, \mathcal{W}, \mathcal{C})\) is a Quillen model structure for \( \mathcal{A}^{\text{op}} \).

The homotopy category of a Quillen model category can be described either
as the category of fractions \( \mathcal{A}[\mathcal{W}^{-1}] \), or (equivalently, but not isomorphically)
as the result of quotienting a full subcategory of \( \mathcal{A} \) by a congruence [10]. The full subcategory in question consists of those \( x \) such that: the unique map
\( x \to 1 \) belongs to \( \mathcal{F} \) (such objects are called fibrant; and, the unique map
\( 0 \to x \) belongs to \( \mathcal{C} \) (such objects are called cofibrant).

It is worth noting at this stage that if either \((\mathcal{C} \cap \mathcal{W}, \mathcal{F})\) or \((\mathcal{C}, \mathcal{W} \cap \mathcal{F})\)
is a strong factorisation system (that is, a factorisation system in the usual
sense, see [2]), then the congruence obtained is trivial, so that the homotopy
category is equivalent to the full subcategory of fibrant and cofibrant objects
of \( \mathcal{A} \). It is this seemingly degenerate case which shall be the focus of the
present paper. Note also that any single factorisation system can be made
into a Quillen model structure by choosing \( \mathcal{W} \) to be the class of all morphisms
in \( \mathcal{A} \).

A monoidal model category is a monoidal closed Quillen model category
satisfying extra axioms designed to guarantee that the homotopy category has
an induced monoidal closed structure. We shall treat only the symmetric case,
which requires no further axioms, from here on in.

The key lemma, which follows from the definition of symmetric monoidal
model category and which is basically all that is needed to prove that the
homotopy category is monoidal closed, is that the adjunctions

\[
\begin{align*}
\mathcal{A} & \xrightarrow{\cdot x} \mathcal{A} \\
\mathcal{A}^{\text{op}} & \xleftarrow{\cdot x} \mathcal{A}
\end{align*}
\]

should be Quillen adjunctions, not necessarily for all objects \( x \) and \( z \), but at
least when $z$ is fibrant and when $x$ is cofibrant.

A Quillen adjunction is an adjunction between Quillen model categories whose left part (i.e., the left adjoint) preserves the ‘left parts’ of the Quillen model structure: $C$ and $C \cap W$. This is equivalent to the preservation of the ‘right parts’ of the Quillen model structure, $F$ and $W \cap F$, by the right part of the adjunction. In the case of the second adjunction, this equivalence is trivial! Both statements require that $(-) \to z$ should map $C$ to $F$ and $C \cap W$ to $W \cap F$.

It is the (eery) similarity between these conditions and Barr’s axioms FS–2 through FS–4, which persuaded me to investigate the situation.

2.2 Chu spaces

Given a symmetric monoidal category $V$ with pullbacks, and an object $d$ in $V$, we define a category $\text{Chu} = \text{Chu}(V, d)$ as follows.

The objects of $\text{Chu}$, alias Chu spaces, are triples $A = (a^+, a^-, \alpha)$ where $a^+$, $a^-$ are objects in $V$ and $\alpha$ is a morphism $a^+ \otimes a^- \to d$ in $V$. A morphism $A \xrightarrow{\Theta} B$ in $\text{Chu}$ is a pair of arrows $a^+ \xrightarrow{\theta^+} b^+$ and $b^- \xrightarrow{\theta^-} a^-$ such that

Now suppose we apply the Chu construction to a monoidal closed category $V$ equipped with a factorisation system $(\mathcal{E}, \mathcal{M})$. Then, given a Chu space $A = (a^+, a^-, \alpha)$, we can consider the two transposes of $\alpha$:

$A$ is said to be separated if $\hat{\alpha} \in \mathcal{M}$ and extensional if $\check{\alpha} \in \mathcal{M}$. We write $\text{Chu}_s$ for the full subcategory of $\text{Chu}$ consisting of separated spaces, $\text{Chu}_e$ for that of extensional spaces, and $\text{chu}$ for their intersection.

