ANNALS OF MATHEMATICAL LOGIC – Volume 2, No. 2 (1970) pp. 143–178

INTERNAL COHEN EXTENSIONS

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Received 2 December 1969

Introduction

Cohen [1, 2] has shown that the continuum hypothesis (CH) cannot be proved in Zermelo-Fraenkel set theory. Levy and Solovay [9] have subsequently shown that CH cannot be proved even if one assumes the existence of a measurable cardinal. Their argument in fact shows that no large cardinal axiom of the kind presently being considered by set theorists can yield a proof of CH (or of its negation, of course). Indeed, many set theorists – including the authors – suspect that CH is false. But if we reject CH we admit ourselves to be in a state of ignorance about a great many questions which CH resolves. While CH is a powerfull assertion, its negation is in many ways quite weak. Sierpinski [15] deduces propositions there called $C_1 - C_{82}$ from CH. We know of none of these propositions which is decided by the negation of CH and only one of them (C_{78}) which is decided if one assumes in addition that a measurable cardinal exists. Among the many simple questions easily decided by CH and which cannot be decided in ZF (Zermelo-Fraenkel set theory, *including the axiom of choice*) plus the negation of CH are the following: Is every set of real numbers of cardinality less than that of the continuum of Lebesgue measure zero? Is $2^{\aleph_0} < 2^{\aleph_1}$? Is there a non-trivial measure defined on all sets of real numbers? (This third question could be decided in ZF + not CH only in the unlikely event

[1] The second author received support from a Sloan Foundation fellowship and the National Science Foundation Grant (GP-8746).

that the existence of a measurable cardinal can be refuted in ZF.)

We are then very much in need of an alternative to CH. The aim of this paper is to study one such alternative. We introduce an "axiom" A which (1) is demonstrably consistent with ZF, (2) allows the continuum to be (loosely speaking) any regular cardinal, (3) follows from CH and implies many of the important consequences of CH, and (4) implies, when $2^{\aleph_0} > \aleph_1$, several interesting statements. The following theorem gives some of the main consequences of A. (For a statement of A, see § 1.2.)

Theorem. *If* A *then*

1) $2^{\aleph_0} > \aleph_1 \rightarrow$ *Souslin's hypothesis* [22];

2) *If* $\aleph$ *is an infinite cardinal number* $< 2^{\aleph_0}$, *then* $2^{\aleph} = 2^{\aleph_0}$;

3) *If* $2^{\aleph_0} > \aleph_1$, *every set of real numbers of cardinality* $\aleph_1$ *is* $\mathbf{\Pi}^1_1$ *if and only if every union of* $\aleph_1$ *Borel sets is* $\mathbf{\Sigma}^1_2$ *if and only if there is a real t with* $\aleph_1^{L[t]} = \aleph_1$;

4) *The union of* $< 2^{\aleph_0}$ *sets of reals of Lebesgue measure zero (respectively, of the first category) is of Lebesgue measure zero (of the first category)*;

5) *If* $2^{\aleph_0} > \aleph_1$, *every* $\mathbf{\Sigma}^1_2$ *set of reals is Lebesgue measurable and has the Baire property*;

6) $2^{\aleph_0}$ *is not a real valued measurable cardinal* (see also [8]).

The axiom arose from the consistency problem for Souslin's hypothesis. Souslin's hypothesis states that there are no "Souslin trees". Now if $\mathfrak{M}$ is a countable standard model of ZF and T is a Souslin tree in $\mathfrak{M}$, there is an easy method for finding a Cohen extension $\mathfrak{M}^T$ of $\mathfrak{M}$ such that $\mathfrak{M}^T$ has the same cardinals as $\mathfrak{M}$ and T is not a Souslin tree in any model $\mathfrak{N}$ of set theory with $\mathfrak{M}^T \subseteq \mathfrak{N}$. Solovay and Tennenbaum found a method for constructing a Cohen extension $\mathfrak{N}$ of any model $\mathfrak{M}_0$ of ZF with the property that, if T is a Souslin tree in some submodel $\mathfrak{M}$ of $\mathfrak{N}$, then some Cohen extension $\mathfrak{M}^T$ is a submodel of $\mathfrak{N}$ (so T is not a Souslin tree in $\mathfrak{N}$). That is, all the Souslin tree destroying Cohen extensions $\mathfrak{M} \rightarrow \mathfrak{M}^T$ can be carried out *inside* the model $\mathfrak{N}$. (This account is slightly inaccurate.)

