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Universal central extensions of twisted forms of split simple Lie algebras over rings

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ABSTRACT

We give sufficient conditions for the descent construction to be the universal central extension of a twisted form of a split simple Lie algebra over a ring. In particular, the universal central extensions of twisted multiloop Lie tori are obtained by the descent construction.

1. Introduction

Central extensions play a crucial role in physics as they can reduce the study of projective representations to the study of true representations. An important example of this is the Witt and Virasoro algebras which are infinite-dimensional Lie algebras with many applications to physics. They often appear in problems with conformal symmetry where the essential spacetime is one or two-dimensional and space is periodic, i.e. compactified to a circle. An example of such a setting is string theory where the string worldsheet is two-dimensional and cylindrical in the case of closed strings (see §4.3 in [8]). Such worldsheets are Riemann surfaces which are invariant under conformal transformations. The algebra of infinitesimal conformal transformations is the direct sum of two copies of the Witt algebra. The Virasoro algebra is a one-dimensional central extension (in this case, the universal central extension) of the Witt algebra.

The study of projective representations of the Witt algebra can be reduced to the study of true representations of the Virasoro algebra. The representations of the Virasoro algebra that are of interest in most physical applications are the unitary irreducible highest weight representations. These are completely characterized by the central charge and the conformal weight corresponding to the highest weight vector (see §3.2 in [15]). To each affine Kac–Moody algebra there is an associated Vi-

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rasoro algebra by Sugawara's construction (see §3.2.3 in [8]). A given unitary representation of the Kac–Moody algebra then naturally transforms into a unitary representation of the associated Virasoro algebra.

Kac's loop construction realizes all affine Kac–Moody algebras as the universal central extensions of loop algebras based on finite-dimensional simple Lie algebras [13]. Extended affine Lie algebras (EALAs), which arose in the work of K. Saito and P. Slodowy on elliptic singularities and in the paper by the physicists R. Høegh-Krohn and B. Torresani [12] on Lie algebras of interest to quantum gauge field theory, are natural generalizations of affine Kac–Moody algebras. A mathematical foundation of the theory of EALAs is provided in [1]. Kac's loop construction gives inspiration to the study of EALAs. The centreless cores of extended affine Lie algebras have been characterized axiomatically as centreless Lie tori. In [20] E. Neher realizes all EALAs as central extensions of centreless Lie tori. Almost all centreless Lie tori, namely those which are finitely generated over their centroids (f.g.c. for short), can be realized as multiloop Lie algebras based on finite-dimensional simple Lie algebras [2,3]. Using Grothendieck's descent formalism allows us to view multiloop Lie algebras as twisted forms [9,10,22]. This new perspective presents a beautiful bridge between infinite-dimensional Lie theory and descent theory. In [24] a natural construction for central extensions of twisted forms of split simple Lie algebras over rings is given by using Galois descent.

The purpose of this article is to study the universal central extensions of infinite-dimensional Lie algebras. In the affine Kac–Moody case, the universal central extension is one-dimensional. For the "higher nullity" EALAs, the universal central extensions are infinite-dimensional [18,7]. In [14] C. Kassel constructs the universal central extensions of untwisted multiloop Lie algebras by using Kähler differentials. It is much more complicated in the twisted case. Kassel's model has been generalized in [6] under certain conditions. Unfortunately twisted multiloop Lie tori do not satisfy these conditions. In [20] E. Neher constructs central extensions of centreless Lie tori by using centroidal derivations and states that the graded dual of the algebra of skew centroidal derivations gives the universal central extension of a centreless Lie torus. Since the centroidal derivations are essentially given by the centroid, to calculate Neher's construction of universal central extensions of centreless Lie tori depends on a good understanding of the centroid. In this article, we give sufficient conditions for the descent construction in [24] to give the universal central extensions of twisted forms of split simple Lie algebras over rings. In particular, the universal central extensions of twisted multiloop Lie tori are given by the descent construction and a good understanding of the centre is provided.

Throughout k will denote a field of characteristic 0, and $\mathfrak{g}$ a finite-dimensional split simple Lie algebra over k. Let R and S be commutative, associative, unital k-algebras. We write $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$ and $\mathfrak{g}_S = \mathfrak{g} \otimes_k S$.

2. Descent constructions for central extensions

In this section we will recall Kassel's construction for the universal central extension of $\mathfrak{g}_R$ and the descent construction for central extensions of twisted forms of $\mathfrak{g}_R$.

Let $\mathcal{L}$ be a Lie algebra over k and V a k-space. Any cocycle $P \in Z^2(\mathcal{L}, V)$, where V is viewed as a trivial $\mathcal{L}$-module, leads to a central extension

$$0 \to V \to \mathcal{L}_P \xrightarrow{\pi} \mathcal{L} \to 0$$

of $\mathcal{L}$ by V. As a space $\mathcal{L}_P = \mathcal{L} \oplus V$, and the bracket $[\,,]_P$ on $\mathcal{L}_P$ is given by

$$[x \oplus u, y \oplus v]_P = [x, y] \oplus P(x, y) \quad \text{for } x, y \in \mathcal{L} \text{ and } u, v \in V.$$

The equivalence class of this extension depends only on the class of P in $H^2(\mathcal{L}, V)$, and this gives a parametrization of all equivalence classes of central extensions of $\mathcal{L}$ by V (see for example [17] or [28] for details). In this situation, we will henceforth naturally identify V with a subspace of $\mathcal{L}_P$. Assume $\mathcal{L}$ is perfect. We fix once and for all a universal central extension $0 \to V \to \widehat{\mathcal{L}} \xrightarrow{\pi} \mathcal{L} \to 0$ (henceforth referred to as *the* universal central extension of $\mathcal{L}$). We will find it useful at times to think of this extension as being given by a (fixed in our discussion) "universal" cocycle $\widehat{P}$, thus $\widehat{\mathcal{L}} = \mathcal{L}_{\widehat{P}} = \mathcal{L} \oplus V$. This cocycle is of course not unique, but the class of $\widehat{P}$ in $H^2(\mathcal{L}, V)$ is unique.

We view $\mathfrak{g}_R$ as a Lie algebra *over k* (in general infinite-dimensional) by means of the unique bracket satisfying

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab \tag{2.1}$$

for all $x, y \in \mathfrak{g}$ and $a, b \in R$. Of course $\mathfrak{g}_R$ is also naturally an R-Lie algebra (which is free of finite rank). It will be clear at all times which of the two structures are being considered.