Now the whole point of the Chu construction is that $\text{Chu}$ is a $\ast$-autonomous category (see 4.1 below); but $\text{chu}$ is not generally a monoidal subcategory, let alone a sub-$\ast$-autonomous category, of $\text{Chu}$. However, [2] shows it is possible to induce a $\ast$-autonomous structure on $\text{chu}$ if $(\mathcal{E}, \mathcal{M})$ satisfies either FS–1 and FS–2, or FS–2 and FS–3.

FS–1. $\mathcal{E} \subseteq \{\text{epis}\}$. 
FS–2. For every $x$, the functor $x \to (\cdot)$ preserves $\mathcal{M}$; equivalently,
\[
\mathcal{V} \xleftrightarrow{x \to (\cdot)} \mathcal{V}
\]
is a Quillen adjunction, where, as suggested in 2.1, we choose $\mathcal{C} = \mathcal{E}$, $\mathcal{F} = \mathcal{M}$
and $\mathcal{W} =$ the class of all arrows in $\mathcal{V}$.

FS–3. For every $z$, the functor $(-) \to z$ maps $\mathcal{E}$ to $\mathcal{M}$; equivalently,
\[
\mathcal{V}^{\text{op}} \xleftrightarrow{(-) \to z} \mathcal{V}
\]
is a Quillen adjunction, with respect to the same choice of $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{F}$.

Moreover, the method of inducing this $\ast$-autonomous structure on chu appears
to be a simplified version of the method of inducing a monoidal closed
structure on a monoidal model category. Thus, one is naturally led to con-
sider the possibility that the factorisation system on $\mathcal{V}$ has made Chu into a
monoidal model category whose homotopy category is chu.

Now although Barr makes a persuasive case, based on the sort of applica-
tions he has in mind, that FS–3 is a less desirable axiom than FS–1, it should
be clear that our bias will be in favour of FS–3 over FS–1. This is partly
because of the obvious parallel which we hope to exploit, but also because
we would be interested to generalise the results to weak factorisation systems,
mentioned earlier, and possibly further; but a weak factorisation system sat-
sifying FS–1 is necessarily strong! Indeed, in abstract homotopy theory, it is
typical for $\mathcal{C}$ to be a class of monics.

2.3 Reflexive factorisation systems

Let us, for a moment, restrict our attention to finitely-complete categories
$\mathcal{A}$ admitting arbitrary intersections of subobjects. [3] calls such a category
finitely well-complete, whereas [2] uses the term wide-complete for such a cat-
egory that also has (small) infinite limits.

Then there exists a bijective\(^3\) correspondence between full reflective sub-
categories of $\mathcal{A}$ and factorisation systems $(\mathcal{W}_i, \mathcal{F})$ on $\mathcal{A}$ such that $\mathcal{W}_i$ satisfies
2-out-of-3.

One direction of this correspondence is easy to describe: first, note that the
type of factorisation system we are dealing with can be made into a Quillen
model structure for $\mathcal{A}$, by taking $\mathcal{C}$ to be the class of all maps in $\mathcal{A}$; then,
we take the full subcategory of fibrant objects (which equals the homotopy

\(^3\) Here we consider two full subcategories of $\mathcal{A}$ to be the same if they differ only by which
representatives of an isomorphism class they contain.
category since there is no congruence and all objects are cofibrant), as the corresponding subcategory—it is reflective because of the strongness of the factorisation system.

One half of the other direction is also easy to describe: given a reflective subcategory of $A_l$ of $A$, with reflector $l$, we define $W_l$ to consist of those arrows $\omega$ in $A$ such that $l\omega$ is an isomorphism in $A_l$. $F$ is therefore determined as those arrows which satisfy the appropriate lifting axiom with respect to $W_l$—but $F$ is not easy to describe, in general.

