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On the topology of components of some Springer fibers and their relation to Kazhdan–Lusztig theory

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Abstract

We describe the irreducible components of Springer fibers for hook and two-row nilpotent elements of $\mathfrak{gl}_n(\mathbb{C})$ as iterated bundles of flag manifolds and Grassmannians. We then relate the topology (in particular, the intersection homology Poincaré polynomials) of the pairwise intersections of these components with the inner products of the Kazhdan–Lusztig basis elements of irreducible representations of the rational Iwahori–Hecke algebra of type A corresponding to the hook and two-row Young shapes.

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1. Introduction

Let V be a finite-dimensional complex vector space. A nilpotent linear map $N : V \to V$ is said to fix a flag $F = \{F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset V\}$ if $NF_i \subseteq F_{i-1}$ for each i. The variety $\mathscr{B}_N$ of all flags in the flag manifold $\mathrm{Fl}(V)$ fixed by a nilpotent map N is a *Springer fiber*. Such varieties arise as fibers of Springer's resolution of singularities of the nilpotent cone of a reductive algebraic group G.

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Springer [25,27] discovered a method of constructing irreducible representations of Weyl groups on the top homology of $\mathscr{B}_N$. For each irreducible representation, his construction yields a distinguished basis given by homology classes of the components of the Springer fiber $\mathscr{B}_N$. However, since their mere existence yields the distinguished basis, it seems that efforts to understand them have not focused on computations of their internal topological structure or that of their pairwise intersections. Only a few papers (such as [24,28,30,21,14]) have studied the topology of the components of the Springer fibers $\mathscr{B}_N$ and their pairwise intersections. We extend some of these results to describe the homological structure of components of $\mathscr{B}_N$ and their pairwise intersection for certain types of nilpotent maps N (those corresponding to hook and two-row shape partitions). In these cases, the components are nonsingular, and in fact are iterated bundles of flag manifolds and Grassmannians. For more general nilpotent maps N, the components can be singular (see [28,24]) and much more complicated.

We also relate our computations to the structure of the Kazhdan–Lusztig bases of certain representations of Iwahori–Hecke algebras of type A. The inner products of these basis vectors, suitably normalized, are polynomials in t and t^{-1} that are invariant under the map $t \to t^{-1}$. We show that for irreducible representations labeled by a hook or two-row shape, the (suitably normalized) inner products equal the intersection homology [10,12] Poincaré polynomials of pairwise intersections of irreducible components of Springer fibers of the general linear group. We believe it would be very interesting to understand how our results might generalize to bases of other Kazhdan–Lusztig representations of type A.

2. Some properties of nilpotent maps and Springer fibers

We record some properties of nilpotent maps and of the space of all flags $\mathscr{B}_N$ fixed by a nilpotent map N, which is called the *Springer fiber* of N. A theorem of Vargas and Spaltenstein [24,28] decomposes the space $\mathscr{B}_N$ into a disjoint union of locally closed subspaces, whose closures are the irreducible components of the space $\mathscr{B}_N$.

Let $N : V \to V$ be a nilpotent map of a vector space V over $\mathbb{C}$. Let b be the least positive integer for which $N^b = 0$. Then we have two filtrations of subspaces on V: the image filtration $\operatorname{im} N^b = 0 \subset \operatorname{im} N^{b-1} \subset \operatorname{im} N^{b-2} \subset \cdots \subset \operatorname{im} N^1 \subset V = \operatorname{im} N^0$ and the kernel filtration $\ker N^0 = 0 \subset \ker N \subset \ker N^2 \subset \cdots \subset \ker N^{b-1} \subset V = \ker N^b$ (with proper inclusions).

Lemma 2.1. *For a nilpotent map N, we have $N^{-1}(\operatorname{im} N^k) = \ker N + \operatorname{im} N^{k-1}$.*

Proof. If $N(v) \in \operatorname{im} N^k$ then $N(v) = N^k(w)$ so $N(v - N^{k-1}w) = 0$.

Note that if $\operatorname{im} N^{k-1}$ contains $\ker N$ then $N^{-1}(\operatorname{im} N^k) = \operatorname{im} N^{k-1}$; otherwise $N^{-1}(\operatorname{im} N^k)$ is strictly larger. Also note that $N(\ker N^{i+1}) \subseteq \operatorname{im} N \cap \ker N^i$.

Definition 2.1. Let F_i be a subspace of V that is taken into itself by the map N. Then there is a map $N_i : V/F_i \to V/F_i$ induced by N. We call the map N_i a *quotient map* of N.

Lemma 2.2. *The image* $\operatorname{im} N_i$ *of the quotient map* $N_i : V/F_i \to V/F_i$ *is equal to* $(\operatorname{im} N + F_i)/F_i$. *Similarly*, $(\operatorname{im} N^k + F_i)/F_i = \operatorname{im} N_i^k$.

Proof. If $N(v) \in \operatorname{im} N$ then $N(v) + F_i \in \operatorname{im} N_i$. On the other hand, if $w + F_i \in \operatorname{im} N_i$ then the coset $w + F_i$ equals the coset $N(v) + F_i$ for some $v \in V$, so $w + F_i$ is clearly in $\operatorname{im} N + F_i$. Then a subspace of V that contains F_i and whose projection to V/F_i is $\operatorname{im} N_i$ must be $\operatorname{im} N + F_i$. The same holds true for the nilpotent map N^k. □

Lemma 2.3. *The kernel of the quotient map* $\ker N_i$ *equals* $N^{-1}(F_i)$.

Proof. The kernel $\ker N_i$ is given by those vectors whose image under N_i is $0 + F_i$, which is exactly $N^{-1}(F_i)$. □

Lemma 2.4. *We have the containment* $\ker N \supseteq \operatorname{im} N^{b-1}$, *but* $\ker N \not\supseteq \operatorname{im} N^{b-j}$ *for* $j > 1$.

Proof. This is obvious from definition of b. □

Lemma 2.5. *If j is the largest integer for which* $\operatorname{im} N^j$ *is not contained in* F_i, *then j is the largest integer for which* $(\operatorname{im} N^j + F_i)/F_i$ *is a nonzero image of the quotient map* N_i.

Proof. Suppose $v \in \operatorname{im} N^j$ but is not in F_i. Then $v + F_i$ is not zero in V/F_i, so $(\operatorname{im} N^j + F_i)/F_i$ is a nonzero image of N_i. Similarly, if $v + F_i$ is a nonzero element of $(\operatorname{im} N^j + F_i)/F_i$, then there exists an element $0 \neq w \in V$ with $w \in \operatorname{im} N^j$ and $w \in v + F_i$. □

Now we discuss the Springer fiber $\mathcal{B}_N$ of N, which is the variety of flags fixed by the nilpotent element N. The ranks of the Jordan blocks of the nilpotent map N determine a partition of n. We form a Young shape from this partition by using this partition as the lengths of the rows (opposite to the conventions of Vargas [28]). Let the number of columns of the Young shape be b; then $N^b = 0$ and $N^{b-1} \neq 0$.

Definition 2.2. Given a flag F with subspaces $\{0\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = V$, we say that N *fixes* F if NF_i is contained in F_{i-1} for all i.

Definition 2.3. We denote by $\mathcal{B}_N$ the set of all flags fixed by N, and call it the Springer fiber of N. It is an algebraic subvariety of the flag manifold $\mathcal{B}$.

Recall that *Young shape* on n boxes is a collection of n boxes arranged in left justified rows of lengths $t_1 \geqslant \cdots \geqslant t_k$. A *standard tableau* (or *Young tableau*) on a Young shape τ is constructed by filling in the n boxes with the numbers $1, \dots, n$ such that the numbers are decreasing from left to right in each row, and decreasing from top to bottom in each column. Note that many authors use increasing rows and columns. We typically use A and B to denote standard Young tableaux. Denote by A_i the tableau obtained by deleting the numbers $1, \dots, i$ in the tableau A and subtracting i from the remaining numbers.

The following theorem of Vargas and Spaltenstein gives a decomposition of the Springer fiber $\mathscr{B}_N$ into a disjoint union of locally closed subsets, whose closures comprise the irreducible components of $\mathscr{B}_N$.

Theorem 2.1 (Vargas [28], Spaltenstein [24]). *Let N be a nilpotent map. Then given a standard tableau A on the Young shape of N, we construct a locally closed subset $\mathrm{SV}(A)$ of the Springer fiber $\mathscr{B}_N$, whose closure $\overline{\mathrm{SV}(A)}$ is an irreducible component of $\mathscr{B}_N$. We have a partition $\mathscr{B}_N = \bigcup_A \mathrm{SV}(A)$ of the Springer fiber into disjoint locally closed subsets. Thus the number of irreducible components of $\mathscr{B}_N$ is equal to the number of standard tableaux on the Young shape of N. In addition, the components are all of the same dimension. In fact, if the lengths of the columns of the Young shape of N are $n_1, n_2, \dots, n_b$, then the dimension of each component is*

$$\sum_i \frac{n_i(n_i-1)}{2}.$$

Proof. Suppose A is a Young tableau on the Young shape of N. Then we inductively specify a subset of $\mathscr{B}_N$ corresponding to A, which we denote $\mathrm{SV}(A)$ (for Spaltenstein–Vargas), by describing how to choose F_1, then F_2/F_1, and so forth. A flag F is in the subset $\mathrm{SV}(A)$ if each subspace of F satisfies the following conditions.

Suppose the number i appears in the $c(i)$th column in A. Then (recalling that $F_0 = 0$) the first subspace F_1 must satisfy $F_1 \subset (N^{-1}(F_0) \cap (\mathrm{im}\, N^{c(1)-1} - \mathrm{im}\, N^{c(1)})$. In other words, F_1 must be in the kernel of N, and it must be in the $(c(1)-1)$st image of N but not in any higher image.

Then, for any F_1 satisfying the above condition, the induced map $N_1 : V/F_1 \to V/F_1$ will have the same Young shape; this shape is the shape obtained by deleting the number 1 in the tableau A.

We now choose $F_2/F_1 \subset V/F_1$, using the above procedure with N_1 in place of N and A_1 in place of A. Note that $N_1^{-1}(0 + F_1) = N^{-1}(F_1)/F_1$. We continue inductively, choosing F_{i+1}/F_i so that $F_{i+1}/F_i \subset (N^{-1}(F_i)/F_i) \cap (\mathrm{im}\, N_i^{c(i)-1} - \mathrm{im}\, N_i^{c(i)})$. Note that any such choice of F_{i+1} yields a quotient map N_{i+1} with Young shape A_{i+1}.

See Vargas [28] and Spaltenstein [24] for the proof that this constructs a locally closed subset of $\mathscr{B}_N$ with the properties claimed in the theorem. The proof is essentially an explicit calculation with the Jordan form of N.

Vargas [28, Proposition 2.2] shows that this set $\mathrm{SV}(A)$ is exactly the set of flags such that

$$F_i \subset N^{-1}(F_{i-1}),$$
$$F_i \subset F_{i-1} + \mathrm{im}\, N^{c(i)-1}. \qquad \square$$

3. Determination of the topology of the irreducible components of Springer fibers for nilpotent maps of hook type for $\mathbf{GL}_n(\mathbb{C})$

Suppose N is a nilpotent map $V \to V$ whose Jordan form has at most one Jordan block of rank >1. Then N is said to be of hook type. For hook type nilpotent maps N, we can characterize the components of the Springer fiber $\mathscr{B}_N$ entirely in terms of the image and kernel filtrations of V. The components and their pairwise intersections turn out to be nonsingular. In fact their homology Poincaré polynomials factor as products of Poincaré polynomials of Grassmannians and flag manifolds.

We will describe each component of the Springer fiber of a nilpotent map N of hook type by expressing the component as a sequence of fiber bundles with progressively simpler bases and fibers.

Definition 3.1. A space X_1 is an *iterated fiber bundle* of base type $(B_1, \ldots, B_n)$ if there exist spaces $X_1, B_1, X_2, B_2, \ldots, X_n, B_n, X_{n+1} = pt$ and maps $p_1, p_2, \ldots, p_n$ such that $p_j : X_j \to B_j$ is a fiber bundle with typical fiber X_{j+1}.

The following two lemmas are straightforward.

Lemma 3.1. *The flag manifold* $\mathrm{Fl}(V)$ *admits a map to the Grassmannian* $G_i(V)$ *via* $F \mapsto F_i$. *The fiber of this map is a product* $\mathrm{Fl}(F_i) \times \mathrm{Fl}(V/F_i)$.