Martin observed that the construction of $\mathfrak{N}$ depended only on very general properties of the Cohen extensions $\mathfrak{M} \rightarrow \mathfrak{M}^T$. He and, independently, Rowbottom, suggested an "axiom" which asserts that all Cohen extensions having these very general properties can be carried out inside the universe of sets: that the universe of sets is – so to speak – *closed* under a large class of Cohen extensions. The methods of [22] show this axiom to be consistent, and the consistency proof is given in [22].

The method of [22] is to construct a transfinite sequence $\mathfrak{M}_\alpha$, $\alpha < \theta$, of models, with $\mathfrak{M}_\beta$ a Cohen extension of $\mathfrak{M}_\alpha$ whenever $\alpha < \beta$. The "limit" $\mathfrak{N}$ of the $\mathfrak{M}_\alpha$ is the desired Cohen extension of $\mathfrak{M}_0$. Several consistency proofs have subsequently been found using this method of *iterated Cohen extensions.* Almost all of these consistency proofs can be simplified as follows: If Φ is the proposition to be shown consistent, one deduces Φ from A (or A + $2^{\aleph_0} > \aleph_1$) and concludes that Φ is consistent since A (A + $2^{\aleph_0} > \aleph_1$) is.

Although this paper is *about* forcing, almost the whole paper can be read without any knowledge of forcing. For the reader not familiar with forcing, § 1 will not be as enlightening, some of the theorems and proofs of § 2 will appear strange and ingenious, and various remarks made here and there in the paper will be unintelligible.

In § 1 we introduce the notion of a generic filter and state the axiom A. § 2 is devoted to two other versions of A: The Boolean algebraic version and a formulation in terms of ideals in the Borel sets of reals. To prove the equivalence of A and this latter version, we introduce the method of "almost disjoint sets", which is perhaps the main tool used in this paper. We assume in § 2 some facts about Boolean algebras, all of which can be found in Halmos [5] or Sikorski [16]. In § 3 we prove parts 2) and 3) of the theorem stated above. Some familiarity with projective sets is assumed in § 3.2. § 4 is concerned with parts 4) and 5) of the theorem. In § 5, we discuss the ways in which A is very close to the continuum hypothesis. We indicate how most consequences in [15] of CH can also be deduced from A (in particular, the non-existence of a real-valued measurable cardinal). (These topics are also discussed in [8].) Finally we consider the problem of the truth of A in light of Gödel's remarks [4] on the truth of CH.

This paper is complementary to [22], where our axiom is proved consistent and where Souslin's hypothesis is deduced from A + $2^{\aleph_0} > \aleph_1$. We have mostly tried to keep the same notation and terminology as [22], and we indicate our departures from [22]. Another study of the consequences of the axiom is § § 11–14 of Kunen's dissertation [8], which we recommend to the reader. Some other papers directly or indirectly related to the axiom are [23], [24], [25], and [26].

§ 1. The axiom

1.1. *$\mathcal{M}$-generic filters.* In using the forcing method of Cohen, one begins with a transitive standard model $\mathcal{M}$ of ZF and a partially ordered set $\mathcal{P}$ belonging to $\mathcal{M}$. If $p_1 \lesssim p_2$ we say that p_2 *extends* p_1. $p_1, p_2 \in \mathcal{P}$ are *compatible* if there is a $p_3 \in \mathcal{P}$ which extends them both; otherwise p_1 and p_2 are *incompatible*. A subset X of $\mathcal{P}$ is *dense open* if
1) $p \in X$, $q \in \mathcal{P}$, and $p \lesssim q \to q \in X$;
2) $p \in \mathcal{P} \to (\exists q \in X)(p \lesssim q)$.