If, however, $A_l$ is what [3] call a semi-left-exact full reflective subcategory of $A$, then $F$ can be described as those arrows $x \xrightarrow{\psi} y$ in $A$ such that the naturality square

\[
\begin{array}{ccc}
lx & \xrightarrow{\eta_x} & x \\
l\psi \downarrow & & \downarrow \psi \\
l\psi & \xrightarrow{\eta_y} & y
\end{array}
\]

is a pullback square. Moreover, in this case, one can dispense with the hypothesis that $A$ admits arbitrary intersections.

3 A Quillen model structure for Chu

Applying the observations of 2.3 to $\text{Chu}_s$, which is always a reflective subcategory of $\text{Chu}$, and dually to $\text{Chu}_c$, we see that, modulo ‘niceness’ conditions, one should be able to derive a pair of factorisation systems $(W_l, F)$ and $(C, W_r)$ such that both $W_l$ and $W_r$ satisfy 2-out-of-3. What remains to show is the existence of a class $W$, satisfying 2-out-of-3, and with $W_l = C \cap W$ and $W_r = W \cap F$. By construction, one would then have $\text{Chu}_s$ equalling the full subcategory of fibrant objects, $\text{Chu}_c$ that of cofibrant objects, and therefore $\text{chu}$ as the homotopy category.

But without a concrete description of either $F$ or $C$, this is quite difficult. It would therefore make sense to try to find extra conditions on $(\mathcal{E}, \mathcal{M})$ which guarantee that $\text{Chu}_s$ is a semi-left-exact full subcategory of $\text{Chu}$. However, semi-left-exactness is a rather difficult condition to prove. So we instead take the approach of showing directly that, with a single further axiom—namely that $\mathcal{E}$ is closed under pullback, we get factorisation systems of the correct type.

For the remainder of this paper $(\mathcal{E}, \mathcal{M})$ shall denote a strong factorisation system on $\mathcal{V}$, which is in turn an arbitrary symmetric monoidal closed category with finite limits and coproducts. $d$ is an arbitrary object of $\mathcal{V}$, and $\text{Chu}$ is the corresponding Chu category $\text{Chu}(\mathcal{V}, d)$. 
**Definition 3.1** Let $A = (a^+, a^-, \alpha)$ and $B = (b^+, b^-, \beta)$ be Chu spaces and $A \xrightarrow{\Theta} B$ a Chu morphism.

(i) We write $a^l$ for the $(\mathcal{E}, \mathcal{M})$-factorisation of $a^+ \rightarrow a^- \rightarrow d$, and $\theta^l$ for the unique lift

$$
\begin{array}{c}
a^+ \\
\theta^+ \\
b^+
\end{array} \xrightarrow{\sim} 
\begin{array}{c}
a^b \\
a^- \\
b^-
\end{array} \xrightarrow{\sim} 
\begin{array}{c}
a^+ \\
\theta^- \circ d \\
b^+
\end{array}
$$

(ii) We define $\mathcal{W}_l$ to be the class of all $\Theta$ such that $\theta^-$ and $\theta^l$ are invertible.

(iii) We define $\mathcal{F}$ to be the class of all $\Theta$ such that

$$
\begin{array}{c}
a^+ \\
\theta^+ \\
b^+
\end{array} \xrightarrow{\sim} 
\begin{array}{c}
a^l \\
\theta^l \\
b^l
\end{array}
$$

is a pullback square.

Dually,

(iv) We write $a^r$ for the $(\mathcal{E}, \mathcal{M})$-factorisation of $a^- \rightarrow a^+ \rightarrow d$, and $\theta^r$ for the unique lift

$$
\begin{array}{c}
a^- \\
\theta^- \\
b^-
\end{array} \xrightarrow{\sim} 
\begin{array}{c}
a^r \\
a^+ \\
b^+
\end{array} \xrightarrow{\sim} 
\begin{array}{c}
b^+ \\
\theta^+ \circ d \\
\theta^r
\end{array}
$$

(v) We define $\mathcal{W}_r$ to be the class of all $\Theta$ such that $\theta^+$ and $\theta^r$ are invertible.