Lemma 3.2. *Consider the variety of flags* $X(I, K)$ *in an n-dimensional vector space* V *such that* F_i *contains an a-dimensional subspace* I *and is contained in a b-dimensional subspace* K. *This variety* $X(I, K)$ *admits a map* $p : X(I, K) \to G_{i-a}(K/I)$ *via* $F \mapsto F_i/I \subset K/I$ *that makes* $X(I, K)$ *the total space of a fiber bundle. The typical fiber of the map p is a product* $\mathrm{Fl}(F_i) \times \mathrm{Fl}(V/F_i)$. *In particular, the variety* $X(I, K)$ *is nonsingular.*

We define some notation to simplify our intersection homology computations. Define $[n]$ by

$$[n] := t^{-(n-1)}(1 + t^2 + t^4 + \cdots + t^{2(n-1)}).$$

Then we define

$$[n]! := [1][2]\dots[n] \quad \text{and} \quad \binom{[n]}{[k]} := \frac{[n]!}{[k]![n-k]!}.$$

The polynomial $[n]$ is essentially the t-analogue of the number n, but shifted to be symmetric around the degree 0 term. Thus the flag manifold $\mathrm{Fl}(V)$ has intersection homology Poincaré polynomial $[n]!$ and $G_k(V)$ has intersection homology Poincaré polynomial $\binom{[n]}{[k]}$.

Corollary 3.1. *Let $X(I)$ be the variety of flags in $\mathrm{Fl}(V)$ such that the subspace F_i contains an a-dimensional subspace I. Then $X(I)$ has intersection homology Poincaré polynomial $\binom{[n-a]}{[i-a]}[i]![n-i]!$.*

Proof. Clear from Lemma 3.2, since complex flag manifolds and complex Grassmannians have only even-dimensional homology so the Leray–Serre spectral sequence for $p : X(I) \to G_{i-a}(V/I)$ with fiber $\mathrm{Fl}(F_i) \times \mathrm{Fl}(V/F_i)$ collapses. □

Lemma 3.3. *Let N be a nilpotent map of hook type with Jordan blocks of size $(b, 1, \dots, 1)$. Then for all $0 < i < b$, we have $\mathrm{im}\, N^{b-i} = \ker N^i \cap \mathrm{im}\, N$, which implies $N(\ker N^{i+1}) \subset \mathrm{im}\, N^{b-i}$.*

Proof. This follows from inspection of the Jordan form of N. □

We now decompose each component of the Springer fiber $\mathscr{B}_N$ for a hook type nilpotent map N as an iterated bundle with nonsingular bases and fibers. Recall that the number b is the least positive integer with $N^b = 0$. Let $b(i)$ be the least positive integer with $N_i^{b(i)} = 0$.

Theorem 3.1. *Suppose we are given a nilpotent map N of hook type and a Young tableau A on the Young shape of N with $n, i_{b-1}, \dots, i_1$ on the top row (where by convention $i_b = n$ and $i_0 = 0$). Then the component K_A of the Springer fiber $\mathscr{B}_N$ is an iterated bundle with $B_{2j-1} = G_{i_j - i_{j-1} - 1}(\ker N_{i_j} / \mathrm{im}\, N_{i_j}^{b(i_j)-1})$ and $B_{2j} = \mathrm{Fl}(F_{i_j}/F_{i_{j-1}})$, where $j = 1, 2, \dots, b-1$, and B_{2b-1} is a full flag manifold $\mathrm{Fl}(V/F_{i_{b-1}})$.*

The proof will be broken up into a series of lemmas and propositions.

Proposition 3.1 (Vargas [28]). *Suppose we are given a nilpotent map N of hook type and a Young tableau A on the Young shape of N with $n, i_{b-1}, \dots, i_1$ on the top row. Then the component $K_A = \overline{\mathrm{SV}(A)}$ of the Springer fiber $\mathscr{B}_N$ consists of all flags F in*

Fl(V) *such that*

$$\begin{aligned}
&\operatorname{im} N^{b-1} \subseteq F_{i_1} \subseteq \ker N,\\
&\operatorname{im} N^{b-2} \subseteq F_{i_2} \subseteq \ker N^2,\\
&\operatorname{im} N^{b-3} \subseteq F_{i_3} \subseteq \ker N^3,\\
&\qquad\cdots\\
&\operatorname{im} N^{1} \subseteq F_{i_{b-1}} \subseteq \ker N^{b-1}.
\end{aligned}$$

Proof. This is Vargas [28, Theorem 4.1]. The proof consists of an explicit limiting argument using the structure of the Spaltenstein–Vargas subset SV(A).

Lemma 3.4. *Let N be a nilpotent map of hook type, and let F_{i_1} be a subspace of V with $NF_{i_1} \subset F_{i_1}$ and* $\operatorname{im} N^{b-1} \subset F_{i_1} \subset \ker N$. *Then we have*

$$\begin{aligned}
\ker N_{i_1}^d &= \ker N^{d+1}/F_{i_1},\\
\operatorname{im} N_{i_1}^d &= (\operatorname{im} N^d + F_{i_1})/F_{i_1}.
\end{aligned}$$

Proof. We describe $\ker N_{i_1}^d \subseteq V/F_i$ as follows. If $v \in \ker N^{d+1}$ then $N^d(v) \in \operatorname{im} N \cap \ker N = \operatorname{im} N^{b-1}$. Since $\operatorname{im} N^{b-1} \subseteq F_{i_1}$, we have $N^d(v) \in F_{i_1}$, so $v + F_{i_1} \in \ker N_{i_1}^d$. On the other hand, if $N_{i_1}^d(v + F_{i_1}) = 0 + F_{i_1}$ then $N^d(v) \in F_{i_1}$. Now by Lemma 3.3, the subspace F_{i_1} contains $\operatorname{im} N^{b-1}$ but no other element of $\operatorname{im} N$. Thus $N^{d+1}(v) = 0$, and so $v \in \ker N^{d+1}$. Thus if a subspace $W \subset V$ contains F_{i_1} then $W/F_{i_1} \subseteq \ker N_{i_1}^d$ iff $W \subseteq \ker N^{d+1}$.

By Lemma 2.2 $\operatorname{im}\,(N^d + F_{i_1})/F_{i_1} = \operatorname{im} N_{i_1}^d$. Thus if $W \subset V$ contains F_{i_1}, then W/F_{i_1} contains $\operatorname{im} N_{i_1}^d$ iff W contains $\operatorname{im} N^d$. $\square$

Lemma 3.5. *Suppose we are given a nilpotent N of hook type and a Young tableau A on the Young shape of N with $n, i_{b-1}, \ldots, i_1$ on the top row (where by convention $i_b = n$ and $i_0 = 0$). Then the component K_A of the Springer fiber $\mathscr{B}_N$ admits a map p_1 to the Grassmannian $G_{i_1-1}(\ker N/\operatorname{im} N^{b-1})$. The fiber X_2 of the map $p_1 : K_A \to G_{i_1-1}(\ker N/\operatorname{im} N^{b-1})$ admits a map $p_2 : X_2 \to \mathrm{Fl}(F_{i_1})$. The fiber of p_2 can be identified with a component of a Springer fiber of the quotient map $N_{i_1} : V/F_{i_1} \to V/F_{i_1}$, where the component is associated to the standard tableau A_{i_1}.*

Proof. The existence of the map p_1 follows from Lemma 3.2 with $I = \operatorname{im} N^{b-1}$ (which is a one-dimensional space) and $K = \ker N$ (which is an $n - b+$

1-dimensional space containing $\operatorname{im} N^{b-1}$). Let B_1 be the Grassmannian $G_{i_1-1}(\ker N/\operatorname{im} N^{b-1})$. The fiber X_2 of the map $p_1 : K_A \to B_1$ consists of all flags in the component with a fixed subspace F_{i_1}. We define the map $p_2 : X_2 \to \mathrm{Fl}(F_{i_1})$ by taking $F \in X_2$ and forgetting all subspaces of F larger than F_{i_1}. By inspecting Proposition 3.1 we see that p_2 is surjective and indeed a fiber bundle projection.

The fiber X_3 of this map p_2 is the set of all flags in the component K_A with fixed subspaces $F_1, \dots, F_{i_1}$. Then X_3 maps bijectively to a subset of $\mathrm{Fl}(V/F_{i_1})$ via the map $F \to F'$, where $F'_j = F_{j+i_1}/F_{i_1}$. We now show that X_3 is the component $K_{A_{i_1}}$ of the Springer fiber of the quotient map N_{i_1} on V/F_{i_1}, by showing that X_3 satisfies the characterization of Proposition 3.1.

By Lemma 3.4, the fiber X_3 of p_2 is in bijection with the set of flags $F' \in \mathrm{Fl}(V/F_{i_1})$ such that

$$\begin{aligned}
&\operatorname{im} N_{i_1}^{b-2} \subseteq F'_{i_2-i_1} \subseteq \ker N_{i_1},\\
&\operatorname{im} N_{i_1}^{b-3} \subseteq F'_{i_3-i_1} \subseteq \ker N_{i_1}^{2},\\
&\operatorname{im} N_{i_1}^{b-4} \subseteq F'_{i_3-i_1} \subseteq \ker N_{i_1}^{3},\\
&\qquad\cdots\\
&\operatorname{im} N_{i_1}^{1} \subseteq F'_{i_{b-1}-i_1} \subseteq \ker N_{i_1}^{b-2}.
\end{aligned}$$

Thus X_3 is the component $K_{A_{i_1}}$ of the Springer fiber $\mathscr{B}_{N_1}$. $\square$

Proof of Theorem 3.1. A typical fiber X_2 of the map $p_2 : K_A \to \mathrm{Fl}(F_{i_1})$ consists of flags $F \in K_A$ with fixed subspaces $F_1, \dots, F_{i_1}$. This fiber X_2 is in bijection with the set of flags in V/F_{i_1} that are fixed by $N_{i_1} : V/F_{i_1} \to V/F_{i_1}$.

So we have exhibited the component $K_A = X_1$ of the Springer fiber $\mathscr{B}_N$ as the total space of a bundle $p_1 : X_1 \to B_1$ with base $B_1 = G_{i_1-1}(\ker N/\operatorname{im} N^{b-1})$. The fiber X_2 of p_1 is the total space of another bundle $p_2 : X_2 \to B_2$ with base $B_2 = \mathrm{Fl}(V/F_{i_1})$.

Successive applications of Lemma 3.5 prove that if X_{2j+1} is a component of the Springer fiber of $N_{i_j} : V/F_{i_j} \to V/F_{i_j}$, then X_{2j+3} is the corresponding component of the Springer fiber for $N_{i_{j+1}} : V/F_{i_{j+1}} \to V/F_{i_{j+1}}$.

Finally, since $\operatorname{im} N \subset F_{i_{b-1}}$, we see that the map $N_{i_{b-1}} : V/F_{i_{b-1}} \to V/F_{i_{b-1}}$ is the zero map. Thus the (unique) component of the Springer fiber for $N_{i_{b-1}}$ (which is X_{2b-1}) is the flag manifold $\mathrm{Fl}(V/F_{i_{b-1}})$. $\square$

Theorem 3.2. *Let N be a nilpotent map of hook type, and let A be a standard tableau on the hook shape of N, with top row $n, i_{b-1}, \dots, i_1$. Then the component K_A of the*

Springer fiber $\mathscr{B}_N$ has intersection homology Poincaré polynomial equal to

$$[i_1]!\binom{[n-b]}{[i_1-1]}[i_2-i_1]!\binom{[(n-i_1)-(b-1)]}{[i_2-i_1-1]}$$

$$\dots[i_{b-1}-i_{b-2}]!\binom{[n-i_{b-2}-2]}{[i_{b-1}-i_{b-2}-1]}[n-i_{b-1}]!.$$

This polynomial equals

$$[n-b]![i_1][i_2-i_1][i_3-i_2]\dots[i_{b-1}-i_{b-2}][n-i_{b-1}].$$

Proof. We have already proven that the component in question is an iterated fiber bundle with B_{2j} a complex flag manifold, with B_{2j-1} a complex Grassmannian for $1 \leqslant j \leqslant b-1$, and with $X_{2b-1} = B_{2b-1}$ a complex flag manifold. So the bundle $p_{2b-2}: X_{2b-2} \to B_{2b-2}$ with fiber X_{2b-1} has only even-dimensional homology in base and fiber. Therefore the Leray-Serre spectral sequence for p_{2b-2} collapses and the Poincaré polynomial of X_{2b-2} is the product of those for X_{2b-1} and B_{2b-2}; in particular, X_{2b-1} has only even-dimensional homology. Then we do the same for $p_{2b-3}: X_{2b-3} \to B_{2b-3}$ with fiber X_{2b-2}; since B_{2b-3} is a complex Grassmannian, the space X_{2b-3} also has only even-dimensional homology. So $X_{2b-3}, X_{2b-5}, \dots$ have only even-dimensional homology and their homology Poincaré polynomials are products of Poincaré polynomials of flag manifolds and Grassmannians. This recursion thus unravels to give us the homology of X_1 as stated above. Finally, since the space X_1 is nonsingular, we need only shift the homology Poincaré polynomial until it is invariant under $t \mapsto t^{-1}$, in order to obtain the intersection homology Poincaré polynomial. □

Remark 3.1. We have proven that the closed subvariety of $\mathscr{B}_N$ associated by Vargas' description to the tableau A is irreducible because this subvariety is a bundle of irreducible varieties; it also contains a Spaltenstein–Vargas set $\mathrm{SV}(A)$; hence it must be exactly the closure K_A of the Spaltenstein–Vargas set $\mathrm{SV}(A)$. This is an alternative proof that Vargas' descriptions indeed yield the components of the Springer fiber $\mathscr{B}_N$.