Let $(\Omega_{R/k}, d_R)$ be the R-module of Kähler differentials of the k-algebra R. When no confusion is possible, we will simply write (Ω_R, d). Following Kassel [14], we consider the k-subspace dR of Ω_R, and the corresponding quotient map $^- : \Omega_R \to \Omega_R/dR$. We then have a unique cocycle $\widehat{P} = \widehat{P}_R \in Z^2(\mathfrak{g}_R, \Omega_R/dR)$ satisfying

$$\widehat{P}(x \otimes a, y \otimes b) = (x|y)\overline{adb}, \tag{2.2}$$

where $(\cdot \mid \cdot)$ denotes the Killing form of $\mathfrak{g}$.

Let $\widehat{\mathfrak{g}_R}$ be the unique Lie algebra over k with the underlying space $\mathfrak{g}_R \oplus \Omega_R/dR$, and the unique bracket satisfying

$$[x \otimes a, y \otimes b]_{\widehat{P}} = [x, y] \otimes ab \oplus (x|y)\overline{adb}. \tag{2.3}$$

As the notation suggests,

$$0 \to \Omega_R/dR \to \widehat{\mathfrak{g}_R} \xrightarrow{\pi} \mathfrak{g}_R \to 0$$

is the universal central extension of $\mathfrak{g}_R$. There are other different realizations of the universal central extension (see [19,17,28] for details on three other different constructions), but Kassel's model is perfectly suited for our purposes.

We now turn our attention to twisted forms of $\mathfrak{g}_R$ for the flat topology of R, i.e. we look at R-Lie algebras $\mathcal{L}$ for which there exists a faithfully flat and finitely presented extension S/R such that

$$\mathcal{L} \otimes_R S \simeq \mathfrak{g}_R \otimes_R S \simeq \mathfrak{g} \otimes_k S, \tag{2.4}$$

where the above are isomorphisms of S-Lie algebras.

Let $\mathbf{Aut}(\mathfrak{g})$ be the k-algebraic group of automorphisms of $\mathfrak{g}$. The R-group $\mathbf{Aut}(\mathfrak{g})_R$ obtained by base change is clearly isomorphic to $\mathbf{Aut}(\mathfrak{g}_R)$. It is an affine, smooth, and finitely presented group scheme over R whose functor of points is given by

$$\mathbf{Aut}(\mathfrak{g}_R)(S) = \mathrm{Aut}_S(\mathfrak{g}_R \otimes_R S) \simeq \mathrm{Aut}_S(\mathfrak{g} \otimes_k S). \tag{2.5}$$

By Grothendieck's theory of descent (see Chapter I, §2 in [16], Chapter XXIV in [25] and §17.6 in [27]), we have a natural bijective map

$$\text{Isomorphism classes of twisted forms of } \mathfrak{g}_R \leftrightarrow H^1_{\acute{e}t}\big(R, \mathbf{Aut}(\mathfrak{g}_R)\big). \tag{2.6}$$

The descent construction for central extensions of twisted forms of $\mathfrak{g}_R$ relies on the following fundamental fact about lifting automorphisms to central extensions.

Proposition 2.7. *Let $\mathcal{L}$ be a perfect Lie algebra over k. Then*

(1) *There exists a (unique up to equivalence) universal central extension*

$$0 \to V \to \widehat{\mathcal{L}} \xrightarrow{\pi} \mathcal{L} \to 0.$$

(2) *If $\mathcal{L}$ is centreless, the centre $\mathfrak{z}(\widehat{\mathcal{L}})$ of $\widehat{\mathcal{L}}$ is precisely the kernel V of the projection homomorphism $\pi : \widehat{\mathcal{L}} \to \mathcal{L}$ above. Furthermore, the canonical map $\mathrm{Aut}_k(\widehat{\mathcal{L}}) \to \mathrm{Aut}_k(\mathcal{L})$ is an isomorphism.*

Proof. (1) This result was proved in [26, Proposition 1.3(ii) and (iii)]. The existence of an initial object in the category of central extensions of $\mathcal{L}$ is due to Garland [11, §5, Remark 5.11 and Appendix III]. (See also Theorem 1.14 in [19], §1.9, Proposition 2 in [17] and §7.9, Theorem 7.9.2 in [28] for details.)

(2) This result goes back to van der Kallen (see §11 in [26]). Other proofs can be found in [19, Theorem 2.2] and in [21, Proposition 2.2, Proposition 2.3 and Corollary 2.1]. □

We recall the following important observation of lifting automorphisms of $\mathfrak{g}_R$ to its central extensions in [24, Proposition 3.11].

Proposition 2.8. *Let $\theta \in \mathrm{Aut}_k(\mathfrak{g}_R)$, and let $\widehat{\theta}$ be the unique lift of θ to $\widehat{\mathfrak{g}_R}$ (see Proposition 2.7). If θ is R-linear, then $\widehat{\theta}$ fixes the centre Ω_R/dR of $\widehat{\mathfrak{g}_R}$ pointwise. In particular, every R-linear automorphism of $\mathfrak{g}_R$ lifts to every central extension of $\mathfrak{g}_R$.*

When S/R is a finite Galois ring extension with Galois group G, the descent data corresponding to $\mathcal{L}$, which a priori is an element of $\mathbf{Aut}(\mathfrak{g})(S \otimes_R S)$, can now be thought as being given by a cocycle $u = (u_g)_{g\in G} \in Z^1(G, \mathrm{Aut}_S(\mathfrak{g}_S))$ (usual non-abelian Galois cohomology), where the group G acts on $\mathrm{Aut}_S(\mathfrak{g}_S) = \mathrm{Aut}_S(\mathfrak{g} \otimes_k S)$ via ${}^g\theta = (1 \otimes g) \circ \theta \circ (1 \otimes g^{-1})$. Then

$$\mathcal{L} \simeq \mathcal{L}_u = \{X \in \mathfrak{g}_S\colon\ u_g{}^gX = X \text{ for all } g \in G\}.$$

As above, we let (Ω_S, d) be the module of Kähler differentials of S/k and let $\widehat{\mathfrak{g}_S} = \mathfrak{g}_S \oplus \Omega_S/dS$ be the universal central extension of $\mathfrak{g}_S$. The Galois group G acts naturally both on Ω_S and on the quotient k-space Ω_S/dS, in such a way that ${}^g(\overline{sdt}) = \overline{{}^gsd{}^gt}$. This leads to an action of G on $\widehat{\mathfrak{g}_S}$ for which

$${}^g\big((x \otimes s) \oplus z\big) = \big(x \otimes {}^gs\big) \oplus {}^gz$$

for all $x \in \mathfrak{g}$, $s \in S$, $z \in \Omega_S/dS$, and $g \in G$. One verifies immediately that the resulting maps are automorphisms of the k-Lie algebra $\widehat{\mathfrak{g}_S}$. We henceforth identify G with a subgroup of $\mathrm{Aut}_k(\widehat{\mathfrak{g}_S})$, and let G act on $\mathrm{Aut}_k(\widehat{\mathfrak{g}_S})$ by conjugation, i.e., ${}^g\theta = g\theta g^{-1}$. Let $\widehat{u}_g$ be the unique lift of u_g. We recall the descent construction for central extensions of twisted forms of $\mathfrak{g}_R$ in [24, Proposition 4.22].