(vi) We define $\mathcal{C}$ to be the class of all $\Theta$ such that

$$
\begin{array}{c}
a^- \\
\theta^- \\
b^-
\end{array} \xrightarrow{\sim} 
\begin{array}{c}
a^r \\
\theta^r \\
b^r
\end{array}
$$

is a pullback square.

Finally,

(vii) We define $\mathcal{W}$ to be the class of all $\Theta$ such that $\theta^l$ and $\theta^r$ are invertible.

**Theorem 3.2** If $\mathcal{E}$ is stable under pullbacks in $\mathcal{V}$, then both $(\mathcal{W}_l, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W}_r)$ are factorisation systems on $\text{Chu}$.

**Proof.** By symmetry, we need only show that $(\mathcal{W}_l, \mathcal{F})$ is a factorisation system.

The factorisation axiom is the easiest to verify: given Chu spaces $X$ and
Y and a Chu morphism $X \xrightarrow{\Theta} Y$, let $a^+, \lambda^+$ and $\phi^+$ be as indicated:

Let also $a^- = x^-$, $\lambda^-$ the identity on $x^-$, and $\phi^- = \theta^-$. Then $A = (a^+, a^-, \alpha)$ is a Chu space where $\alpha$ is the transpose of the composite

$$a^+ \xrightarrow{\lambda^+} x^l \xrightarrow{\phi^+} y^l \xrightarrow{\phi^-} y$$

Then $\Lambda = (\lambda^+, \lambda^-)$ is a Chu morphism belonging to $\mathcal{W}_l$, and $\Phi = (\phi^+, \phi^-)$ is a Chu morphism belonging to $\mathcal{F}$.

Next the lifting axiom: given a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\Theta} & X \\
\downarrow{\Lambda} & & \downarrow{\Phi} \\
B & \xrightarrow{\Omega} & Y
\end{array}
$$

of Chu spaces and Chu morphisms, with $\Lambda \in \mathcal{W}_l$ and $\Phi \in \mathcal{F}$, we define $B \xrightarrow{\Delta} X$ as follows.

The fact that we can invert $\lambda^l$ in the following diagram

allows us to construct an arrow

$$
\begin{array}{ccc}
b^+ & \xrightarrow{\delta^+} & x^+ \\
\downarrow{\omega^+} & & \downarrow{\phi^+} \\
y^+ & \xrightarrow{\phi^-} & y^l
\end{array}
$$

—since the right-hand square is, by hypothesis, a pullback.

Now if we define $\delta^-$ to be the composite $x^- \xrightarrow{\theta^-} a^- \xrightarrow{(\lambda^-)^{-1}} b^-$, then $\Delta =$
$(\delta^+, \delta^-)$ is a Chu morphism $B \longrightarrow X$, as demonstrated below.

Moreover, it is easy to check that $\Delta$ is the unique Chu morphism such that

For example,

follows from the universal property of the pullback above.

Finally, it should be clear that $\mathcal{W}_l$ and $\mathcal{F}$ are closed under isomorphisms and composition.

**Theorem 3.3** If $\mathcal{E}$ is stable under pullbacks in $\mathcal{V}$ and the functor $(-) \rightarrow d$ maps $\mathcal{E}$ to $\mathcal{M}$, then $\mathcal{W}_r = \mathcal{W} \cap \mathcal{F}$ and $\mathcal{W}_l = \mathcal{C} \cap \mathcal{W}$. Moreover, $\mathcal{W}$ always satisfies 2-out-of-3. Hence, under these hypotheses, $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ forms a Quillen model structure on $\Chu$.