4. Structure of pairwise intersections of two components of hook type

Theorem 4.1. *Let N be a nilpotent map of hook type. Suppose we have two standard tableaux A and B on the Young shape of N, where the standard tableau A has top row $n, i_{b-1}, \dots, i_1$ and the standard tableau B has top row $n, i'_{b-1}, \dots, i'_1$. Then the intersection of the two components $K_A \cap K_B$ is nonempty iff*

$\beta_j = \max\{i_j, i'_j\} < \min\{i_{j+1}, i'_{j+1}\} = \alpha_{j+1}$, *in which case* $K_A \cap K_B$ *is an iterated fiber bundle with*

$$B_{2j-1} = G_{\beta_j - \beta_{j-1} - 1}(\ker N_{\beta_j} / \operatorname{im} N_{\beta_j}^{b(\beta_j)}),$$

$$B_{2j} = \{F \in \mathrm{Fl}(F_{\beta_j}/F_{\beta_{j-1}}) \mid \operatorname{im} N_j^{b-j-1} \in F_{\alpha_j - \beta_{j-1}}\}.$$

Proof. We proceed as in the proof of Theorem 3.1. Let $\alpha_j = \min\{i_j, i'_j\}$ and $\beta_j = \max\{i_j, i'_j\}$. By superimposing the characterizations of the components K_A and K_B, we deduce that the intersection $K_A \cap K_B$ is given by those flags F in $\mathrm{Fl}(V)$ for which

$$\operatorname{im} N^{b-1} \subseteq F_{\alpha_1} \subseteq F_{\beta_1} \subseteq \ker N,$$

$$\operatorname{im} N^{b-2} \subseteq F_{\alpha_2} \subseteq F_{\beta_2} \subseteq \ker N^2,$$

$$\operatorname{im} N^{b-3} \subseteq F_{\alpha_3} \subseteq F_{\beta_3} \subseteq \ker N^3,$$

$$\cdots$$

$$\operatorname{im} N^1 \subseteq F_{\alpha_{b-1}} \subseteq F_{\beta_{b-1}} \subseteq \ker N^{b-1}.$$

If there exists j for which $\max\{i_j, i'_j\} \geqslant \min\{i_{j+1}, i'_{j+1}\}$ then the flag $F_{\max\{i_j, i'_j\}}$ would have to contain $\operatorname{im} N^{b-j-1}$ yet be contained in $\ker N^j$, which is impossible by Lemma 3.3. This proves the emptiness assertion of the lemma.

Now we exhibit the intersection $K_A \cap K_B = X_1$ as an iterated bundle. Define a map $p_1 : X_1 \to B_1 = G_{\beta_1 - 1}(\ker N / \operatorname{im} N^{b-1})$ by $F \mapsto F_{\beta_1}$. The typical fiber X_2 of the map p_1 consists of all flags F with a fixed F_{β_1} such that F_{α_1} contains the one-dimensional space $\operatorname{im} N^{b-1}$. Then there is a map taking X_2 to the space B_2, which consists of all flags inside F_{β_1} such that F_{α_1} contains $\operatorname{im} N^{b-1}$, given by $F_1 \subset F_2 \subset \cdots \subset F_n \mapsto F_1 \subset F_2 \subset \cdots \subset F_{\beta_1}$. We described the structure and homology of the space B_2 in Lemma 3.2 above.

Then the fiber of the map $p_2 : X_2 \to B_2$ is in bijection with a certain space of flags in V/F_{β_1} satisfying (as in the previous theorem) a list of conditions with respect to the quotient map $N_{\beta_1} : V/F_{\beta_1} \to V/F_{\beta_1}$. As before, these conditions are exactly the ones that specify the intersection of two components of the Springer fiber for N_{β_1} whose tableaux A_{β_1} and B_{β_1} have top rows $n - \beta_1, i_{b-1} - \beta_1, \ldots, i_2 - \beta_1$ and $n - \beta_1, i'_{b-1} - \beta_1, \ldots, i'_2 - \beta_1$, respectively. Thus by descending induction we have our result. $\square$

Corollary 4.1. *The intersection of the two components in the above theorem has intersection homology Poincaré polynomial equal to*

$$\begin{pmatrix}[n-b]\\ [\beta_1-1]\end{pmatrix}[\beta_1-\alpha_1]!\begin{pmatrix}[\beta_1-1]\\ [\alpha_1-1]\end{pmatrix}[\alpha_1]!$$
$$\begin{pmatrix}[n-\beta_1-b+1]\\ [\beta_2-\beta_1-1]\end{pmatrix}[\beta_2-\alpha_2]!\begin{pmatrix}[\beta_2-\beta_1-1]\\ [\alpha_2-\beta_1-1]\end{pmatrix}[\alpha_2-\beta_1]!$$
$$\ldots\begin{pmatrix}[n-\beta_{j-1}-b+j-1]\\ [\beta_j-\beta_{j-1}-1]\end{pmatrix}[\beta_j-\alpha_j]!\begin{pmatrix}[\beta_j-\beta_{j-1}-1]\\ [\alpha_j-\beta_{j-1}-1]\end{pmatrix}[\alpha_j-\beta_{j-1}]!$$
$$\ldots\begin{pmatrix}[n-\beta_{b-2}-2]\\ [\beta_{b-1}-\beta_{b-2}-1]\end{pmatrix}[\beta_{b-1}-\alpha_{b-1}]!\begin{pmatrix}[\beta_{b-1}-\beta_{b-2}-1]\\ [\alpha_{b-1}-\beta_{b-2}-1]\end{pmatrix}[\alpha_{b-1}-\beta_{b-2}]![n-\beta_{b-1}]!$$

This polynomial equals

$$[n-b]![\alpha_1][\alpha_2-\beta_1][\alpha_3-\beta_2]\ldots[\alpha_{b-1}-\beta_{b-2}][n-\beta_{b-1}].$$

5. Determination of the topology of components of Springer fibers for nilpotent maps of two-row type for $\mathbf{GL}_n(\mathbb{C})$

In this section we study the Springer fibers of nilpotent maps N whose Young shapes have at most two rows. Thus N has at most two Jordan blocks. We will find that the components are iterated bundles with $\mathbb{CP}^1$ as base spaces, and we will relate the intersection homology Poincaré polynomials of their pairwise intersections to the inner products of the Kazhdan–Lusztig basis. In doing so, we will extend some results of Lorist [21] on the topology of the components of two-row shapes with two boxes in the lower row. We also correct a result of Wolper [30].

Let $N : V \to V$ be a nilpotent map of two-row type. Recall that a flag F with subspaces $0 = F_0 \subset F_1 \subset \cdots \subset F_n = V$ is fixed by N if for all i, we have $NF_i \subseteq F_{i-1}$. Recall that b is defined to be the least positive integer with $N^b = 0$. Similarly let $b(i)$ be the least positive integer such that $N_i^{b(i)} = 0$.

Definition 5.1. Suppose $N_i : V/F_i \to V/F_i$ is a quotient map of a nilpotent map N. Suppose that the subspace F_j/F_i contains $\operatorname{im} N_i^k$ but not $\operatorname{im} N_i^{k-1}$. Then we call $\operatorname{im} N_i^k$ the *lowest image* contained in F_j/F_i and we denote this lowest image $\operatorname{im} N_i^k$ by $\operatorname{Lowim}_i(F_j)$. (Note that the image of a higher power of N_i is a smaller subspace of V/F_i.) Similarly, we denote the lowest image of N that is not contained in F_j by $\operatorname{Lowim}(F_j)$.

Lemma 5.1. *Let N be a nilpotent map of two-row type. Let F_i be a subspace of V such that $F_i \subseteq \operatorname{im} N$ and $NF_i \subset F_i$. Then the quotient space $N^{-1}(F_i)/F_i$ is two dimensional.*

Proof. Since $NF_i \subset F_i$, we see that indeed $F_i \subset N^{-1}(F_i)$. Then the dimensionality of $N^{-1}(F_i)/F_i$ is clear from the Jordan form of the map N, since $F_i \subseteq \operatorname{im} N$. □

Lemma 5.2. *Suppose N is a nilpotent map corresponding to a two-row Young shape τ. Let A be a standard tableau of shape τ with top row $n, i_{b-1}, \ldots, i_1$. Then every flag F in the Spaltenstein–Vargas subset* SV(A) *defined in Theorem 2.1 satisfies the following conditions*: $F_i \subset N^{-1}(F_{i-1})$ *and* $\operatorname{im} N^{b-j} \subseteq F_{i_j}$.

Proof. Let the flag F be in the Spaltenstein–Vargas subset SV(A). Then by construction of SV(A), every subspace clearly satisfies the first condition.

Now we prove the second condition by inspecting the procedure used to specify the flag subspaces of F. Recall that if i is in the $c(i)$th column of A, then F_i/F_{i-1} must lie in $\ker N_{i-1} \cap (\operatorname{im} N_{i-1}^{c(i)-1} - \operatorname{im} N_{i-1}^{c(i)})$.

Now we show that if i is on the bottom row of A then, for any flag F in SV(A), $\operatorname{Lowim}(F_i) = \operatorname{Lowim}(F_{i-1})$; in other words, the subspace F_i will never contain a lower image than F_{i-1} contains. There are two cases. First suppose the highest nonzero image $\operatorname{im} N_{i-1}^{b_{i-1}-1}$ of N_{i-1} is two dimensional. Then F_i/F_{i-1} cannot exhaust $\operatorname{im} N_{i-1}^{b_{i-1}-1}$. On the other hand, if the highest image $\operatorname{im} N_{i-1}^{b_{i-1}-1}$ is one dimensional, then since the number i is not on the top row of the tableau A, the subspace F_i/F_{i-1} must not equal $\operatorname{im} N_{i-1}^{b_{i-1}-1}$.

Now note that $F_1, \ldots, F_{i_1-1}$ do not contain wholly any image of N. To stress our line of argument, note that these subspaces contain the same image of N as $F_0 = \{0\}$ does. Therefore $\operatorname{im} N_{i_1-1}^{b-1} \neq 0$. Then by construction, F_{i_1}/F_{i_1-1} must contain $\operatorname{im} N_{i_1-1}^{b-1}$. Since F_{i_1} also contains F_{i_1-1}, we see by Lemma 2.2 that F_{i_1} must contain $\operatorname{im} N^{b-1}$.

Similarly, $F_{i_j}, \ldots, F_{i_{j+1}-1}$ all contain $\operatorname{im} N^{b-j}$ and no lower image of N, because each of these subspaces is constructed not to contain the highest image of the previous quotient map; and, as before, $F_{i_{j+1}}$ must contain $\operatorname{im} N^{b-j-1}$. Thus the lemma is proved. □

Remark 5.1. Note that the conditions of the lemma are closed and so are satisfied by the closure of the Spaltenstein–Vargas subset SV(A), which is the entire component K_A of the Springer fiber $\mathscr{B}_N$.