Proposition 2.9. *Let $u = (u_g)_{g\in G}$ be a cocycle in $Z^1(G, \mathrm{Aut}_S(\mathfrak{g}_S))$. Then*

(1) *$\widehat{u} = (\widehat{u}_g)_{g\in G}$ is a cocycle in $Z^1(G, \mathrm{Aut}_k(\widehat{\mathfrak{g}_S}))$.*
(2) *$\mathcal{L}_{\widehat{u}} = \{x \in \widehat{\mathfrak{g}_S}\colon\ \widehat{u}_g{}^gx = x$ for all $g \in G\}$ is a central extension of the descended algebra $\mathcal{L}_u$ corresponding to u.*
(3) *There exist canonical isomorphisms $\mathfrak{z}(\mathcal{L}_{\widehat{u}}) \simeq (\Omega_S/dS)^G \simeq \Omega_R/dR$.*

The following proposition in [24, Proposition 4.23] gives equivalent conditions for $\mathcal{L}_{\widehat{u}} = \mathcal{L}_u \oplus \Omega_R/dR$.

Proposition 2.10. *With the above notation, the following conditions are equivalent.*

(1) *$\mathcal{L}_{\widehat{u}} = \mathcal{L}_u \oplus \Omega_R/dR$ and $\mathcal{L}_u$ is stable under the action of the Galois group G.*
(2) *$\widehat{u}_g(\mathcal{L}_u) \subset \mathcal{L}_u$ for all $g \in G$.*

If these conditions hold, then every $\theta \in \mathrm{Aut}_R(\mathcal{L}_u)$ lifts to an automorphism $\widehat{\theta}$ of $\mathcal{L}_{\widehat{u}}$ that fixes the centre of $\mathcal{L}_{\widehat{u}}$ pointwise.

Remark 2.11. Multiloop Lie algebras provide special examples of twisted forms of $\mathfrak{g}_R$ in the sense of Galois descent. Given a finite-dimensional split simple Lie algebra $\mathfrak{g}$ over k and commuting finite order automorphisms $\sigma_1, \dots, \sigma_n$ of $\mathfrak{g}$ with $\sigma_i^{m_i} = 1$, the n-step multiloop Lie algebra of $(\mathfrak{g}, \sigma_1, \dots, \sigma_n)$ is defined by

$$L(\mathfrak{g}, \sigma_1, \dots, \sigma_n) := \bigoplus_{(i_1,\dots,i_n)\in\mathbb{Z}^n} \mathfrak{g}_{\bar{i}_1,\dots,\bar{i}_n} \otimes t_1^{i_1/m_1} \dots t_n^{i_n/m_n},$$

where $^- : \mathbb{Z} \to \mathbb{Z}/m_j\mathbb{Z}$ is the canonical map for $1 \leqslant j \leqslant n$ and

$$\mathfrak{g}_{\bar{i}_1,\dots,\bar{i}_n} = \{x \in \mathfrak{g}\colon\ \sigma_j(x) = \zeta_{m_j}^{i_j} x \ \text{ for } 1 \leqslant j \leqslant n\}$$

is the simultaneous eigenspace corresponding to the eigenvalues ζ_{m_j} (the primitive m_jth roots of unity) for $1 \leqslant j \leqslant n$. A multiloop Lie algebra $L(\mathfrak{g}, \sigma_1, \dots, \sigma_n)$ is infinite-dimensional over the given base field k, but is finite-dimensional over its centroid $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Let S/R be the finite Galois ring extension with $S = k[t_1^{\pm 1/m_1}, \dots, t_n^{\pm 1/m_n}]$, then the following S-Lie algebra isomorphism

$$L(\mathfrak{g}, \sigma_1, \dots, \sigma_n) \otimes_R S \simeq \mathfrak{g}_R \otimes_R S$$

tells that $L(\mathfrak{g}, \sigma_1, \dots, \sigma_n)$ is a twisted form of $\mathfrak{g}_R$. This perspective of viewing multiloop Lie algebras as twisted forms, which is developed in [9,10,22], provides a new way to look at their structure through the lens of descent theory. Thus a multiloop Lie algebra $L(\mathfrak{g}, \sigma_1, \dots, \sigma_n)$ as a twisted form of $\mathfrak{g}_R$ must be isomorphic to an R-Lie algebra $\mathcal{L}_u$ for some cocycle u in $Z^1(G, \mathrm{Aut}_S(\mathfrak{g}_S))$. From the general theory about the nature of multiloop Lie algebras as twisted forms (see [22, Theorem 2.1] for loop algebras, and [10, §5] for multiloop algebras in general), the cocycle $u = (u_g)_{g\in G}$ is constant (i.e., it has trivial Galois action) with $u_g = v_g \otimes id$ for all $g \in G$. The multiloop Lie algebra $\mathcal{L}_u$ then has a basis consisting of eigenvectors of the u_g's, and therefore the second equivalent condition of Proposition 2.10 holds. Thus for multiloop Lie algebras, we have $\mathcal{L}_{\widehat{u}} = \mathcal{L}_u \oplus \Omega_R/dR$.

3. Universal central extensions

Since $\mathcal{L}_u$ in the above section is perfect (see §5.1 and §5.2 of [10] for details), it admits a universal central extension $\widehat{\mathcal{L}_u}$. By Proposition 2.7, there exists a canonical map $\widehat{\mathcal{L}_u} \to \mathcal{L}_{\widehat{u}}$. In this section, we give a sufficient condition for $\mathcal{L}_{\widehat{u}} \simeq \widehat{\mathcal{L}_u}$. As an application we show that if $\mathcal{L}_u$ is a multiloop Lie torus, then $\mathcal{L}_{\widehat{u}}$ is the universal central extension of $\mathcal{L}_u$.