**Proof.** Firstly observe that $(\psi \circ \omega)^l = \psi^l \circ \omega^l$, by the uniqueness of lifts under the $(\mathcal{E}, \mathcal{M})$ factorisation. [I.e., $(-)^l$ is actually a functor $\Chu \longrightarrow \mathcal{V}$.] Hence if any two of $\omega^l$, $\psi^l$, $(\psi \circ \omega)^l$ are invertible, then so is the third. Similarly $(\psi \circ \omega)^r = \omega^r \circ \psi^r$; hence $\mathcal{W}$ satisfies 2-out-of-3.

Now suppose $\Phi = (\phi^+, \phi^-) \in \mathcal{W}_r$. Then applying the functor $(-) \rightarrow d$ to the diagram
we obtain a diagram

\[
\begin{array}{c}
\bullet & \xleftarrow{(\phi^-\circ d)} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xleftarrow{(\phi^r\circ d)} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xleftarrow{(\phi^+\circ d)} & \bullet \\
\end{array}
\]

By lifting, we get

\[
\begin{array}{c}
\bullet & \xleftarrow{(\phi^+)} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xleftarrow{(\phi^r)} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xleftarrow{(\phi^-)} & \bullet \\
\end{array}
\]

By uniqueness of factorisation, we get that both dotted arrows are in \(\mathcal{M}\).

But both \(\phi^+\) and \((\phi^r\circ d)\) are invertible, so the uniqueness of factorisation [again!] implies that \(\phi^r\) is invertible; i.e., \(\phi \in \mathcal{W}\). But the leftmost square is also, trivially, a pullback; i.e. \(\phi \in \mathcal{F}\).

Conversely, the pullback of an iso being an iso shows that \(\mathcal{W} \cap \mathcal{F} \subseteq \mathcal{W}_r\).

**Theorem 3.4** A Chu space is separated if and only if it is fibrant with respect to the Quillen model structure given above, and extensional if and only if it cofibrant.

**Proof.** Let 0 and 1 denote the initial and terminal objects, respectively, of \(\mathcal{V}\). Then the terminal Chu space is \(T = (1, 0, !)\) where ! is the unique map \(1 \otimes 0 \to d\). Note that \(0 \to d \cong 1\). So for \(X \to T\) to be a fibration means that we have a pullback square

\[
\begin{array}{c}
x^+ \ar[r] & x^r \\
1 \ar[r] \ar[u] & 1 \ar[u] \\
\end{array}
\]

and the pullback of an iso is an iso. \(\square\)

### 4 Monoidal structure

In this section, we investigate under what circumstances the Quillen model structure defined above, \((\mathcal{C}, \mathcal{W}, \mathcal{F})\), satisfies the key lemma—discussed in 2.1 above, and proven as Corollary 4.3 below—needed to conclude that its homotopy category is symmetric monoidal closed. The hypotheses of Theorem 3.3 will be assumed throughout this section; note that one of them is a special case of Barr’s axiom FS–3.

First let us recall the \(*\)-autonomous structure on \(\text{Chu}\).

**Definition 4.1** Let \(A = (a^+, a^-, \alpha)\) and \(X = (x^+, x^-, \chi)\) be Chu spaces.
Then

(i) $A \otimes X$ is defined to be the Chu space $((a \cdot x)^+, (a \cdot x)^-, \alpha \cdot \chi)$, where
- $(a \cdot x)^+$ is defined to be $a^+ \otimes x^+$;
- $(a \cdot x)^-$ is defined to be the pullback below:

\[
\begin{array}{ccc}
(a \cdot x)^- & \to & (a^+ \to x^-) \\
\downarrow & & \downarrow \\
(x^+ \to a^-) & \to & (x^+ \to (a^+ \to d))
\end{array}
\]

and
- $\alpha \cdot \chi$ is defined to be the transpose of the map

\[
\begin{array}{ccc}
(a \cdot x)^- & \to & ((a \cdot x)^+ \to d)
\end{array}
\]

in the diagram above.

(ii) $A^*$ is defined to be $(a^-, a^+, \alpha \circ \sigma)$ where $\sigma$ is the symmetry map $a^- \otimes a^+ \to a^+ \otimes a^-$.  