Theorem 5.1. *Suppose that N is a nilpotent map of two-row type and that A is a standard tableau on the Young shape of N, with top row $n, i_{b-1}, \ldots, i_1$. For any i between* 1 *and n, denote by $T(i)$ and $B(i)$ the lengths of the top and bottom rows of the tableau obtained from A by deleting* $1, \ldots, i$. *Suppose the flag F is contained in the Spaltenstein–Vargas subset* SV(A). *Then the subspace F_i contains* $\operatorname{im} N^{T(i)}$ *and is contained in* $\operatorname{im} N^{B(i)}$.

Proof. The assertion about $T(i)$ is proven above. As to the assertion about $B(i)$, we proceed by induction on i. First note that F_1 is contained in $N^{-1}(F_0)$ and therefore must be contained in the lowest image that has nontrivial intersection with the kernel. By inspecting the Jordan form of N we see that this image is exactly $\operatorname{im} N^{B(1)}$.

Now suppose F_i is contained in $\operatorname{im} N^{B(i)}$. Then there are two possibilities for F_{i+1}. If $i+1$ is on the bottom, then F_{i+1} is within $N^{-1}(F_i) \subseteq N^{-1}(\operatorname{im} N^{B(i)})$. Now by the Jordan form, we see that $B(i)$ cannot be greater than the power of the lowest image that contains $\ker N$, so $\operatorname{im} N^{B(i)}$ contains $\ker N$. Thus, by Lemma 2.1,

$$N^{-1}(\operatorname{im} N^{B(i)}) = \operatorname{im} N^{B(i)-1} = \operatorname{im} N^{B(i+1)}.$$

Now if $i+1$ is on the top row of the tableau A, then by Theorem 2.1, F_{i+1}/F_i is equal to $(\operatorname{im} N^{T(i+1)} + F_i)/F_i$. Since $T(i+1) \geqslant B(i)$, the subspace F_{i+1} is still contained in $\operatorname{im} N^{B(i)}$. $\square$

Theorem 5.2. *Let N be a nilpotent map of two-row type, and let A be a standard tableau on the Young shape of A with top row $n, i_{b-1}, \ldots, i_1$. Then the component K_A of the Springer fiber $\mathscr{B}_N$ consists of all flags whose subspaces satisfy the following conditions*:

$$F_i \subset N^{-1}(F_{i-1}) \quad \textit{for each } i$$

and if i is on the top row of the tableau A and $i-1$ is on the bottom row, *then*

$$F_i = N^{-1}(F_{i-2});$$

if i and $i-1$ are both in the top row of A, *then if $F_{i-1} = N^{-d}(F_r)$ where r is on the bottom row then*

$$F_i = N^{-d-1}(F_{r-1})$$

and if $F_{i-1} = N^{-d}(\operatorname{im} N^{b-i})$ where $0 \leqslant i < n-b$ then

$$F_i = N^{-d}(\operatorname{im} N^{b-i-1}).$$

The subspaces that are specified as inverse images of other spaces will be called dependent; *note that they are exactly the subspaces whose indices are on the top row of the tableau A. The other subspaces are called* independent.

Proof. Denote by $K(A)$ the closed subset of flags that satisfy the conditions of the theorem. Note that $K(A) \subseteq \mathscr{B}_N$. Let F be a flag in the Spaltenstein–Vargas subset $\mathrm{SV}(A)$. Then we prove by induction on i that each subspace F_i of F satisfies the above conditions, so that the flag F lies in $K(A)$. Then we will show below that the closed subset of the theorem is in fact irreducible and of the same dimension as the (nonempty) subset $\mathrm{SV}(A)$. Thus $K(A)$ is exactly the closure of the Spaltenstein–Vargas subset $\mathrm{SV}(A)$ and is thus the component K_A of the Springer fiber $\mathscr{B}_N$.

First we settle the $i=1$ case. Suppose that the number 1 is in the bottom row of A. Then for all F in the Spaltenstein–Vargas subset $\mathrm{SV}(A)$, it is the case that F_1 must be in $\ker N$, which is exactly $N^{-1}(F_0)$. If 1 is in the top row then F_1 must equal the highest image $\operatorname{im} N^{b-1}$; note that $F_0 = 0 = N^{-0}(\operatorname{im} N^b)$ so $F_1 = N^{-0}(\operatorname{im} N^{b-1})$. Also note that the highest image must indeed be one-dimensional in order for 1 to be in the top row. This settles the $i = 1$ case.

Now suppose that $F_1, \dots, F_i$ satisfy the conditions of the theorem. Now the number $i+1$ is either on the bottom row of the tableau A, or on the top row of A. If $i+1$ is on the bottom row of A, then the Spaltenstein–Vargas procedure requires only that F_{i+1}/F_i be contained in $\ker N_i$, which by Lemma 2.3 equals $N^{-1}(F_i)$.

Now suppose $i+1$ is on the top row of the tableau A. Then the number i is either on the bottom row of A or on the top row. First we will prove that if i is on the bottom row, then $F_{i+1} = N^{-1}(F_{i-1})$.

Recall that $F_i \subset N^{-1}(F_{i-1})$, and F_{i+1}/F_i must be the one-dimensional subspace of V/F_i which is the highest nonzero image of N_i. (Since i was on the bottom row, it is clear that the tableau obtained by deleting $1, \dots, i$ is not rectangular so the highest image of N_i is not two dimensional.)

So F_i is in $N^{-1}(F_{i-1}) = N_{i-1}^{-1}(0 + F_{i-1})$ and by construction F_i/F_{i-1} cannot be all of the highest nonzero image of N_{i-1}. Therefore there must be other vectors $v \in N^{-1}(F_{i-1})$ such that $v + F_i \neq 0 + F_i$, and also $v + F_{i-1}$ is in the highest image of N_{i-1}, hence v is in the highest image of N which is in not in F_{i-1}. Then any such v must be in the highest image of N which is not contained in F_i. Since $N^{-1}(F_{i-1})/F_i$ is one-dimensional, this proves that $N^{-1}(F_{i-1})/F_i$ must be the highest image of $N_i : V/F_i \to V/F_i$.

Thus we have proven that if i is in the bottom row and $i+1$ is in the top row, then $F_{i+1} = N^{-1}(F_{i-1})$.

Now suppose $i+1$ is on the top row, and i is also on the top row. Then by induction either $F_i = N^{-d}(F_r)$ where r is on the bottom row, or else $F_i = N^{-d}(\operatorname{im} N^a)$ (where $n - b < a \leqslant b$). If r is on the bottom row then, by Theorem 5.1, the lowest image that F_r contains is exactly the same as the lowest image that F_{r-1} contains. Therefore $N^{-1}(F_{r-1})$ contains one lower image and is of dimension exactly one larger than that of F_r. Thus $N^{-d-1}(F_{r-1})$ contains exactly one lower image and is of one larger dimension than $N^{-d}(F_r)$.

Otherwise $F_i = N^{-d}(\operatorname{im} N^a)$ for some $a > n - b$. Since $a > n - b$, the dimension of $\operatorname{im} N^{a-1}$ is exactly one larger than the dimension of $\operatorname{im} N^a$. Thus $N^{-d}(\operatorname{im} N^{a-1})$ contains one lower image and is one dimension larger than $\operatorname{im} N^a$. So $N^{-d}(\operatorname{im} N^{a-1})/F_i$ equals the highest nonzero image $\operatorname{im} N_i^{b(i)}$ and so $F_{i+1} = N^{-d}(\operatorname{im} N^{a-1})$. Finally, we prove below that $K(A) = \overline{\mathrm{SV}(A)} = K_A$. $\square$

Proposition 5.1. *The closed subset $K(A)$ is irreducible, and $K(A)$ is an iterated bundle of base type $(\mathbb{CP}^1, \dots, \mathbb{CP}^1)$, where there are as many terms as there are numbers in the bottom row of the tableau A.*

Proof. Suppose that the shape of N has exactly two rows (if it has only one row, then K_A is a point). Let F be a flag in the component K_A of the Springer fiber $\mathscr{B}_N$.

Suppose F_{j_1} is the smallest independent subspace of the flag F. Then F_{j_1-1} is some fixed subspace of V (necessarily in im N), and F_{j_1}/F_{j_1-1} can be any point in the fixed space $\mathbb{P}(N^{-1}(F_{j_1-1})/F_{j_1-1}) = \mathbb{CP}^1$. Set $B_1 = N^{-1}(F_{j_1-1})/F_{j_1-1}$. The map $p_1 : K_A \to B_1$ given by $F \mapsto F_{j_1}/F_{j_1-1}$ is then a fiber bundle (it is clearly a proper submersion). The typical fiber X_2 of the map p_1 consists of all flags F in K_A with the subspace F_i fixed, as well as with all subspaces of F that are dependent on F_{j_1} fixed. Now find the smallest independent subspace F_{j_2} in X_2. Again, all subspaces smaller then F_{j_2} are dependent, and thus fixed. So we see that F_{j_2}/F_{j_2-1} can be any point in the fixed space $\mathbb{P}(N^{-1}(F_{j_2-1})/F_{j_2-1}) = B_2$, which defines the bundle projection $p_2 : X_2 \to B_2$, with fiber X_3. We continue until all independent subspaces are exhausted and the fiber consists of one flag with all subspaces fixed. $\square$

Theorem 5.3. *Every component K_A of the Springer fiber for a nilpotent map of two-row type is an iterated bundle of base type* $(\mathbb{CP}^1, \ldots, \mathbb{CP}^1)$, *where there are as many terms as there are numbers in the bottom row of the tableau A.*

Proof. The closed subset $K(A)$ is irreducible, contained in $\mathscr{B}_N$, and clearly has dimension $n - b$, which is the dimension of the nonempty Spaltenstein–Vargas subset $\mathrm{SV}(A)$. Therefore, by standard algebraic geometry, $K(A)$ must equal the component $K_A = \overline{\mathrm{SV}(A)}$. $\square$

Example 5.1. Consider the standard tableau

$$\begin{matrix} 5 & 4 & 1 \\ 3 & 2 & \end{matrix}$$

Then the corresponding component of $\mathscr{B}_N$ is the set of flags with $F_i \subset N^{-1}(F_{i-1})$, and the conditions $F_1 = \mathrm{im}\, N^2$ and $F_4 = N^{-1}(F_2)$, which we can write compactly as

$$\mathrm{im}\, N^2 \subset F_2 \subset F_3 \subset N^{-1}(F_2) \subset V.$$

Similarly, the standard tableau

$$\begin{matrix} 5 & 3 & 2 \\ 4 & 1 & \end{matrix}$$

corresponds to the component

$$F_1 \subset N^{-1}(F_0) \subset N^{-1}(\mathrm{im}\, N^2) \subset F_4 \subset V.$$

Example 5.2. Consider the standard tableau

$$\begin{matrix} 5 & 4 & 3 \\ 2 & 1 & \end{matrix}$$

Then the corresponding component of $\mathcal{B}_N$ is the set of flags with $F_i \subset N^{-1}(F_{i-1})$ and

$$F_1 \subset F_2 \subset N^{-1}(F_1) \subset N^{-2}(F_0) \subset V.$$

Similarly, the standard tableau

$$\begin{matrix} 5 & 4 & 1 \\ 3 & 2 & \end{matrix}$$

corresponds to the component with $F_i \subset N^{-1}(F_{i-1})$ and

$$\operatorname{im} N^2 \subset F_2 \subset F_3 \subset N^{-1}(F_2) \subset V.$$

Their intersection is the flag with $F_i \subset N^{-1}(F_{i-1})$ and

$$\operatorname{im} N^2 \subset N^{-1}(F_0) \subset N^{-1}(\operatorname{im} N^2) \subset N^{-2}(F_0) \subset V.$$

In particular, the intersection is not empty, in contrast to the assertions of Wolper [30].