Throughout this section S/R is a finite Galois ring extension with Galois group G. We identify R with a subring of S and Ω_R/dR with $(\Omega_S/dS)^G$ through a chosen isomorphism. Let $u = (u_g)_{g\in G} \in Z^1(G, \mathrm{Aut}_S(\mathfrak{g}_S))$ be a constant cocycle with $u_g = v_g \otimes id$ for all $g \in G$. Then the descended Lie algebra corresponding to u is

$$\begin{aligned}\mathcal{L}_u &= \{X \in \mathfrak{g}_S\colon\ u_g{}^g X = X \text{ for all } g \in G\}\\ &= \{\Sigma_i x_i \otimes a_i \in \mathfrak{g}_S\colon\ \Sigma_i v_g(x_i) \otimes {}^g a_i = \Sigma_i x_i \otimes a_i \ \text{ for all } g \in G\}.\end{aligned}$$

Let $\mathfrak{g}_0 = \{x \in \mathfrak{g}\colon\ v_g(x) = x \text{ for all } g \in G\}$. Then $\mathfrak{g}_0$ is a k-Lie subalgebra of $\mathfrak{g}$. We write $\mathfrak{g}_{0R} = \mathfrak{g}_0 \otimes_k R$. Clearly $\mathfrak{g}_{0R}$ is a k-Lie subalgebra of $\mathcal{L}_u$. Assume $\mathfrak{g}_0$ is central simple and let $\widehat{\mathfrak{g}_{0R}} = \mathfrak{g}_{0R} \oplus \Omega_R/dR$ be the universal central extension of $\mathfrak{g}_{0R}$.

We first prove a useful lemma and then generalize C. Kassel's proof in [14] that $\widehat{\mathfrak{g}_R}$ is the universal central extension of $\mathfrak{g}_R$.

Lemma 3.12. *Let $\mathcal{L}$ be a Lie algebra over k and let V be a trivial $\mathcal{L}$-module. If $\mathfrak{s} \subset \mathcal{L}$ is a finite-dimensional semisimple k-Lie subalgebra and $\mathcal{L}$ is a locally finite $\mathfrak{s}$-module, then every cohomology class in $H^2(\mathcal{L}, V)$ can be represented by an $\mathfrak{s}$-invariant cocycle.*

Proof. For any cocycle $P \in Z^2(\mathcal{L}, V)$, our goal is to find another cocycle $P' \in Z^2(\mathcal{L}, V)$ such that $[P] = [P']$ and $P'(\mathcal{L}, \mathfrak{s}) = \{0\}$. Note that $\mathrm{Hom}_k(\mathcal{L}, V)$ is a $\mathcal{L}$-module given by $y.\beta(x) = \beta(-[y, x])$.

Define a k-linear map $f : \mathfrak{s} \to \mathrm{Hom}_k(\mathcal{L}, V)$ by $f(y)(x) = P(x, y)$. We claim that $f \in Z^1(\mathfrak{s}, \mathrm{Hom}_k(\mathcal{L}, V))$. Indeed, since $P \in Z^2(\mathcal{L}, V)$, we have

$$P(x, y) = -P(y, x) \quad \text{and} \quad P\big([x, y], z\big) + P\big([y, z], x\big) + P\big([z, x], y\big) = 0$$

for all $x, y, z \in \mathcal{L}$. Then $P(x, [y, z]) = P([x, y], z) + P([z, x], y)$, namely $f([y, z])(x) = f(z)([x, y]) + f(y)([z, x])$ for all $x, y, z \in \mathcal{L}$. Thus

$$f\big([y, z]\big) = y.f(z) - z.f(y)$$

implies $f \in Z^1(\mathfrak{s}, \mathrm{Hom}_k(\mathcal{L}, V))$.

By our assumption that $\mathfrak{s}$ is finite-dimensional and semisimple, the Whitehead's first lemma (see §7.8 in [28]) yields $H^1(\mathfrak{s}, \mathrm{Hom}_k(\mathcal{L}, V)) = 0$. Note that the standard Whitehead's first lemma holds for finite-dimensional $\mathfrak{s}$-modules. However, $\mathrm{Hom}_k(\mathcal{L}, V)$ is a direct sum of finite-dimensional $\mathfrak{s}$-modules when $\mathcal{L}$ is a locally finite $\mathfrak{s}$-module and V is a trivial $\mathcal{L}$-module, so the result easily extends. So $f = d^0(\tau)$ for some $\tau \in \mathrm{Hom}_k(\mathcal{L}, V)$, where d^0 is the coboundary map from $\mathrm{Hom}_k(\mathcal{L}, V)$ to $C^1(\mathfrak{s}, \mathrm{Hom}_k(\mathcal{L}, V))$.

Let $P' = P + d^1(\tau)$, where d^1 is the coboundary map from $\mathrm{Hom}_k(\mathcal{L}, V)$ to $C^2(\mathcal{L}, V)$. Then $[P'] = [P]$. For all $x \in \mathcal{L}$ and $y \in \mathfrak{s}$ we have

$$\begin{aligned} P'(x, y) &= P(x, y) + d^1(\tau)(x, y) \\ &= P(x, y) - \tau\big([x, y]\big) \\ &= P(x, y) - f(y)(x) = 0. \end{aligned}$$

Thus P' is an $\mathfrak{s}$-invariant cocycle. □

Remark 3.13. A version of this lemma was proved in [4, §3.3–3.5] for Lie algebras graded by finite root systems.

The following proposition follows from C. Kassel's result on the universality of the central extension $\widehat{\mathfrak{g}_{0R}}$. To make this paper more self-contained, a proof is included here by constructing the maps φ and ψ explicitly which are to be used in the proof of Proposition 3.25.

Proposition 3.14. *Let $\mathcal{L}_u$ be the descended algebra corresponding to a constant cocycle $u = (u_g)_{g \in G} \in Z^1(G, \mathrm{Aut}_S(\mathfrak{g}_S))$. Let $\mathcal{L}_P$ be a central extension of $\mathcal{L}_u$ with cocycle $P \in Z^2(\mathcal{L}_u, V)$. Assume $\mathfrak{g}_0$ is central simple, then there exist a k-Lie algebra homomorphism $\psi : \widehat{\mathfrak{g}_{0R}} \to \mathcal{L}_p$ and a k-linear map $\varphi : \Omega_R/dR \to V$ such that the following diagram is commutative.*

$$\begin{CD} 0 @>>> \Omega_R/dR @>>> \widehat{\mathfrak{g}_{0R}} @>>> \mathfrak{g}_{0R} @>>> 0 \\ @. @V{\varphi}VV @V{\psi}VV @VVV @. \\ 0 @>>> V @>>> \mathcal{L}_P @>>> \mathcal{L}_u @>>> 0. \end{CD}$$