(iii) $A \to X$ is defined to be $(X^* \otimes A)^*$.

It is well-known that the operations defined above do indeed make $\text{Chu}$ into a $\ast$-autonomous category, in particular, into a symmetric monoidal closed category with unit $E = (e, d, \nu)$, where $\nu$ is the canonical isomorphism $e \otimes d \to d$.

Now it is clear from all the definitions involved that the functor $(-)^*$ preserves and reflects $\mathcal{W}$ while swapping $\mathcal{C}$ and $\mathcal{F}$. As we shall see below, this observation greatly simplifies our proof of Corollary 4.3; it is also a sufficient, though far from necessary, condition for the homotopy category to be not merely symmetric monoidal closed but $\ast$-autonomous.

Also, note that $E$ is cofibrant, since $\tilde{\nu}$ is the canonical isomorphism $d \to e \to d$. This eliminates the need for an annoying technical condition. For a full treatment of $\ast$-autonomous model categories, see [4].

**Theorem 4.2** If $(\mathcal{E}, \mathcal{M})$ satisfies Barr’s axiom FS–2 and $\mathcal{M} \subseteq \{\text{monos}\}$, then $X$ cofibrant implies $X \otimes (-)$ preserves cofibrations.
\textbf{Proof.} To understand \((a \cdot x)^r\), note that we have

\[
\begin{array}{ccc}
(a \cdot x)^r & \rightarrow & (a^+ \circ x^-) \\
\downarrow & & \downarrow \\
(a^+ \circ (x^+ \circ d)) & \rightarrow & (a^+ \circ (x^+ \circ d)) \\
\end{array}
\]

since \((x^- \rightarrow x^+ \circ d) \in \mathcal{M}\) and \(a^+ \circ (-)\) preserves \(\mathcal{M}\).

Similarly, we obtain the dotted arrows below from the hypothesis that \(x^+ \circ (-)\) preserves \(\mathcal{M}\).

\[
\begin{array}{ccc}
(a \cdot x)^r & \rightarrow & (a^+ \circ x^-) \\
\downarrow & & \downarrow \\
(a^+ \circ (x^+ \circ d)) & \rightarrow & (a^+ \circ (x^+ \circ d)) \\
\end{array}
\]

We need to show that the large rectangle at the left (the one with dashed verticals) is a pullback. But the little square at the centre-left is a pullback by hypothesis, and because the functor \(x^+ \circ (-)\), being a right adjoint, preserves pullbacks.

Now it suffices that the two medium-sized trapezoids at the top left and bottom left are pullbacks.

But this follows from the fact that the large rectangles at the top and bottom are pullbacks by an easy diagram chase — if the \(\rightarrow\)s are indeed monic. \(\square\)

\textbf{Corollary 4.3} Under the hypotheses of Theorem 4.2, if \(X\) is cofibrant and \(Z\) is fibrant, then both of the adjunctions

\[
\begin{array}{ccc}
\text{Chu} & \xrightarrow{X \circ (-)} & \text{Chu} \\
\xleftarrow{X \otimes (-)} & & \xrightarrow{(-) \circ Z} \\
\end{array}
\]

\text{Chu}^\text{op} \xrightarrow{(-) \circ Z} \text{Chu}

are Quillen adjunctions.
Proof. We have shown that the functor $X \otimes (-)$ preserves cofibrations, whenever $X$ is cofibrant. It therefore follows that the functor $X \otimes (-)^*$ maps $\mathcal{F}$ to $\mathcal{C}$, and hence that $X \to (-) \cong (X \otimes (-)^*)^*$ maps $\mathcal{F}$ to $\mathcal{F}$. But this is equivalent to the assertion that $X \otimes (-)$ preserves $\mathcal{C} \cap \mathcal{W}$. Hence the first adjunction is a Quillen adjunction.