Example 5.3. Consider the standard tableau

$$\begin{matrix} 10 & 9 & 8 & 7 & 4 & 3 \\ 6 & 5 & 2 & 1 & & \end{matrix}$$

Then the corresponding component of $\mathcal{B}_N$ is the set of flags with $F_i \subset N^{-1}(F_{i-1})$ and

$$F_1 \subset F_2 \subset N^{-1}(F_1) \subset N^{-2}(F_0) \subset F_5 \subset F_6$$
$$\subset N^{-1}(F_5) \subset N^{-4}(F_0) \subset N^{-4}(\operatorname{im} N^5) \subset V.$$

Example 5.4. We show how to recover Lorist's description of the structure of components of Springer fibers of 2-regular nilpotent maps (that is, those nilpotent maps N whose Young shapes have two boxes in the second row). There are two types of components K_A, corresponding to whether the numbers on the bottom row of A are consecutive (yielding a non-trivial bundle) or not consecutive (yielding a trivial bundle). If the numbers in the bottom rows A are not consecutive, say $i < j$, then we get

$$\operatorname{im} N^{b-1} \subset \operatorname{im} N^{b-2} \subset \operatorname{im} N^{b-i+1} \subset F_i \subset N^{-1}(\operatorname{im} N^{b-i+1}) \subset \cdots$$
$$\subset N^{-1}(\operatorname{im} N^{b-j+3}) \subset F_j \subset N^{-2}(\operatorname{im} N^{b-j+3}) \subset N^{-2}(\operatorname{im} N^{b-j+2}) \subset \cdots \subset F_n.$$

If the numbers are consecutive, say $i, i+1$, then we get

$$\operatorname{im} N^{b-1} \subset \operatorname{im} N^{b-2} \subset \operatorname{im} N^{b-i+1} \subset F_i \subset F_{i+1} \subset N^{-1}(F_i) \subset N^{-2}(\operatorname{im} N^{b-i+1}) \subset \cdots \subset F_n.$$

Finally, we derive the scholium that the "dependence" on one subspace on another can be thought of as symmetric: if $F_i = N^{-d}(F_r)$ in the above theorem, then indeed F_i also determines F_r; so given either subspace, we can obtain the other. So "independent" can be thought of as "smallest in the chain of dependencies."

Proposition 5.2. *Suppose F_i is specified as $N^{-d}(F_r)$ in Theorem 5.2. Then the map $N^d : F_i \to F_r$ is surjective. Thus the subspace F_i determines the subspace F_r.*

Proof. In general, $N(N^{-1}(W)) = W \cap \operatorname{im} N$. Thus we need only ensure that if F_i is specified as $N^{-d}(F_r)$ then in fact $F_r \subseteq \operatorname{im} N^d$. First note that the process described in the proof of Theorem 5.2 never takes an inverse image of a subspace F_i unless $F_i \subset \operatorname{im} N$. Furthermore, if $F_i = N^{-d}(F_r)$ where F_r is independent and $r > 0$, then (for F in $\mathrm{SV}(A)$) the subspace F_{r-1} is contained in one higher image of N than F_r. So if $N^{-d+1}(F_r) \subset \operatorname{im} N^k$, then also $N^{-d}(F_{r-1}) \subset \operatorname{im} N^k$. This last statement holds in the entire component K_A. Hence if $F_i = N^{-d}(F_r)$ then $F_r \subseteq \operatorname{im} N^d$, so $N^d(F_i) = F_r$. □

6. Relationship with Kazhdan–Lusztig theory

Let W be a Coxeter group with simple reflections S. Denote the Chevalley–Bruhat order by $<$. Recall [5,16,18,19] that the Kazhdan–Lusztig construction yields elements C'_w in the Iwahori–Hecke algebra of W, which give distinguished bases for certain representations of the Iwahori–Hecke algebra, called left cell representations, which are associated to certain subsets $\mathscr{C}$ of W called left cells. In particular, this construction yields a distinguished basis for each irreducible representation of the Iwahori–Hecke algebra $\mathscr{H}_n$ of the symmetric group S_n.

Now recall [7,22] that every irreducible representation M of the Iwahori–Hecke algebra $\mathscr{H}_n$ possesses a unique (up to a scalar) nondegenerate symmetric bilinear form $\langle\, ,\ \rangle$ that is invariant under $\mathscr{H}_n$, in the sense that for any $v, v' \in M$, we have $\langle T_w v, v' \rangle = \langle v, T_w^* v' \rangle$. (Here, the involution $^* : \mathscr{H}_n \to \mathscr{H}_n$ is defined by sending $T_w \mapsto T_{w^{-1}}$ and then extending linearly.)

Thus, we can consider the inner products of the Kazhdan–Lusztig basis vectors of an irreducible representation with respect to this inner product. We find that they satisfy equations very reminiscent of a possible application of the Beilinson–Bernstein–Deligne–Gabber Decomposition Theorem. To state these relations, first we determine the eigenvectors and eigenvalues of the elements C'_s (for s a simple reflection) acting by left multiplication on $\mathscr{H}_n$.

Lemma 6.1. *The eigenvectors of the map $\mathscr{H}_n \to \mathscr{H}_n$ given by left multiplication by C'_s are given by*

$$C'_w \quad \textit{where } sw < w$$

and

$$C'_s C'_w - (t + t^{-1}) C'_w \quad \textit{where } sw > w.$$

Proof. The first follows immediately from the formula for $C'_s C'_w$ (see [26]). To see the second, we calculate

$$C'_s C'_s C'_w - (t + t^{-1}) C'_s C'_w = 0.$$

Now note that these eigenvectors span the Iwahori–Hecke algebra $\mathscr{H}_n$. □

Lemma 6.2. *The eigenvectors for C'_s on a left cell representation $M_{\mathscr{C}}$ are*

$$c_w \quad \textit{where } sw < w$$

and

$$C'_s c_w - (t + t^{-1}) c_w \quad \textit{where } sw > w.$$

Proof. Immediate from Lemma 6.1. □

Our equations follow from the following

Lemma 6.3. *Let V be a representation of an algebra A. Suppose V is equipped with an invariant symmetric bilinear form $\langle , \rangle$. Suppose we have an element $a \in A$ with $a = a^*$, and let x and y be eigenvectors of a with different eigenvalues. Then x and y are orthogonal*; *i.e.* $\langle x, y \rangle = 0$.

Proof. Suppose x has eigenvalue λ and y has eigenvalue ρ under $a \in A$. Then $\langle ax, y \rangle = \lambda \langle x, y \rangle = \langle x, ay \rangle = \rho \langle x, y \rangle$ so if $\lambda \neq \rho$ then $\langle x, y \rangle = 0$. □

Theorem 6.1. *Let $\mathscr{C}$ be a left cell yielding a left cell representation $M_{\mathscr{C}}$. Let $\langle \ , \ \rangle$ be an invariant nondegenerate symmetric bilinear form on $M_{\mathscr{C}}$. Let s be a simple reflection for W. Then for each pair (x, w) with $s \in L(x)$ and $s \notin L(w)$ (so s descends x but not w), we have an equation between inner products*

$$(t + t^{-1}) \langle c_x, c_w \rangle = \langle c_x, C'_s c_w \rangle.$$

Proof. Given a simple reflection s, we have a large supply of eigenvectors for C_s' in $M_{\mathscr{C}}$ given by Lemma 6.2, namely c_x where $sx<x$ and $(t+t^{-1})c_w - C_s'c_w$ where $sw>w$. Then for each pair (x,w), we have two eigenvectors of C_s' with different eigenvalues, so the eigenvectors are orthogonal. Therefore we have the equations as claimed. □

In fact, these equations are equivalent to $\mathscr{H}_n$-invariance:

Proposition 6.1. *If a bilinear form on a left cell representation satisfies the equations*

$$(t+t^{-1})\langle c_x, c_w \rangle = \langle c_x, C_s'c_w \rangle$$

for each pair c_x, c_w where $sw>w$ and $sx<x$, then the inner product is invariant under the action of C_s'.

Proof. The inner product of any two vectors can be expressed as a linear combination of inner products of basis vectors, so we are reduced to proving $\langle C_s'c_x, c_w \rangle = \langle c_x, C_s'c_w \rangle$ for each pair (x,w).

There are three cases. If both c_x and c_w are descended by s, then clearly

$$\langle (t+t^{-1})c_x, c_w \rangle = \langle c_x, (t+t^{-1})c_w \rangle.$$

If c_x is descended by s but c_w is not, then by the equations above

$$\langle C_s'c_x, c_w \rangle = \langle (t+t^{-1})c_x, c_w \rangle = \langle c_x, C_s'c_w \rangle.$$

Finally, suppose neither basis vector in $\langle c_x, c_w \rangle$ is descended by C_s'. Then, recalling that $C_s'c_x = \sum_{\substack{y \simeq_L x \\ sy<y}} \mu(x,y)c_y$, we have

$$(t+t^{-1})\langle C_s'c_x, c_w \rangle = \langle C_s'C_s'c_x, c_w \rangle = \langle C_s'c_x, C_s'c_w \rangle,$$

since each term in $C_s'c_x$ is descended by s. Then by a symmetric argument

$$\langle C_s'c_x, C_s'c_w \rangle = \langle c_x, C_s'C_s'c_w \rangle = (t+t^{-1})\langle c_x, C_s'c_w \rangle.$$

This completes the proof. □

The form of these equations is very similar to the conclusion of the Decomposition Theorem [1,4,11]. This analogy suggests an interpretation in terms of Poincaré polynomials: $\langle c_x, c_w \rangle$ as the intersection homology Poincaré polynomial of the intersection of two spaces K_A and K_B, $(t+t^{-1})$ as a $\mathbb{CP}^1$, the left side as a $\mathbb{CP}^1$ bundle over $K_A \cap K_B$, and the right side as terms from the Decomposition Theorem, applied to some map from the total space of the bundle to some space.

Thus, given a W-graph for a left cell representation, we can write down a set of equations that the Gram matrix entries of an $\mathscr{H}_n$-invariant inner product must satisfy, and which determine them up to a common scalar. If we can prove that the

Poincaré polynomials of some collection of spaces satisfy those equations, and we can calculate the Poincaré polynomial of one of the spaces, then we can calculate the Poincaré polynomials of all the spaces in the collection. We will now do this for Springer fibers of hook and two-row type, where the W-graphs are known explicitly.

The W-graphs for left cells with hook shapes are very easily determined because no two standard tableaux have the same descent set. (See [9, Fact 14] or [6].) Since all left cell representations of $\mathscr{H}_n$ for a given Young shape are isomorphic [18, Theorem 1.4], we can label the Kazhdan–Lusztig basis vectors by their tableaux.

Definition 6.1. A standard tableau of hook shape B is *adjacent* to the standard tableau A via k if B can be obtained from A by exchanging k with either $k+1$ or $k-1$ in the tableau A. Note that exactly one of A and B will have k as a descent.

Note that for each standard tableau A and number k, there are at most two standard tableaux adjacent to A. We denote by $(k-1\ k)A$ the tableau obtained by switching k and $k-1$ in A; similarly, $(k\ k+1)A$ is the tableau obtained from A by switching k and $k+1$.

Therefore, in the case of hook shapes, we can explicitly exhibit the set of equations that the inner products of the Kazhdan–Lusztig basis vectors must satisfy. Using the above description of the Poincaré polynomials of the intersection homology of the components of the Springer fibers and their intersections, we will show that the intersection homology Poincaré polynomials of the intersections of the components satisfy the same equations. Since the equations determine the inner product up to a scalar by Proposition 6.1, we will be able to show that the inner product matrix of the Kazhdan–Lusztig basis vectors computes the intersection homology Poincaré polynomials for the pairwise intersections of the components of the Springer fibers for hook shapes.

Theorem 6.2. *Suppose A is a hook shape tableau with top row $n, i_{b-1}, \dots, k, \dots, i_1$, so that the number k is not a descent of the tableau A. Then the Kazhdan–Lusztig basis vector c_A transforms as*

$$T_{(k\ k+1)}c_A = -c_A + tc_{(k\ k-1)A} + tc_{(k\ k+1)A}.$$

If $(k\ k+1)A$ or $(k-1\ k)A$ is not standard, then omit the corresponding term in the formula above.

If B is a standard tableau that does not have k in the top row then

$$T_{(k\ k+1)}c_B = t^2 c_B.$$

Proof. Given two different Young tableaux A and A' of hook type, there exists a simple reflection that descends A but not A', and a different simple reflection that descends A' but not A, because their first columns are distinct. Given a left cell $\mathscr{C}$ of hook type, there is a W-graph for the left cell representation, indexed by elements of $\mathscr{C}$. By Humphreys [16, Proposition 7.15], the only edges in the W-graph are those that connect elements of the form x and sx, with s a simple reflection. So in particular the only possible W-graph neighbors to A with k as a descent are $(k-1\ k)A$ and $(k\ k+1)A$. Depending on which of $k-1, k+1$ can be interchanged with k in the tableau A, we arrive at the above possibilities. □

Lemma 6.4. *Given the above tableaux A, B the following are eigenvectors for the action of $C'_{(k\ k+1)}$:*

$$(t+t^{-1})c_A - c_{(k\ k-1)A} - c_{(k\ k+1)A} \text{ with eigenvalue } 0$$

$$c_B \quad \text{with eigenvalue } (t+t^{-1}).$$

If $(k\ k+1)A$ or $(k-1\ k)A$ is not standard, then omit the corresponding term in the formula above.

