Frattini-based starters in 2-groups

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Abstract

Let $G$ be a group of order $2^t$, with $t \geq 4$. We prove a sufficient condition for the existence of a one-factorization $\mathcal{F}$ of a complete graph, admitting $G$ as an automorphism group acting sharply transitively on the vertex-set.

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1. Introduction

Let $G$ be a finite group of even order. Can $G$ be represented as an automorphism group of a one-factorization $\mathcal{F}$ of a complete graph acting sharply transitively on vertices? If $G$ is cyclic of order $2^t$, with $t \geq 3$, then the answer to this question is negative due to a result of Hartman and Rosa [5]. For all other classes of groups of even order which have been investigated in this respect, the answer has been affirmative: e.g. for the class of finite abelian groups of even order [3] and for the class of finite 2-groups admitting a cyclic subgroup of index 2 [1]. So one might conjecture that the cyclic groups of 2-power order are the only exceptions.

In the present paper we restrict our attention to the class of finite 2-groups and prove that if $G$ is a finite 2-group whose Frattini subgroup is elementary abelian, then there exists a one-factorization $\mathcal{F}$ of a complete graph admitting $G$ as an automorphism group acting sharply transitively on vertices. This condition is based on the notion of a starter developed in [3]; it has been shown that a group $G$ can be represented as an automorphism group of a one-factorization $\mathcal{F}$ of a complete graph acting sharply transitively on vertices if and only if the group $G$ admits a starter. We also prove that, within the class of 2-groups of order $\leq 64$, the cyclic groups are the only ones which do not admit a starter. In order to fix notation for the subsequent section, we recall from [3] the definition of a starter in a group of even order.

Let $G$ be an additive group and let $K_G = (G, \left( \frac{G}{2} \right))$ be the complete graph on $G$. An edge $e = [x, y] \in \left( \frac{G}{2} \right)$ is said to be short if $x - y$ is an involution in $G$, otherwise $e = [x, y]$ is said to be long. We define

$$\check{e} = \begin{cases} \{x - y\} & \text{if } e \text{ is short}, \\ \{x - y, y - x\} & \text{if } e \text{ is long} \end{cases}$$

$$\phi(e) = \begin{cases} \{x\} & \text{if } e \text{ is short}, \\ \{x, y\} & \text{if } e \text{ is long}. \end{cases}$$
Theorem 3. Each finite group $G$ of even order is a set $\Sigma = \{S_1, \ldots, S_r\}$ of subsets of $\left(\frac{G}{2}\right)$ satisfying the following conditions:

- $\partial S_1 \cup \cdots \cup \partial S_r = G - \{0\}$;
- for every $i = 1, \ldots, r$, the set $\phi(S_i)$ is a system of distinct representatives (left transversal) for the left cosets of a suitable subgroup $H_i$ of $G$ containing all the involutions arising from the short edges in $S_i$.

2. A condition involving the Frattini subgroup

In this section we give a sufficient condition for the existence of a one-factorization with an automorphism group acting sharply transitively on vertices.

Definition 1. A starter in a group $G$ of even order is a set $\Sigma = \{S_1, \ldots, S_r\}$ of subsets of $\left(\frac{G}{2}\right)$ such that:

- $\partial S_1 \cup \cdots \cup \partial S_r = G - \{0\}$;
- for every $i = 1, \ldots, r$, the set $\phi(S_i)$ is a left transversal for a suitable subgroup $M_i$ of $G$ containing all the involutions fixing the short edges in $S_i$, for every $i = 1, \ldots, m$. Then $G$ admits a starter.

Using Proposition $\Phi$ we can state the following theorem.

Theorem 3. Each finite group $G$ of even order is a set $\Sigma = \{S_1, \ldots, S_r\}$ of subsets of $\left(\frac{G}{2}\right)$ satisfying the following conditions:

- $\partial S_1 \cup \cdots \cup \partial S_r = G - \{0\}$;
- for every $i = 1, \ldots, r$, the set $\phi(S_i)$ is a left transversal for a suitable subgroup $M_i$ of $G$ containing all the involutions fixing the short edges in $S_i$, for every $i = 1, \ldots, m$. Then $G$ admits a starter.

References