Similarly, if $Z$ is fibrant, then $Z^*$ is cofibrant. So $(Z^* \otimes (-))$ preserves both $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$, and hence $(-) \to Z \cong (Z^* \otimes (-))^*$ maps $\mathcal{C}$ to $\mathcal{F}$ and $\mathcal{C} \cap \mathcal{W}$ to $\mathcal{W} \cap \mathcal{F}$. So the second adjunction is also a Quillen adjunction.

5 Conclusions

The intended value of this work lies not so much in the enlightenment of the theory of Chu spaces as the demonstration of the implicitness of abstract homotopy theoretic concepts in the existing linear logic literature.

Future work includes the study of Ehrhard’s serial and parallel hypercoherence spaces [5]. It again seems evident, from the manner of Ehrhard’s presentation, that one ought to be able to put a Quillen model structure on the category of hypercoherence spaces in such a way that the fibrant objects coincide with the parallel hypercoherences, and cofibrant objects with the serial hypercoherences. Moreover the concept of a ‘parallel unfolding’ of a hypercoherence ought to coincide with that of a ‘fibrant replacement’—i.e, a factorisation of the unique map $x \to 1$ as an acyclic cofibration followed by a fibration.

Since parallel unfoldings are not unique, this opens the possibility of a non-trivial homotopy relation, and hence a homotopy category which is a non-trivial quotient of the full subcategory of serial and parallel hypercoherence spaces.

Another question which can be seen as arising from this research is the following. Suppose that $\mathcal{A}$ is a Quillen model category and that its homotopy category, $\mathcal{B}$, also admits a Quillen model structure. Is it possible to describe the homotopy category of $\mathcal{B}$ as a homotopy category of $\mathcal{A}$? I.e., does there exist another Quillen model structure for $\mathcal{A}$ whose homotopy category coincides with that of $\mathcal{B}$?

[Note that $\mathcal{B}$ is not generally finitely complete or cocomplete, but it does have finite products and coproducts. And, while the most intuitive proof of transitivity of the homotopy relation uses pushouts and pullbacks, there exists an alternate proof which does not use this extra structure. In fact, we show in [4] that it is possible to effect the construction of a homotopy theory in this more general context.]

This question arises because $\text{chu}$ is a co-reflective subcategory, and hence a
homotopy category of, \( \text{Chu}_s \)—while \( \text{Chu}_s \) is a reflective subcategory, hence a homotopy category of \( \text{Chu} \). Effectively, we have shown that, in certain cases, the ‘double homotopy’ category \( \text{chu} \) can be expressed directly as a homotopy category of \( \text{Chu} \).

What would be even more interesting would be if any of the known homotopy categories can be understood as ‘double homotopy’ categories. I.e., if there were interesting intermediates between, for example, the category of simplicial sets and its homotopy category.

6 Appendix

The notion of weak factorisation system, though important in algebraic topology, seems to have been ignored by the categorical mainstream until very recently. For completeness’ sake we include its definition and a couple of important lemmas.

**Definition 6.1** A weak factorisation system for a category \( \mathcal{A} \) consists of two subclasses \( \mathcal{L} \) and \( \mathcal{R} \) of the arrows of \( \mathcal{A} \) such that

(i) every arrow \( \alpha \) in \( \mathcal{A} \) can be factored \( \alpha = \rho \circ \lambda \) where \( \lambda \in \mathcal{L} \) and \( \rho \in \mathcal{R} \);  

(ii) for every commutative square

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\alpha} & \bullet \\
\downarrow{\beta} & & \downarrow{\rho} \\
\bullet & \xrightarrow{\lambda} & \bullet 
\end{array}
\]

with \( \lambda \in \mathcal{L} \) and \( \rho \in \mathcal{R} \), there should exist a (not necessarily unique) map \( \delta \)

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\alpha} & \bullet \\
\downarrow{\beta} & & \downarrow{\rho} \\
\bullet & \xrightarrow{\lambda} & \bullet 
\end{array}
\]

which makes both triangles commute; and

(iii) for every retract, \( \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \), in \( \mathcal{A} \):