Proof. This is immediate from Lemma 6.2 and the multiplication formula above. □

Theorem 6.3. *Suppose we are given a nilpotent map N on V with a Young shape τ of hook type. Let* TOP *be the standard tableau on the shape τ with top row $n, b-1, b-2, \ldots, 1$. Normalize the inner products of the Kazhdan–Lusztig basis vectors so that the norm $\langle c_{\mathrm{TOP}}, c_{\mathrm{TOP}} \rangle$ has the intersection homology Poincaré polynomial of the Springer fiber component K_{TOP}. Then the inner product $\langle c_A, c_B \rangle$ is equal to the intersection homology Poincaré polynomial of the intersection $K_A \cap K_B$.*

To accomplish this, we define maps between certain spaces, from which the Decomposition Theorem asserts that these intersection homology Poincaré polynomials satisfy the equations of the Kazhdan–Lusztig inner products. We need some geometric preliminaries.

Definition 6.2. A subvariety X of the flag manifold $\mathrm{Fl}(V)$ is a *union of lines of type k* if whenever X contains a flag $F_1 \subset \cdots \subset F_{k-1} \subset F_k \subset F_{k+1} \subset \cdots \subset F_n$, then X contains all flags of the form $F_1 \subset \cdots \subset F_{k-1} \subset F'_k \subset F_{k+1} \subset \cdots \subset F_n$ where F'_k is between the given subspaces F_{k-1} and F_{k+1}. We will, by analogy with Weyl groups, also say that k is a *descent* of X. Some authors say that X is *k-vertical*.

Definition 6.3. Let X be a subvariety of the flag manifold of $\mathrm{GL}_n(\mathbb{C})$ and $1 \leqslant k \leqslant n-1$. We will denote by $\mathbb{CP}^1 \bigstar X$ the variety of pairs

$$\{(F'_k, F) \text{ where } F \in X \text{ and } F'_k \text{ lies between } F_{k-1} \text{ and } F_{k+1}\}.$$

This variety admits a map $\phi : \mathbb{CP}^1 \bigstar X \to \mathrm{Fl}(V)$ given by mapping (F'_k, F) to the flag $F' = F_1 \subset \cdots \subset F_{k-1} \subset F'_k \subset F_{k+1} \subset \cdots \subset F_n$.

Definition 6.4. The image $\phi(\mathbb{CP}^1 \bigstar X) \subseteq \mathrm{Fl}(V)$ is called the *k-saturation* of X; this image is denoted $S_k(X)$.

So the k-saturation $S_k(X)$ is obtained roughly by taking all flags in X and allowing F_k to vary freely within them. Note that $\phi^{-1}(F)$ is a $\mathbb{CP}^1$ exactly when F is contained in a line of type k in X.

Lemma 6.5. *The space $\mathbb{CP}^1 \bigstar X$ is a locally trivial fiber bundle over X with fiber $\mathbb{CP}^1$ via the obvious map $(F'_k, F) \mapsto F$.*

Proof. This is clear. □

Remark 6.1. We can also rephrase this using a minimal parabolic subgroup P_k corresponding to the simple reflection $(k\ k+1)$. Then the map $\pi : G/B \to G/P_k$ forgets about F_k, so the preimage $\pi^{-1}(\pi(X))$ is equal to $\phi(\mathbb{CP}^1 \bigstar X)$.

Suppose we have two hook-type standard tableaux A with top row $n, i_{b-1}, \dots, k = i_j, \dots, i_1$ and B with top row $n, i'_{b-1}, \dots, \hat{k}, \dots, i'_1$ (that is, k is in the jth position from the right in A, but is not in the top row in B). Then k is a descent of the component K_B but not of the component K_A. Let the position of k in the top row of A be i_j. Then we have the adjacent components $K_{(k\ k-1)A}$ and $K_{(k\ k+1)A}$ of the Springer fiber $\mathscr{B}_N$ that have k as a descent. Then we shall show that the intersection homology Poincaré polynomial of the k-saturation of the intersection $K_B \cap K_A$ equals the sum of the intersection homology Poincaré polynomials of $K_B \cap K_{(k\ k-1)A}$ and $K_B \cap K_{(k\ k+1)A}$, and we shall show how this equality can be interpreted in terms of the Decomposition Theorem.

Theorem 6.4. *Let N be a nilpotent map of hook type. Let A and B be two standard tableau on the Young shape of N such that k is a descent of B but not of A. Then the intersection $K_A \cap K_B$ of the two components K_A and K_B of the Springer fiber $\mathscr{B}_N$ is not k-saturated. Then the map ϕ from $\mathbb{CP}^1 \bigstar (K_A \cap K_B)$ to the k-saturation $S_k(K_A \cap K_B)$ yields an equation of intersection homology Poincaré polynomials*

$$\mathrm{IP}(\mathbb{CP}^1 \bigstar (K_B \cap K_A)) = \mathrm{IP}(K_{(k\ k+1)A} \cap K_B) + \mathrm{IP}(K_{(k-1\ k)A} \cap K_B).$$

Remember that the intersection homology Poincaré polynomial IP is normalized so that the sum is centered around the degree 0 *term.*

If either $(k-1\ k)A$ or $(k\ k+1)A$ is not standard, then omit the corresponding term in the above. If both are not standard then the intersection $K_A \cap K_B$ is empty.

Proof. Suppose that we have the intersection $K_A \cap K_B$ of two components K_A and K_B where k is a descent of B but not of A. Let $k = i_j$ in A. First, if neither $(k-1\ k)A$ or $(k\ k+1)A$ is standard, then we can check by Theorem 4.1 that the intersection $K_A \cap K_B$ is empty. In fact, suppose $K_A \cap K_B$ is nonempty. Then if A has top row

$i_{j+1} = k + 1$, $i_j = k$ and $i_{j-1} = k - 1$, then $\beta_{j-1} = k - 1$ and $\alpha_{j+1} = k + 1$. Then in the tableau B, the entry i'_j must satisfy $k - 1 < i'_j < k + 1$, but $i'_j \neq k$ because k is a descent of B, which is a contradiction.

Suppose first that $(k\ \ k+1)A$ is standard. Then $i_j = k$, $i_{j-1} < k - 1$ and $i_{j+1} > k + 1$. Now the term i'_j in B must satisfy either $i'_j > k$ or $i'_j < k$. Suppose now that $i'_j > k$ (which is only possible when $(k\ \ k+1)A$ is standard), so that $k = \alpha_j$ and $i'_j = \beta_j$. Then for any $F \in K_A \cap K_B$, the subspaces of F satisfy

$$\operatorname{im} N^{b-j} \subset F_k \subset F_{k+1} \subset \cdots \subset F_{i'_j} \subset \ker N^j.$$

Note that $K_A \cap K_B$ has a (Zariski) open subset of flags with $\operatorname{im} N^{b-j} \subset F_k$ but $\operatorname{im} N^{b-j} \not\subset F_{k-1}$. In such flags, F_k is determined by F_{k-1} (and $\operatorname{im} N^{b-j}$), so the intersection $K_A \cap K_B$ is not a union of lines of type k.

So the k-saturation of the intersection $K_A \cap K_B$ consists of all flags $\cdots F_{k-1} \subset F'_k \subset F_{k+1} \cdots$, where F'_k is any subspace between F_{k-1} and F_{k+1}. In particular, F'_k no longer need contain $\operatorname{im} N^{b-j}$. However, the subspace F_{k+1} must still contain $\operatorname{im} N^{b-j}$ in all of the resulting flags. Therefore the k-saturation of $K_A \cap K_B$ is all flags with

$$\operatorname{im} N^{b-j} \subseteq F_{k+1} \subset F_{i_j} \subseteq \ker N^j,$$

and all the other conditions unaffected. Therefore the k-saturation $S_k(K_A \cap K_B)$ is indeed $K_{(k\ \ k+1)A} \cap K_B$.

If $(k - 1\ \ k)A$ is also standard, then there will be a nonempty subset of $K_A \cap K_B$ consisting of those flags in the intersection for which

$$\operatorname{im} N^{b-j} \subset F_{k-1} \subset F_{i_j} \subset \ker N^j$$

which clearly corresponds to the intersection $K_B \cap K_{(k-1\ \ k)A}$; this subset is a union of lines of type k. The subvariety $K_B \cap K_{(k-1\ \ k)A}$ is of codimension 2 in $K_B \cap K_{(k\ \ k+1)A}$.

Therefore we have a map $\phi : \mathbb{CP}^1 \bigstar (K_B \cap K_A) \to K_{(k\ \ k+1)A} \cap K_B$. This map is generically $1 - 1$. The map ϕ is a semismall resolution, because the subvariety where ϕ has a $\mathbb{CP}^1$ fiber is exactly $K_{(k-1\ \ k)A} \cap K_B$, which is of codimension 2 in the image space $K_{(k\ \ k+1)A} \cap K_B$, and the domain is nonsingular.

Therefore when we invoke the Decomposition Theorem for semismall maps [3,4], we find that the intersection homology Poincaré polynomial $\mathrm{IP}(\mathbb{CP}^1 \bigstar (K_A \cap K_B))$ equals the intersection homology Poincaré polynomial $\mathrm{IP}(K_{(k\ \ k+1)A} \cap K_B)$ of the range, plus the intersection homology $\mathrm{IP}(K_{(k-1\ \ k)A} \cap K_B)$ of the smaller intersection. (Remember that the intersection homology Poincaré polynomial IP is normalized so that all of these sums will be centered around 0.)

If $(k\ \ k+1)A$ is standard but $(k - 1\ \ k)$ is not, then $K_{(k-1\ \ k)A} \cap K_B$ is empty, so its term is omitted. Finally, if $i'_j < k$ so $k = \beta_j$ (which is only possible if $(k - 1\ \ k)A$ is standard), then the roles of $K_{(k-1\ k)A}$ and $K_{(k\ k+1)A}$ will be reversed in the above argument. $\square$

Example 6.1. We have an equality of IP's of the following spaces (where we denote the space K_A by its tableau A):

$$\mathbb{CP}^1 \bigstar \begin{pmatrix} 7 & 6 & 3 & 2 & & 7 & 5 & 3 & 1 \\ 5 & & & & & 6 & & & \\ 4 & & & & \bigcap & 4 & & & \\ 1 & & & & & 2 & & & \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 6 & 3 & 2 & & 7 & 6 & 3 & 1 \\ 5 & & & & & 5 & & & \\ 4 & & & & \bigcap & 4 & & & \\ 1 & & & & & 2 & & & \end{pmatrix} + \begin{pmatrix} 7 & 6 & 3 & 2 & & 7 & 4 & 3 & 1 \\ 5 & & & & & 6 & & & \\ 4 & & & & \bigcap & 5 & & & \\ 1 & & & & & 2 & & & \end{pmatrix}.$$

For the first intersection, we have

$$\alpha_1 = 1, \quad \beta_1 = 2, \quad \alpha_2 = 3, \quad \beta_2 = 3, \quad \alpha_3 = 5, \quad \beta_3 = 6,$$

so the computation of the polynomials is

$$\begin{aligned} &[2] \times [1][7-4]![3-2][5-3][7-6] \\ &\quad = [1][7-4]![3-2][6-3][7-6] + [1][7-4]![3-2][4-3][7-6]. \end{aligned}$$

7. Structure of intersections of components of two-row type and the relationship with Kazhdan–Lusztig theory

Now we compute the intersection homology Poincarè polynomials of pairwise intersections of components and then relate them to Kazhdan–Lusztig theory.

Let N be a nilpotent map with a two-row Young shape τ. In this section we prove that the intersection homology Poincaré polynomials of the intersections of the components of the Springer fiber $\mathscr{B}_N$ coincide with the (appropriately normalized) inner products of the Kazhdan–Lusztig basis vectors of the left cell representation of $\mathscr{H}_n$ associated to the Young shape τ.

The Kazhdan–Lusztig inner product matrix has been studied for left cell representations of two-row type because they yield the representations of the so-called Temperley–Lieb algebra [29]. We shall understand how the combinatorics of the Temperley–Lieb algebra representations encodes the structure of the intersections of components of the Springer fiber.