The model $\mathcal{M}$ is usually assumed to be countable, and this guarantees the existence of an *$\mathcal{M}$-generic filter on $\mathcal{P}$*, a subset G of $\mathcal{P}$ satisfying
a) $p \in G$ and $q \lesssim p \to q \in G$;
b) $p_1, p_2 \in G \to (\exists p_3)(p_1 \lesssim p_3 \;\&\; p_2 \lesssim p_3 \;\&\; p_3 \in G)$;
c) $X \subseteq \mathcal{P}$ and $X \in \mathcal{M}$ and X dense open $\to X \cap G \neq \phi$,
where ϕ is the empty set. If G is an $\mathcal{M}$-generic filter on $\mathcal{P}$, there is a unique minimal model $\mathcal{M}[G]$ of ZF such that $\mathcal{M} \subseteq \mathcal{M}[G]$ and $G \in \mathcal{M}[G]$ and such that $\mathcal{M}[G]$ has the same ordinals as $\mathcal{M}$.

Remarks. What we call *dense open* is called *dense* in [22]. In § 2.1 we make a partial ordering $\mathcal{P}$ into a topological space. Condition 1) then says that X is an open subset of $\mathcal{P}$ and condition 2) that X is dense in the topological sense.

In [22], the weaker condition
b′) $p_1, p_2 \in G \to p_1$ and p_2 are compatible
appears instead of condition b). This change does not affect the notion of $\mathcal{M}$-generic filters. Indeed, if a), b′), and c) hold of G, then

$$X = \{p : (p_1 \lesssim p \;\&\; p_2 \lesssim p)$$

$$\text{or } (p \text{ is incompatible with } p_1 \text{ or } p_2)\}$$

is a dense open subset of $\mathcal{P}$ belonging to $\mathfrak{M}$. By c), let $p_3 \in X \cap G$. b′) guarantees that p_3 is compatible with p_1 and p_2, so p_3 extends them both. Our use of b) instead of b′) does change the notion of an $\mathcal{F}$-generic filter (§ 1.2) but has no effect on the propositions $\mathsf{A}_{\aleph}$ defined in § 1.2.

"Generic ideal" might be more descriptive than "generic filter". The word "filter" is used because a generic filter on $\mathcal{P}$ is associated with a filter in a related Boolean algebra. (See § 2.1.) Some authors reverse the extension relation in order to make $\preceq$ agree with the partial ordering in this Boolean algebra. We do not do this for historical agreement with Cohen [2] and because we, like Cohen, think of *p extends q* as meaning *p* has *more information* than *q*.

As a final remark, we note that if we replace c) by the condition that G meets every *dense* subset of $\mathcal{P}$, i.e., every subset of $\mathcal{P}$ satisfying 2), then a) implies that the notion of generic filter is unchanged.

Example 1 (essentially that of Cohen [2, Ch. 4, § 3]). Let $\mathfrak{M}$ be a countable standard model of ZF + V = L; let $\mathcal{P}$ be the set of finite functions p with domain$(p) \subseteq \omega$ and range$(p) \subseteq \{0, 1\}$; partially order $\mathcal{P}$ by inclusion. An $\mathfrak{M}$-generic filter on $\mathcal{P}$ is then just the set of finite subsets of the characteristic function of a subset of ω which is generic relative to $\mathfrak{M}$ in the sense of [2, Ch. IV].

When countable models $\mathfrak{M}$ are considered, the existence of $\mathfrak{M}$-generic filters is never a problem, for there are then only countably many dense open subsets of $\mathcal{P}$ which belong to $\mathfrak{M}$. (Let $X_1, X_2, \ldots$ be all these dense open subsets; let $p_0 \in \mathcal{P}$ be arbitrary and p_{n+1} be some extension of p_n belonging to X_n; $\{p : (\exists n)(p \preceq p_n)\}$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}$.) Suppose however that $\mathfrak{M}$ is uncountable or even that $\mathfrak{M}$ is a proper class. For instance consider:

Example 2. Let $\mathcal{P}$ be as in Example 1, but replace the $\mathfrak{M}$ of that example by the whole universe L of constructible sets. If G is an L-generic filter on $\mathcal{P}$, then $\cup G$ is a non-constructible function $f : \omega \to \{0,1\}$ as can easily be seen.