(a) if \( \lambda \in \mathcal{L} \) and

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\alpha} & \bullet \\
\downarrow{\mu} & & \downarrow{\lambda} \\
\bullet & \xrightarrow{\beta} & \bullet \\
\downarrow{\mu} & & \downarrow{\beta} 
\end{array}
\]

commutes, then \( \mu \in \mathcal{L} \);
(b) if \( \rho \in \mathcal{R} \) and

\[
\begin{array}{c}
\bullet \\
\alpha \\
\downarrow \rho \\
\bullet \\
\hline
\sigma \\
\downarrow \sigma \\
\bullet \\
\hline
\beta \\
\downarrow \sigma \\
\bullet \\
\end{array}
\]

commutes, then \( \sigma \in \mathcal{R} \).

The analogue of [2, Proposition 5.4] holds.

**Lemma 6.2** If \((\mathcal{L}, \mathcal{R})\) form a weak factorisation system for \(\mathcal{A}\), and \(\sigma\) has the lifting property with respect to all arrows in \(\mathcal{L}\), then \(\sigma \in \mathcal{R}\).

**Proof.** Factor \(\sigma\) as a map \(\lambda \in \mathcal{L}\) followed by a map \(\rho \in \mathcal{R}\). Therefore we have a commutative square

\[
\begin{array}{c}
\bullet \\
\lambda \\
\downarrow \rho \\
\bullet \\
\hline
\sigma \\
\end{array}
\]

which can be lifted, by hypothesis, as follows:

\[
\begin{array}{c}
\bullet \\
\lambda \\
\downarrow \rho \\
\bullet \\
\hline
\sigma \\
\end{array}
\]

hence we have a retract \(\lambda \to \delta\) and

\[
\begin{array}{c}
\bullet \\
\lambda \\
\downarrow \rho \\
\bullet \\
\hline
\sigma \\
\end{array}
\]

Therefore we have \(\sigma \in \mathcal{R}\), as desired. \(\square\)

This lemma, and its dual, have the usual corollaries: \(\mathcal{L}\) and \(\mathcal{R}\) are closed under composition, contain all the isomorphisms, and preserved by certain sorts of (co-)limits.

To show that our definition of Quillen model category agrees with that of [7,11], it is necessary to prove the following:

**Lemma 6.3** If \(\mathcal{A}\) is a finitely complete and co-complete category with three distinguished classes of morphisms, \(\mathcal{C}, \mathcal{W}, \text{ and } \mathcal{F}\), such that both \((\mathcal{C} \cap \mathcal{W}, \mathcal{F})\) and \((\mathcal{C}, \mathcal{W} \cap \mathcal{F})\) form weak factorisation systems and \(\mathcal{W}\) satisfies the 2-out-of-3 rule, then \(\mathcal{W}\) is closed under all retracts in \(\mathcal{A}^\to\).

**Proof.** Let \(\alpha \in \mathcal{W}\), and \(\beta\) a retract of \(\alpha\). Then we can factor \(\beta\) as \(\phi \circ \lambda\), where \(\lambda \in \mathcal{C} \cap \mathcal{W}\) and \(\phi \in \mathcal{F}\). It suffices to show that \(\phi \in \mathcal{W}\).
Form the following pushout.

Observe that $\mu \in C \cap W$, since it is a pushout of $\lambda$. Hence $\theta \in W$, by 2-out-of-3. Now $\phi$ is a retract of $\theta$.

But $\theta$ can be factored as $\rho \circ \kappa$ with $\kappa \in C \cap W$, and $\rho \in W \cap F$. We can then apply the lifting property, as follows:

——which shows that $\phi$ is also a retract of $\rho$, and therefore also in $W \cap F$. Hence, one last application of 2-out-of-3 yields $\beta = \phi \circ \lambda \in W$. □

References