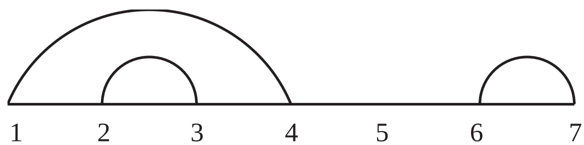
First, let us review some notions from Temperley–Lieb theory [29].

Definition 7.1. Suppose we have the numbers 1 to n on a horizontal line, increasing to the right. Then an (n, p)-*cup diagram* consists of p *cups* on these numbers, where each cup connects two numbers, no two cups intersect each other, and no number is underneath a cup and yet not connected to any cup. The entire cup diagram must lie in one half-plane.

Lemma 7.1. *Suppose τ is a Young shape with n boxes and $n-b$ boxes in the second row. Then there is a bijection between standard Young tableaux on τ and $(n, n-b)$-cup diagrams. This bijection is denoted $A \to \mathrm{CupD}(A)$.*

Proof. Let A be a standard tableau of shape τ. We construct a cup diagram as follows. Begin at the number 1 on the horizontal line. Proceed from left to right, starting a cup at i if the number i is on the bottom row of A, and ending a cup if i is on the top row by matching the number i with the closest started cup that can be matched with i. All unpaired ends of cups are then left blank (these are called *orphaned* numbers). It is easily seen that this procedure produces a bijection between cup diagrams and two-row tableaux. □

Example 7.1. The $(7, 3)$ cup diagram



corresponds to the standard Young tableau

$$\begin{matrix} 7 & 5 & 4 & 3 \\ 6 & 2 & 1 & \end{matrix}$$

Lemma 7.2. *A two-row standard tableau A has a descent at i if and only if the associated cup diagram* $\mathrm{CupD}(A)$ *has a cup connecting i and $i+1$. Such a cup will be called a* minimal *cup.*

Proof. The tableau A has a descent exactly when i is on the bottom row and $i+1$ is on the top row. This is the case exactly when there is cup connecting the numbers i and $i+1$ in the cup diagram $\mathrm{CupD}(A)$. □

We shall exhibit a correspondence between the cup diagram $\mathrm{CupD}(A)$ and the dependencies among the subspaces of the flags in the component K_A. First, let us extend the cup diagram $\mathrm{CupD}(A)$ by adding the numbers $0, -1, \ldots, -(n-b)$ to the left of the numbers $1, \ldots, n$. Now match each orphaned number in the cup diagram, working from left to right, to the closest possible negative number. This creates an extended cup diagram that we label $\mathrm{ECupD}(A)$. Note that there are no additional choices here so these extended cup diagrams are still in bijection with the standard tableaux. For a point i which is an endpoint of a cup, denote by $\sigma(i)$ the other endpoint of the cup.

Definition 7.2. Given a cup diagram $\mathrm{CupD}(A)$ and an index i at which a cup begins or ends, we denote by $\mathrm{Cup}_A(i)$ the cup that begins or ends at i.

Recall that a subspace F_i in a component K_A is called *dependent* if F is specified by Theorem 5.2 as the inverse image of some smaller subspace (equivalently, the number i is on the top row of A), and is said to *depend* on that smaller space. Otherwise the subspace is called *independent*.

Remark 7.1. For $-(n-b) \leqslant i < 0$, we interpret F_i to mean $\operatorname{im} N^{b-i}$.

Definition 7.3. A cup Cup_1 lies *directly beneath* a cup Cup_2 if Cup_1 is beneath Cup_2 and there are no other cups that lie both above Cup_1 and below Cup_2.

Theorem 7.1. *Consider a nilpotent map N and a two-row standard tableau A on the Young shape of N. Then the extended cup diagram* $\mathrm{ECupD}(A)$ *encodes the dependencies among the subspaces of the flags in* K_A *as follows. If a cup begins at i, then* F_i *is independent. If a cup ends at i then* F_i *is an inverse image of* $F_{\sigma(i)-1}$ (*interpreted using Remark* 7.1).

Proof. Let $i > 0$. First, note that F_i is independent iff the number i is on the bottom row of the tableau A. This is true iff i starts a cup in $\mathrm{ECupD}(A)$. The subspace F_i is dependent on a smaller subspace iff i ends a cup in $\mathrm{ECupD}(A)$.

Now suppose F_i is dependent. We apply the characterization of Theorem 5.2. Let us proceed by induction on the length $|\sigma(i) - i|$ of the cup $\mathrm{Cup}_A(i)$. If $i-1$ is independent, then F_i is equal to $N^{-1}(F_{i-2})$. On the other hand, $i-1$ starts a cup and i ends the cup, so there must be a minimal cup connecting $i-1$ and i, and so $i-2 = \sigma(i) - 1$. This proves the $|\sigma(i) - i| = 1$ case.

Now suppose that i is dependent and $i-1$ is also dependent. Then, since cups cannot cross, we see that $\sigma(i) < \sigma(i-1)$, so that the cup $\mathrm{Cup}_A(i-1)$ is shorter than the cup $\mathrm{Cup}_A(i)$. Now note that all numbers under the cup $\mathrm{Cup}_A(i)$ must either begin or end a cup. So if $\sigma(i-1) \neq \sigma(i) + 1$, then there must exist a sequence of adjacent cups directly beneath $\mathrm{Cup}_A(i)$ beginning at $\sigma(i)+1$ and ending at $\sigma(i-1) - 1$. Then by induction, the space F_{i-1} depends on the independent subspace $F_{\sigma(i)}$, so by Theorem 5.2 the subspace F_i depends on the subspace $F_{\sigma(i)-1}$. □

Proposition 7.1. *Suppose* $i > 0$ *begins a cup in* $\mathrm{CupD}(A)$. *If a subspace* F_j *depends on the* (*independent*) *subspace* F_i, *then* $i < j < \sigma(i)$; *that is, j lies strictly under the cup that begins at i. In fact, if j is the end of a cup that lies directly beneath* $\mathrm{Cup}_A(i)$ *then* F_j *depends in* F_i.

Proof. Note first that $j > i$ since F_j depends on the independent subspace F_i. Recall that since F_j is dependent, j ends a cup, and F_j depends on the subspace $F_{\sigma(j)-1}$. Either that subspace is independent, or $\sigma(j) - 1$ ends another cup so $F_{\sigma(j)-1}$ depends on $\sigma(\sigma(j) - 1) - 1$, and so forth.

If $j > i$ and j does not lie strictly under the cup starting at i then either $\sigma(j) < i$ or $\sigma(j) > \sigma(i)$ so $\sigma(j) - 1$ cannot lie strictly under the cup either. If $\sigma(j) - 1 = \sigma(i)$ then F_j depends on F_{i-1} and thus not on F_i. So, $F_{\sigma(j)-1}$ cannot depend on F_i unless $i < j < \sigma(i)$.

As to the last assertion, note that if j ends a cup lying directly beneath $\mathrm{Cup}_A(i)$, then F_j must depend on $F_{\sigma(j)-1}$. Then $\sigma(j) - 1$ must also end a cup lying directly beneath $\mathrm{Cup}_A(i)$ unless $\sigma(j) - 1 = i$. This completes the proof. □

Now we demonstrate that the pairwise intersections of components of the Springer fiber $\mathscr{B}_N$ satisfy the equations for the Kazhdan–Lusztig inner products. Suppose A is a standard tableau on the shape of N. Suppose i is not a descent of A (so it is not the case that i is on the bottom row and $i + 1$ is on top). The assertion that i is not a descent is equivalent to the assertion that there is not a cup joining i and $i + 1$. Then, we can manufacture a cup diagram having i as a descent.

Definition 7.4. Suppose $\mathrm{CupD}(A)$ is a cup diagram that does not have a minimal cup connecting i and $i + 1$. Suppose $\sigma(i) \neq i$ and $\sigma(i + 1) \neq i + 1$. Then the cup diagram $\mathrm{CupD}(u_i A)$ is defined by deleting the cups with endpoints at i and $i + 1$, then connecting i and $i + 1$ with a minimal cup and connecting $\sigma(i)$ to $\sigma(i + 1)$ with another cup. If exactly one of $\sigma(i) = i$ or $\sigma(i + 1) = i + 1$, then we only insert the cup between i and $i + 1$. If both $\sigma(i) = i$ and $\sigma(i + 1) = i + 1$ then $\mathrm{CupD}(u_i A)$ does not exist. Note that this definition also defines a standard tableau $u_i A$.

In [29] it is proven that the tableau $u_i A$ gives the unique W-graph neighbor to A that has i as a descent; if this tableau $u_i A$ does not exist, then there are no neighbors to A in the W-graph with i as a descent. We now show that if we i-saturate the intersection $K_A \cap K_B$ (where K_B is descended by i) then we get the intersection $K_{u_i A} \cap K_B$.

Theorem 7.2. *Let N be a nilpotent map of two-row type. Consider two standard tableaux A and B on the Young shape of N such that i descends B but not A. Suppose $u_i A$ is the unique W-graph neighbor to A that has i as a descent. Then the intersection Poincaré polynomials of the intersections satisfy the following equality*:

$$(t + t^{-1})\mathrm{IP}(K_A \cap K_B) = \mathrm{IP}(K_{u_i A} \cap K_B).$$

If there is no such neighbor $u_i A$ then the intersection $K_A \cap K_B$ is empty.

Proof. We shall show that the i-saturation $\mathbb{CP}^1 \bigstar (K_A \cap K_B)$ of the intersection $K_A \cap K_B$ has F_i independent, but all other dependencies among the other subspaces are the same as in $K_A \cap K_B$. This will prove the theorem. We use the fact that if F_j depends on F_i, then we can also determine F_i from knowledge of F_j (Proposition 5.2).

There are several cases; first note that in all cases, $F_{i+1} = N^{-1}(F_{i-1})$ because this dependency holds in K_B.

1. Suppose $\sigma(i+1) < \sigma(i) < i < i+1$ in A. First, for a flag F in the component K_A, we see that F_{i+1} depends on $F_{\sigma(i+1)-1}$. Because $F_{i+1} = N^{-1}(F_{i-1})$ in K_B, we see that F_{i+1} must depend on F_{i-1}. Now in the component K_A, the number $i-1$ is either equal to $\sigma(i)$, or $i-1$ lies at the end of a cup lying directly under $\mathrm{Cup}_A(i)$. So the subspace F_{i-1} depends on the subspace for the start of the cup $\mathrm{Cup}_A(i)$, namely $F_{\sigma(i)}$. Thus F_{i+1} depends on $F_{\sigma(i)}$. Since F_{i+1} depends on $F_{\sigma(i+1)-1}$ in K_A, we see that $F_{\sigma(i)}$ must depend on $F_{\sigma(i+1)-1}$ in the intersection $K_A \cap K_B$.

Then in the transformed tableau u_iA, we now have a cup connecting $\sigma(i+1)$ to $\sigma(i)$ and one connecting i to $i+1$. This means that for any flag F in the component K_{u_iA}, $F_{\sigma(i)}$ is dependent on $F_{\sigma(i+1)-1}$. Thus the intersection $K_{u_iA} \cap K_B$ will have all the same dependencies between subspaces as $K_A \cap K_B$ does, except that $K_{u_iA} \cap K_B$ will be i-saturated.

2. Suppose $\sigma(i) < i < i+1 < \sigma(i+1)$. Then in K_A, the subspace F_i depends on $F_{\sigma(i)-1}$, and $F_{\sigma(i+1)}$ depends on F_i and thus on $F_{\sigma(i)-1}$.

Now for any flag F in K_{u_iA}, the subspace $F_{\sigma(i+1)}$ depends on $F_{\sigma(i)-1}$ as well. No dependency is imposed on F_{i+1} that was not present in $K_A \cap K_B$.

3. Suppose $i < i+1 < \sigma(i+1) < \sigma(i)$. Then for any flag F in the component K_A, the subspace $F_{\sigma(i)}$ depends on F_{i-1}, and the subspace $F_{\sigma(i+1)}$ depends on F_i. Also $\sigma(i+1)-1$ is either $i+1$ or is the end of a cup directly under $\mathrm{Cup}_A(i+1)$. So, by Proposition 7.1, the subspace $F_{\sigma(i+1)-1}$ depends on F_{i+1}. Thus in the intersection $K_A \cap K_B$, the subspace $F_{\sigma(i+1)-1}$ depends on F_{i-1}.