Proof. Our goal is to find $P_0 \in Z^2(\mathcal{L}_u, V)$ with $[P_0] = [P]$ satisfying

$$P_0(x \otimes a, y \otimes 1) = 0 \quad \text{for all } x, y \in \mathfrak{g}_0 \text{ and } a \in R. \tag{3.15}$$

Applying Lemma 3.12 to $\mathcal{L} = \mathfrak{g}_{0R}$ and $\mathfrak{s} = \mathfrak{g}_0 \otimes_k k$, it is clear that $\mathcal{L}$ is a locally finite $\mathfrak{s}$-module and thus we can find an $\mathfrak{s}$-invariant cocycle $P' \in Z^2(\mathcal{L}, V)$, where $P' = P|_{\mathcal{L}\times\mathcal{L}} + d^1(\tau)$ for some $\tau \in \operatorname{Hom}_k(\mathcal{L}, V)$. We can extend this τ to get a k-linear map $\tau_0 : \mathcal{L}_u = \mathfrak{g}_{0R} \oplus \mathfrak{g}_{0R}^{\perp} \to V$ by $\tau_0|_{\mathfrak{g}_{0R}} = \tau$ and $\tau_0|_{\mathfrak{g}_{0R}^{\perp}} = 0$. Let $P_0 = P + d^1(\tau_0)$, where d^1 is the coboundary map from $\operatorname{Hom}_k(\mathcal{L}_u, V)$ to $C^2(\mathcal{L}_u, V)$. Then $[P_0] = [P]$ and it is easy to check that for all $x, y \in \mathfrak{g}_0$ and $a \in R$ we have

$$\begin{aligned} P_0(x \otimes a, y \otimes 1) &= P(x \otimes a, y \otimes 1) + d^1(\tau_0)(x \otimes a, y \otimes 1) \\ &= P(x \otimes a, y \otimes 1) + d^1(\tau)(x \otimes a, y \otimes 1) \\ &= P'(x \otimes a, y \otimes 1) = 0. \end{aligned}$$

Replace P by P_0. Since $P \in Z^2(\mathcal{L}_u, V)$, we have

$$P(x \otimes a, y \otimes b) = -P(y \otimes b, x \otimes a), \tag{3.16}$$

$$P\big([x \otimes a, y \otimes b], z \otimes c\big) + P\big([y \otimes b, z \otimes c], x \otimes a\big) + P\big([z \otimes c, x \otimes a], y \otimes b\big) = 0 \tag{3.17}$$

for all $x \otimes a, y \otimes b, z \otimes c \in \mathcal{L}_u$. We can define a k-linear map $\Omega_R/dR \to V$ as follows. Fix $a, b \in R$ and define $\alpha : \mathfrak{g}_0 \times \mathfrak{g}_0 \to V$ by $\alpha(x, y) = P(x \otimes a, y \otimes b)$. Then with $c = 1$ in (3.17) we obtain $P([y, z] \otimes b, x \otimes a) + P([z, x] \otimes a, y \otimes b) = 0$ for all $z \in \mathfrak{g}_0$. By (3.16) we have

$$P\big([z, x] \otimes a, y \otimes b\big) = -P\big([y, z] \otimes b, x \otimes a\big) = P\big(x \otimes a, [y, z] \otimes b\big).$$

So $\alpha([z, x], y) = \alpha(x, [y, z])$. This tells us $\alpha([x, z], y) = \alpha(x, [z, y])$, namely α is an invariant bilinear form on $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is central simple by our assumption, $\mathfrak{g}_0$ has a unique invariant bilinear form up to scalars. It follows that there is a unique $z_{a,b} \in V$ such that for all $x, y \in \mathfrak{g}_0$ we have

$$P(x \otimes a, y \otimes b) = \alpha(x, y) = (x|y) z_{a,b}, \tag{3.18}$$

where $(\cdot \mid \cdot)$ denotes the Killing form of $\mathfrak{g}$. From (3.15)–(3.17) and $(\cdot \mid \cdot)$ is symmetric we have

$$\text{(i)} \quad z_{a,1} = 0, \qquad \text{(ii)} \quad z_{a,b} = -z_{b,a}, \qquad \text{(iii)} \quad z_{ab,c} + z_{bc,a} + z_{ca,b} = 0. \tag{3.19}$$

Then by (ii) and (iii) the map $\varphi : \Omega_{R/k} \simeq H_1(R, R) \simeq R \otimes_k R/\langle ab \otimes c - a \otimes bc + ca \otimes b\rangle \to V$ given by $\varphi(adb) = z_{a,b}$ is a well-defined k-linear map. Here H_1 is the Hochschild homology. By (i) φ induces a well-defined k-linear map $\varphi : \Omega_R/dR \to V$ given by $\varphi(\overline{adb}) = z_{a,b}$.

Finally let $\sigma : \mathcal{L}_u \to \mathcal{L}_P$ be any section map satisfying

$$\big[\sigma(x \otimes a), \sigma(y \otimes b)\big]_{\mathcal{L}_P} - \sigma\big([x, y] \otimes ab\big) = P(x \otimes a, y \otimes b) \tag{3.20}$$

for all $x \otimes a, y \otimes b \in \mathcal{L}_u$. Define $\psi : \widehat{\mathfrak{g}_{0R}} \to \mathcal{L}_P$ by $\psi(X \oplus Z) = \sigma(X) \oplus \varphi(Z)$ for all $X \in \mathfrak{g}_{0R}$ and $Z \in \Omega_R/dR$. Clearly ψ is a well-defined k-linear map. We claim that ψ is a Lie algebra homomorphism. Indeed, let $x \otimes a, y \otimes b \in \mathfrak{g}_{0R}$, then

$$\psi\big([x \otimes a, y \otimes b]_{\widehat{\mathfrak{g}_{0R}}}\big) = \psi\big([x, y] \otimes ab \oplus (x|y)\overline{adb}\big) = \sigma\big([x, y] \otimes ab\big) + (x|y)z_{a,b},$$
$$\big[\psi(x \otimes a), \psi(y \otimes b)\big]_{\mathcal{L}_P} = \big[\sigma(x \otimes a), \sigma(y \otimes b)\big]_{\mathcal{L}_P} = \sigma\big([x, y] \otimes ab\big) + P(x \otimes a, y \otimes b).$$

By (3.18) this shows that ψ is a Lie algebra homomorphism. It is easy to check the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_R/dR & \longrightarrow & \widehat{\mathfrak{g}_{0R}} & \longrightarrow & \mathfrak{g}_{0R} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & \mathcal{L}_P & \longrightarrow & \mathcal{L}_u & \longrightarrow & 0. \end{array} \qquad \square$$

Remark 3.21. The above proposition is a special case of [26, Proposition 1.3(v)] since the first row of the diagram is a universal central extension of $\mathfrak{g}_{0R}$ and the second row is a central extension of $\mathcal{L}_u$. When u is a trivial cocycle, we have $\mathcal{L}_u = \mathfrak{g}_R$ and $\mathfrak{g}_0 = \mathfrak{g}$. The above proposition, which is a consequence of C. Kassel's result, shows that $\widehat{\mathfrak{g}_R}$ is the universal central extension of $\mathfrak{g}_R$.

To understand the universal central extensions of twisted forms of $\mathfrak{g}_R$, we need to construct a cocycle P_0 which satisfies a stronger condition than (3.15). For each $a \in S\backslash\{0\}$ define $\mathfrak{g}_a = \{x \in \mathfrak{g}\colon\ v_g(x) \otimes {}^g a = x \otimes a \text{ for all } g \in G\}$. Then $\mathfrak{g}_a$ is a k-subspace of $\mathfrak{g}$. It is easy to check that $\mathfrak{g}_a \subset \mathfrak{g}_{ra}$ for any $r \in R$ and $\mathfrak{g}_a \otimes_k Ra$ is a k-subspace of $\mathcal{L}_u$.

Lemma 3.22.

(1) $\mathfrak{g}_a = \mathfrak{g}_0$ *if* $a \in R\backslash\{0\}$. *In particular,* $\mathfrak{g}_1 = \mathfrak{g}_0$.
(2) $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{ab}$ *for any* $a, b \in S\backslash\{0\}$. $[\mathfrak{g}_a, \mathfrak{g}_0] \subset \mathfrak{g}_a$ *for any* $a \in S\backslash\{0\}$.
(3) $\mathrm{Hom}_k(\mathfrak{g}_a, V)$ *is a* $\mathfrak{g}_0$*-module for any* k*-vector space* V, $a \in S\backslash\{0\}$.

Proof. (1) If $a \in R$, then ${}^g a = a$ for all $g \in G$. Thus $\mathfrak{g}_a = \{x \in \mathfrak{g}\colon\ v_g(x) \otimes a = x \otimes a \text{ for all } g \in G\}$. Clearly $\mathfrak{g}_a \supset \mathfrak{g}_0$. On the other hand, let $x \in \mathfrak{g}_a$ and let $\{x_i \otimes a_j\}_{i \in I, j \in J}$ be a k-basis of $\mathfrak{g}_S$. Assume $x = \Sigma_i \lambda_i x_i$, $v_g(x) = \Sigma_i \lambda_i^g x_i$ and $a = \Sigma_j \mu_j a_j$. Then $v_g(x) \otimes a = x \otimes a$ implies that $\Sigma_{i,j} \lambda_i^g \mu_j (x_i \otimes a_j) = \Sigma_{i,j} \lambda_i \mu_j (x_i \otimes a_j)$. Thus $\lambda_i^g \mu_j = \lambda_i \mu_j$ for all $i \in I$, $j \in J$ and $g \in G$. Since $a \neq 0$, there exists $\mu_{j_a} \neq 0$. By $\lambda_i^g \mu_{j_a} = \lambda_i \mu_{j_a}$ we get $\lambda_i^g = \lambda_i$ for all $i \in I$ and $g \in G$. Thus $x \in \mathfrak{g}_0$, so $\mathfrak{g}_a = \mathfrak{g}_0$.

(2) Let $x \in \mathfrak{g}_a$ and $y \in \mathfrak{g}_b$. Then $v_g([x, y]) \otimes {}^g(ab) = [v_g(x), v_g(y)] \otimes {}^g(ab) = [v_g(x) \otimes {}^g a, v_g(y) \otimes {}^g b] = [x \otimes a, y \otimes b] = [x, y] \otimes ab$. Thus $[x, y] \in \mathfrak{g}_{ab}$. For any $a \in S\backslash\{0\}$ we have $[\mathfrak{g}_a, \mathfrak{g}_0] = [\mathfrak{g}_a, \mathfrak{g}_1] \subset \mathfrak{g}_a$.

(3) Let $y \in \mathfrak{g}_0$ and $\beta \in \mathrm{Hom}_k(\mathfrak{g}_a, V)$. Define $y.\beta(x) = \beta(-[y, x])$. We can check $y.\beta$ is a well-defined $\mathfrak{g}_0$ action. $\square$

Proposition 3.23. *Let* $\mathcal{L}_u$ *be the descended algebra corresponding to a constant cocycle* $u = (u_g)_{g \in G} \in Z^1(G, \mathrm{Aut}_S(\mathfrak{g}_S))$. *Let* $\mathcal{L}_P$ *be a central extension of* $\mathcal{L}_u$ *with cocycle* $P \in Z^2(\mathcal{L}_u, V)$. *Assume* $\mathfrak{g}_0$ *is simple and* $\mathfrak{g}$ *has a basis consisting of simultaneous eigenvectors of* $\{v_g\}_{g \in G}$, *then we can construct a cocycle* $P_0 \in Z^2(\mathcal{L}_u, V)$ *with* $[P_0] = [P]$ *satisfying* $P_0(x \otimes a, y \otimes 1) = 0$ *for all* $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_0$ *and* $a \in S$.

Proof. For each $a \in S\backslash\{0\}$, let $\mathcal{L}_a$ be the k-Lie subalgebra of $\mathcal{L}_u$ generated by the elements in $(\mathfrak{g}_a \otimes_k Ra) \cup (\mathfrak{g}_0 \otimes_k k)$. Let $\mathfrak{s} = \mathfrak{g}_0 \otimes_k k$. By Lemma 3.22(2) we have $[\mathfrak{g}_a, \mathfrak{g}_0] \subset \mathfrak{g}_a$, thus $\mathcal{L}_a$ is a locally finite $\mathfrak{s}$-module. Applying Lemma 3.12 to $\mathcal{L} = \mathcal{L}_a$ and $\mathfrak{s} = \mathfrak{g}_0 \otimes_k k$, we can find an $\mathfrak{s}$-invariant cocycle $P'_a \in Z^2(\mathcal{L}_a, V)$, where $P'_a = P|_{\mathcal{L}_a \times \mathcal{L}_a} + d^1(\tau_a)$ for some $\tau_a \in \mathrm{Hom}_k(\mathcal{L}_a, V)$. Let $\{x_i \otimes a_j\}_{i \in I, j \in J}$

be a k-basis of $\mathcal{L}_u$. For each a_j choose one $\tau_{a_j} \in \mathrm{Hom}_k(\mathcal{L}_{a_j}, V)$. Note that $x_i \otimes a_j \in \mathcal{L}_u$ implies $x_i \in \mathfrak{g}_{a_j}$, thus $x_i \otimes a_j \in \mathcal{L}_{a_j}$. Define $\tau : \mathcal{L}_u \to V$ to be the unique linear map such that $\tau(x_i \otimes a_j) = \tau_{a_j}(x_i)$.