Then in K_{u_iA}, we have that $F_{\sigma(i)}$ depends on $F_{\sigma(i+1)-1}$. By the same chain of dependencies, $F_{\sigma(i+1)-1}$ depends on F_{i+1} and thus on F_{i-1}, so $F_{\sigma(i+1)-1}$ depends on F_{i-1}. Finally, the subspace $F_{\sigma(i+1)}$ (which in K_A depended on F_i) does not depend in K_{u_iA} on F_i.

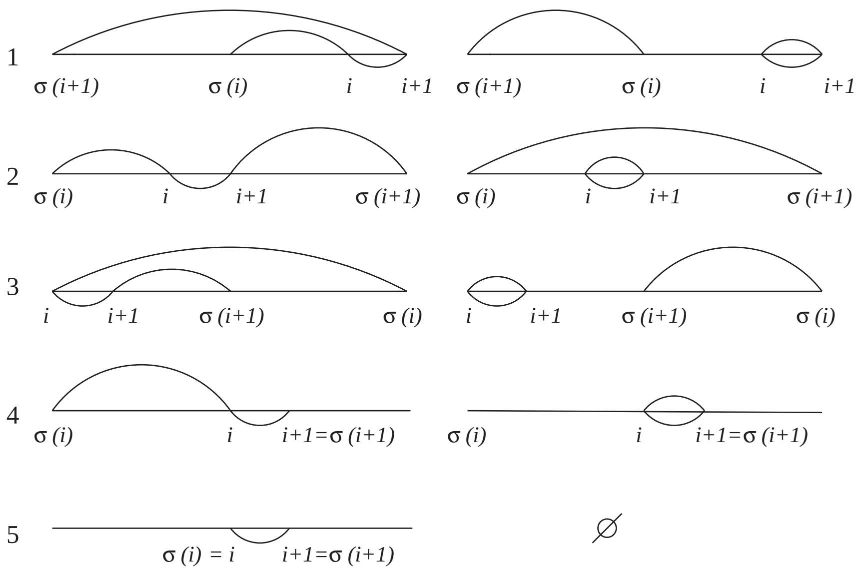
4. Note that the above two arguments are identical if exactly one of $\sigma(i)$ and $\sigma(i+1)$ is negative.

5. Finally, if both $\sigma(i)$ and $\sigma(i+1)$ are negative, then in the original cup diagram $\mathrm{CupD}(A)$, both i and $i+1$ are orphans. This is exactly the case where there is no W-graph neighbor to A having i as a descent (see [29]). So we show that the intersection $K_A \cap K_B$ has to be empty. In the component K_B where i is a descent, $F_{i+1} = N^{-1}(F_{i-1})$ and so F_{i+1} is chosen to contain exactly one lower image than F_{i-1} as well as F_i.

On the other hand, consider a flag F in the component K_A where i is not a descent. In the original cup diagram $\mathrm{CupD}(A)$, both i and $i+1$ are orphans. Therefore F_{i-1} is also dependent (otherwise $i-1$ would have to connect to i). So F_i is chosen to contain exactly one lower image than F_{i-1}, and in fact $F_i = N^{-k}(\mathrm{im}\, N^j)$ for some j, k. Then F_{i+1} is chosen to contain exactly one lower image than F_i and in fact $F_i = N^{-k}(\mathrm{im}\, N^{j-1})$. Thus, in K_A, the subspace F_{i+1} must contain two lower images than

F_{i-1}. But in K_B, the subspace F_{i+1} must contain exactly one lower image then F_{i-1}. Therefore there are no flags in the intersection $K_A \cap K_B$.

The following diagram illustrates the possibilities; in each case, the relevant subdiagrams of the cup diagrams CupD(A) and CupD(u_iA) are on the top left and top right respectively. The relevant subdiagram of CupD(B) is on the bottom in all cases.



Now we have demonstrated in all cases that the i-saturation of $K_A \cap K_B$ is indeed equal to $K_{u_iA} \cap K_B$. Since K_A is not a union of lines of type i, the conclusion of Theorem 5.2 ensures that K_A has no subvariety that consists of lines of type i. Thus the Decomposition Theorem yields the conclusion of the theorem. □

As a complement, we describe the computation of the inner product matrix of the Kazhdan–Lusztig basis for a two-row shape.

Theorem 7.3 (Westbury [29], Graham-Lehrer [13]). *Let τ be a two-row Young shape, and let A and B be two standard tableaux on the shape of τ. Consider the diagram formed by placing the cup diagram* CupD(A) *above the horizontal line, and* CupD(B) *below. Suppose the diagram contains r closed loops, and also that the endpoints of each open arc are pointing in opposite directions. Then the inner product of two Kazhdan–Lusztig basis vectors c_A and c_B is $\langle c_A, c_B \rangle = (t + t^{-1})^r$. If an open arc in the diagram has both ends pointing in the same direction, then the inner product is* 0.

Proof. See Westbury [29, Sections 5 and 7] and Graham-Lehrer [13, Section 6]. Note that their answer differs from ours by a sign because they use the other Kazhdan–Lusztig basis arising from the elements C_w. □

Conjecture 7.1. *Based on the strong evidence of the above calculations, we conjecture that the pairwise intersection of components of Springer fibers of two-row type are also iterated $\mathbb{CP}^1$ bundles. It would suffice to show that each pairwise intersection admits a description of the same form as Theorem 5.2. Of course, Theorem 5.2 shows that the intersection of two components consists of all flags that satisfy the descriptions of both components simultaneously. So it remains to show that there is a single description of the same form for the intersection.*

8. Further speculations

Much research into the relation between the Kazhdan–Lusztig basis and the Springer fibers for $GL_n(\mathbb{C})$ has been stimulated by the conjecture in Kazhdan–Lusztig [18, 6.3] which states that with the tableau labelings of the Springer fiber components and the Kazhdan–Lusztig basis vectors, the codimension 1 pairwise intersections of the components yield the edges of the left cell W-graphs. It was also conceivable that the Springer basis and the Kazhdan–Lusztig basis were the same at the level of S_n, and that perhaps there was a way to get an Iwahori–Hecke algebra action on the Springer basis [25].

Recent work of Kashiwara–Saito [17] disproved a conjecture concerning irreducibility of a certain characteristic variety, which implies that the Springer and Kazhdan–Lusztig bases are indeed different at the S_n level, and disproves Conjecture 6.3 in general.

For hook shapes, the Kazhdan–Lusztig basis is known to coincide with the Springer basis for S_n (see [14]); the Conjecture [18, 6.3] also holds for two-row shapes, because of the equations established in Section 5 and the work of Lascoux–Schützenberger [20] (see also [29,30]). In fact, for representations of S_n labeled by hooks and two-rows (and all left cells for which the Bruhat order coincides with the weak Bruhat order), Kazhdan and Lusztig [18, 6.3] is known in the sense that Hotta's transformation formula (see [2,15]) for the Springer basis coincides with the Kazhdan–Lusztig transformation formula (see [8]).

It is not yet clear how the results of this dissertation fit into the framework of the above results and counterexamples. The Kazhdan–Lusztig inner products, properly normalized, are always polynomials that are symmetric around 0; that is, invariant with respect to the map $t \to t^{-1}$. So we would like them to correspond to a method of associating a symmetric Poincaré polynomial to each component (and to each pairwise intersection of components) of the Springer fiber. In the cases in this work, all components and intersections were nonsingular, so the homology Poincaré polynomials were already symmetric, once shifted appropriately. A natural choice for a symmetric Poincaré polynomial associated to a singular variety is the intersection homology Poincaré polynomial. In the nonsingular case, intersection homology coincides with ordinary homology, except for the shift. Also, intersection homology satisfies the crucial Decomposition Theorem [1]. However, it appears that the natural conjecture extending Theorems 6.2 and 7.2 using intersection homology alone is not correct. The first example of a singular component arises in S_6 (see [24,28]). This component X is specified in our notation by the tableau

$$\begin{matrix} 6 & 4 \\ 5 & 2 \\ 3 & \\ 1 & \end{matrix}$$

and X can be given the following description. The component X consists of flags F such that $F_2 \subset \ker N$, $F_2 \cap \operatorname{im} N$ has dimension at least 1, $F_4 \cap \ker N$ has dimension at least 3, $F_4 \subset N^{-1}(F_2)$, and $\operatorname{im} N \subset F_4$. In the Spaltenstein–Vargas subset for this component, $F_2 \cap \operatorname{im} N$ is a one-dimensional space p and F_4 is chosen in $N^{-1}(p)$ to contain $\operatorname{im} N$. Since F_4 and $\ker N$ are both four-dimensional subspaces of the five-dimensional space $N^{-1}(p)$, their intersection $F_4 \cap \ker N = \mathrm{PL}$ must have at least dimension 3 and contain $\operatorname{im} N$.

The singular set of the component X consists of those flags such that $F_2 = \operatorname{im} N$ and $F_4 = \ker N$. In those cases, the one-dimensional subspace p is no longer uniquely determined, nor is the three-dimensional space PL. If, for each flag in X, we choose a one-dimensional subspace in $\operatorname{im} N \cap F_2$ and a three-dimensional subspace in $F_4 \cap \ker N$, then the resulting space $\tilde{X}$ of such triples is a resolution of singularities of X. Also, the fiber over a point in the singular set is a $\mathbb{CP}^1 \times \mathbb{CP}^1$. The singular set has complex codimension 4. Thus the resolution is semismall.

We can see that space $\tilde{X}$ has homology Poincaré polynomial equal to $[2][2][2][2][2][2][2]$ as follows. The choice of a one-dimensional subspace p in $\operatorname{im} N$ is a $\mathbb{CP}^1$. Then the choice of a three-dimensional space PL in $\ker N$, containing $\operatorname{im} N$, is a $\mathbb{CP}^1$. Then the choice of a two-dimensional space F_2 containing p and contained in PL is a $\mathbb{CP}^1$. The choice of a space F_4 containing PL and contained in the five-dimensional space $N^{-1}(p)$ is another $\mathbb{CP}^1$. Finally, since X is a union of lines of types 1, 3, and 5, the other choices each contribute a $\mathbb{CP}^1$. This exhibits $\tilde{X}$ as an iterated fiber bundle.

The semismall Decomposition Theorem says that the intersection homology Poincaré polynomial $\mathrm{IP}(\tilde{X})$ of the resolution $\tilde{X}$ equals the sum of IP's of strata of X, each with multiplicity equal to the number of components of the fiber over the point in the stratum. So we can compute $\mathrm{IP}(X)$, since we know that the fiber over each point in the stratum $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ is $\mathbb{CP}^1 \times \mathbb{CP}^1$. Thus $[2][2][2][2][2][2][2] = \mathrm{IP}(X) + [2][2][2]$. However, the entry in our normalized inner product matrix is $[2][2][2][2][2][2][2]$, which is larger than the homology Poincaré polynomial of $\tilde{X}$.

However, this suggests that perhaps the inner products correspond to some other semisimple perverse sheaves (see [23]) in the intersection $K_A \cap K_B$, since semisimple perverse sheaves also satisfy the Decomposition Theorem and have symmetric Poincaré polynomials. We also have examples of inner products (from the same shape as the above example) that are the sum of intersection homology Poincaré polynomials of multiple irreducible components of the corresponding intersection of two components of the Springer fiber. This lends further weight to the idea of using semisimple perverse sheaves on the intersection $K_A \cap K_B$ of two components of the Springer fiber, since the appropriate Poincaré polynomial is obtained by summing the Poincaré polynomials of the irreducible components of the intersection $K_A \cap K_B$. The data also suggests that it would be worthwhile to investigate the structure of resolutions of singularities of components of Springer fibers.

There is now W-graph data available up to S_{15} (see [23]) and it would be interesting to compute the inner products of the Kazhdan–Lusztig basis vectors from them. For instance, one could check whether they satisfy the Hard Lefschetz theorem.

There is of course more to be done on the computation of the topology of the components of the Springer fibers. The techniques exposed here exploit the relative simplicity of the structures of the nilpotent maps for hook and two-row types. It would be interesting to understand these components better. Even in the two-row case, it would be worthwhile to gain more information on the structure of the fiber bundles, for instance extensions of Lorist's theorem [21] concerning the e-invariants of the nontrivial $\mathbb{CP}^1$ bundles.

We believe that further study of the Kazhdan–Lusztig inner products and their relation to the components of Springer fibers will prove to be fruitful, and that there are many questions left to be answered here.

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