Let $P_0 = P + d^1(\tau)$, where d^1 is the coboundary map from $\mathrm{Hom}_k(\mathcal{L}_u, V)$ to $C^2(\mathcal{L}_u, V)$. Then $[P_0] = [P]$. For each a_j $(j \in J)$ it is easy to check that for all $x \in \mathfrak{g}_{a_j}$, $y \in \mathfrak{g}_0$ we have

$$\begin{aligned} P_0(x \otimes a_j, y \otimes 1) &= P(x \otimes a_j, y \otimes 1) + d^1(\tau)(x \otimes a_j, y \otimes 1) \\ &= P(x \otimes a_j, y \otimes 1) + d^1(\tau_{a_j})(x \otimes a_j, y \otimes 1) \\ &= P'_{a_j}(x \otimes a_j, y \otimes 1) = 0. \end{aligned}$$

Note that our proof does not depend on the choice of τ_{a_j} because $ker(d^0) = \mathrm{Hom}_k(\mathcal{L}_{a_j}, V)^{\mathfrak{s}}$ and different choices of τ_{a_j} become the same when restricted to $[\mathcal{L}_{a_j}, \mathfrak{s}]$. Thus for any $x \otimes a = \Sigma_{i,j} \lambda_{ij} x_i \otimes a_j \in \mathcal{L}_u$, we have $P_0(x \otimes a, y \otimes 1) = \Sigma_{ij} P_0(x_i \otimes a_j, y \otimes 1) = 0$. □

We have the following important observation when $\mathfrak{g}$ has a basis consisting of simultaneous eigenvectors of $\{v_g\}_{g \in G}$.

Lemma 3.24. *Let $\mathscr{B} = \{x_i \otimes a_j\}_{i \in I, j \in J}$ be a k-basis of $\mathcal{L}_u$ with $\{x_i\}_{i \in I}$ consisting of simultaneous eigenvectors of $\{v_g\}_{g \in G}$. Take $x_i \otimes a_j, x_l \otimes a_k \in \mathcal{L}_u$. If $0 \neq \overline{a_j da_k} \in \Omega_R/dR$, then $a_j a_k \in R$ and $[x_i, x_l] \in \mathfrak{g}_0$.*

Proof. Let $v_g(x_i) = \lambda_g^i x_i$, where $\lambda_g^i \in k$. If $x_i \otimes a_j \in \mathcal{L}_u$, then $x_i \in \mathfrak{g}_{a_j}$. So $v_g(x_i) \otimes {}^g a_j = \lambda_g^i x_i \otimes {}^g a_j = x_i \otimes a_j$. Thus $x_i \otimes {}^g a_j = x_i \otimes (\lambda_g^i)^{-1} a_j$, and therefore $x_i \otimes ({}^g a_j - (\lambda_g^i)^{-1} a_j) = 0$. Since $x_i \neq 0$, we have ${}^g a_j - (\lambda_g^i)^{-1} a_j = 0$, thus ${}^g a_j = (\lambda_g^i)^{-1} a_j$. Similarly, we can show that ${}^g a_k = (\lambda_g^l)^{-1} a_k$. So if $\overline{a_j da_k} \in \Omega_R/dR$, then $\overline{{}^g a_j d {}^g a_k} = \overline{a_j da_k}$ for all $g \in G$. Note that

$$\overline{a_j da_k} = \overline{{}^g a_j d {}^g a_k} = \overline{\left(\lambda_g^i\right)^{-1} a_j d\left(\lambda_g^l\right)^{-1} a_k} = \left(\lambda_g^i\right)^{-1} \left(\lambda_g^l\right)^{-1} \overline{a_j da_k} = \left(\lambda_g^i \lambda_g^l\right)^{-1} \overline{a_j da_k}.$$

So if $\overline{a_j da_k} \neq 0$, then $(\lambda_g^i \lambda_g^l)^{-1} = \lambda_g^i \lambda_g^l = 1$. Thus ${}^g(a_j a_k) = {}^g a_j {}^g a_k = (\lambda_g^i)^{-1} a_j (\lambda_g^l)^{-1} a_k = a_j a_k$. So $a_j a_k \in R$ and $[x_i, x_l] \in [\mathfrak{g}_{a_j}, \mathfrak{g}_{a_k}] \subset \mathfrak{g}_{a_j a_k} = \mathfrak{g}_0$ by Lemma 3.22. □

Now we are ready to prove the main result of this section.

Proposition 3.25. *Let $u = (u_g)_{g \in G} \in Z^1(G, \mathrm{Aut}_S(\mathfrak{g}_S))$ be a constant cocycle with $u_g = v_g \otimes id$. Let $\mathcal{L}_u$ be the descended algebra corresponding to u and let $\mathcal{L}_{\widehat{u}}$ be the central extension of $\mathcal{L}_u$ obtained by the descent construction* (*see Proposition* 2.9)*. Assume $\mathfrak{g}_0$ is central simple and $\mathfrak{g}$ has a basis consisting of simultaneous eigenvectors of $\{v_g\}_{g \in G}$. Assume $\mathcal{L}_{\widehat{u}} = \mathcal{L}_u \oplus \Omega_R/dR$, then $\mathcal{L}_{\widehat{u}}$ is the universal central extension of $\mathcal{L}_u$.*

Proof. First of all, $\mathcal{L}_{\widehat{u}}$ is perfect. Indeed, let $X \oplus Z \in \mathcal{L}_{\widehat{u}}$, where $X \in \mathcal{L}_u$ and $Z \in \Omega_R/dR$. Since $\mathcal{L}_u$ is perfect, we have $X = \Sigma_i [X_i, Y_i]_{\mathcal{L}_u}$ for some $X_i, Y_i \in \mathcal{L}_u$. By the assumption $\mathcal{L}_{\widehat{u}} = \mathcal{L}_u \oplus \Omega_R/dR$ we have $\mathcal{L}_u \subset \mathcal{L}_{\widehat{u}}$, then $X_i, Y_i \in \mathcal{L}_{\widehat{u}}$. Thus $\Sigma_i [X_i, Y_i]_{\mathcal{L}_{\widehat{u}}} = \Sigma_i [X_i, Y_i]_{\mathcal{L}_u} \oplus W$ for some $W \in \Omega_R/dR$. So $X \oplus Z = \Sigma_i [X_i, Y_i]_{\mathcal{L}_{\widehat{u}}} \oplus (Z - W)$, where $Z - W \in \Omega_R/dR \subset [\mathfrak{g}_{0R}, \mathfrak{g}_{0R}]_{\mathcal{L}_{\widehat{u}}} \subset [\mathcal{L}_{\widehat{u}}, \mathcal{L}_{\widehat{u}}]_{\mathcal{L}_{\widehat{u}}}$. Thus $\mathcal{L}_{\widehat{u}}$ is perfect.

Let $\mathcal{L}_P$ be a central extension of $\mathcal{L}_u$ with cocycle $P \in Z^2(\mathcal{L}_u, V)$. By Proposition 3.23, we can assume that $P(x \otimes a, y \otimes 1) = 0$ for all $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_0$ and $a \in S$. Let $\sigma : \mathcal{L}_u \to \mathcal{L}_P$ be any section of $\mathcal{L}_P \to \mathcal{L}_u$ satisfying

$$\left[\sigma(x \otimes a), \sigma(y \otimes b)\right]_{\mathcal{L}_P} - \sigma\left([x, y] \otimes ab\right) = P(x \otimes a, y \otimes b) \tag{3.26}$$

for all $x \otimes a$, $y \otimes b \in \mathcal{L}_u$. Define $\psi : \mathcal{L}_{\widehat{u}} \to \mathcal{L}_P$ by $\psi(X \oplus Z) = \sigma(X) + \varphi(Z)$ for all $X \in \mathcal{L}_u$ and $Z \in \Omega_R/dR$, where $\varphi : \Omega_R/dR \to V$ is the map given by $\varphi(\overline{adb}) = z_{a,b}$ in Proposition 3.14. Clearly ψ is a well-defined k-linear map. We claim that ψ is a Lie algebra homomorphism. Indeed, let $x \otimes a$, $y \otimes b \in \mathcal{L}_{\widehat{u}}$, then

$$\psi\big([x \otimes a, y \otimes b]_{\mathcal{L}_{\widehat{u}}}\big) = \psi\big([x, y] \otimes ab \oplus (x|y)\overline{adb}\big) = \sigma\big([x, y] \otimes ab\big) + (x|y)\varphi(\overline{adb}),$$
$$\big[\psi(x \otimes a), \psi(y \otimes b)\big]_{\mathcal{L}_P} = \big[\sigma(x \otimes a), \sigma(y \otimes b)\big]_{\mathcal{L}_P} = \sigma\big([x, y] \otimes ab\big) + P(x \otimes a, y \otimes b).$$

By (3.18) we have $P(x \otimes a, y \otimes b) = (x|y)z_{a,b}$ for all $x, y \in \mathfrak{g}_0$ and $a, b \in R$. If $a, b \in S\backslash R$, we have two cases. Since ψ is well-defined, we only need to consider basis elements in $\mathcal{L}_u$. Let $\mathscr{B} = \{x_i \otimes a_j\}_{i\in I, j\in J}$ be a k-basis of $\mathcal{L}_u$ with $\{x_i\}_{i\in I}$ consisting of eigenvectors of the v_g's. Take $x_i \otimes a_j, x_l \otimes a_k \in \mathcal{L}_u$. If $0 \neq \overline{a_j da_k} \in \Omega_R/dR$, then $a_j a_k \in R$ and $[x_i, x_l] \in \mathfrak{g}_0$ by Lemma 3.24. Thus $[x_i \otimes a_j, x_l \otimes a_k]_{\mathcal{L}_{\widehat{u}}} \subset \mathfrak{g}_{0R} \oplus \Omega_R/dR$. By Proposition 3.14 ψ is a Lie algebra homomorphism in this case. If $0 = \overline{a_j da_k} \in \Omega_R/dR$, then $[x_i \otimes a_j, x_l \otimes a_k]_{\mathcal{L}_{\widehat{u}}} = [x_i \otimes a_j a_k, x_l \otimes 1]_{\mathcal{L}_{\widehat{u}}}$. By Proposition 3.23 we have $P(x_i \otimes a_j a_k, x_l \otimes 1) = 0$. So ψ is a Lie algebra homomorphism as well in this case. It is easy to check the following diagram is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_R/dR & \longrightarrow & \mathcal{L}_{\widehat{u}} & \longrightarrow & \mathcal{L}_u & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \| & & \\
0 & \longrightarrow & V & \longrightarrow & \mathcal{L}_P & \longrightarrow & \mathcal{L}_u & \longrightarrow & 0.
\end{array}$$ □

Corollary 3.27. *If $\mathcal{L}_u$ is a multiloop Lie torus over an algebraically closed field of characteristic* 0*, then $\mathcal{L}_{\widehat{u}}$ is the universal central extension of $\mathcal{L}_u$ and the centre of $\mathcal{L}_{\widehat{u}}$ is Ω_R/dR.*

Proof. If $\mathcal{L}_u$ is a multiloop Lie torus, by Remark 2.11 we have $\mathcal{L}_{\widehat{u}} = \mathcal{L}_u \oplus \Omega_R/dR$. By the definition of multiloop Lie algebras, $\{v_g\}_{g\in G}$ is a set of commuting finite order automorphisms of $\mathfrak{g}$, thus $\mathfrak{g}$ has a basis consisting of simultaneous eigenvectors of $\{v_g\}_{g\in G}$. By our assumption $\mathcal{L}_u$ is a multiloop Lie torus over an algebraically closed field of characteristic 0, then we have $\mathfrak{g}_0$ is central simple (see §3.2 and §3.3 in [3] for details). Thus for a multiloop Lie torus $\mathcal{L}_u$, our construction $\mathcal{L}_{\widehat{u}}$ gives the universal central extension by Proposition 3.25 and the centre of $\mathcal{L}_{\widehat{u}}$ is Ω_R/dR by Proposition 2.9. □

Remark 3.28. Proposition 3.25 provides a good understanding of the universal central extensions of twisted forms corresponding to constant cocycles. The assumption that $\mathfrak{g}_0$ is central simple is crucial for our proof. As an important application, Corollary 3.27 provides a good understanding of the universal central extensions of twisted multiloop Lie tori. Recently E. Neher calculated the universal central extensions of twisted multiloop Lie algebras by using a result on a particular explicit description of the algebra of derivations of multiloop Lie algebras in [5] or [23]. Discovering more general conditions under which the descent construction gives the universal central extension remains an open problem.

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