We cannot in general prove in ZF that $\mathfrak{M}$-generic filters exist. In Example 2, a proof that an L-generic filter exists would be a refutation of the axiom of constructibility (V = L), which is known [3] to be consistent with ZF.

Nevertheless it is not obviously false that in many instances $\mathcal{M}$-generic filters exist even though $\mathcal{M}$, or even the set of dense open subsets of $\mathcal{P}$ belonging to $\mathcal{M}$, is uncountable. Our axiom will say that this is indeed the case.

1.2. *$\mathcal{F}$-generic filters.* The model $\mathcal{M}$ is involved in the notion of an $\mathcal{M}$-generic filter on $\mathcal{P}$ only via the collection of dense open subsets of $\mathcal{P}$ belonging to $\mathcal{M}$. Accordingly we introduce a more general notion. If $\mathcal{P}$ is a partial ordering and $\mathcal{F}$ is a collection of dense open subsets of $\mathcal{P}$, an *$\mathcal{F}$-generic filter on $\mathcal{P}$* is a subset G of $\mathcal{P}$ satisfying a) and b) in the definition of $\mathcal{M}$-generic filters and

c′) $\qquad X \in \mathcal{F} \to X \cap G \neq \phi .$

If one looks for a proposition asserting the existence of $\mathcal{F}$-generic filters, one naturally thinks of the following: For every partial ordering $\mathcal{P}$ and every collection $\mathcal{F}$ of dense open subsets of $\mathcal{P}$, there is an $\mathcal{F}$-generic filter on $\mathcal{P}$. Now it is possible to accept this strong proposition, provided that one is willing to abandon the power set axiom of ZF. In ZF without the power set axiom, the proposition is equivalent to the assertion that every set is countable. To see that the proposition is inconsistent with ZF, let $\mathcal{P}$ be the set of finite functions p with domain$(p) \in \omega$ and range$(p) \subseteq \omega_1$. Partially order $\mathcal{P}$ by inclusion. For each countable ordinal α, let

$$X_\alpha = \{ p \in \mathcal{P} : \alpha \in \text{range}(p) \} .$$

Let $\mathcal{F} = \{ X_\alpha : \alpha < \aleph_1 \}$. (We always identify cardinals with initial ordinals.) Each X_α is dense open for if $p \in \mathcal{P}$, and n is the least natural number not in domain (p), then

$$p \preceq p \cup \{ \langle n, \alpha \rangle \} .$$

If G were an $\mathcal{F}$-generic filter, it is easy to see that $\cup G$ would be a function mapping ω onto ω_1.

Some restriction is required so that we do not assert the existence of generic filters which "collapse" cardinals in this way. We adopt a restriction on $\mathcal{P}$ to be described below. It is not the weakest restriction on $\mathcal{P}$

which will prevent cardinal collapse, but it has the virtue of being strong enough to permit the proof of Theorem 2 of this section.

An *antichain* in a partially ordered set $\mathcal{P}$ is a collection of elements of $\mathcal{P}$ any two distinct members of which are incompatible. $\mathcal{P}$ satisfies the *countable antichain condition* (the *cac*) if every antichain in $\mathcal{P}$ is countable.

For $\aleph$ an infinite cardinal, let $\mathsf{A}_\aleph$ be the assertion:

If $\mathcal{P}$ is a partial ordering satisfying the cac and $\mathcal{F}$ is a collection of dense open subsets of $\mathcal{P}$ of cardinality $\leq \aleph$, then there is an $\mathcal{F}$-generic filter on $\mathcal{P}$.

Our $\mathsf{A}_\aleph$ is equivalent to $\mathrm{MA}(\aleph^+)$ of [8] and $\mathrm{M}_{\aleph^+}$ of [22], where $\aleph^+$ is the least cardinal greater than $\aleph$. The equivalence between our $\mathsf{A}_\aleph$ and the $\mathrm{MA}(\aleph^+)$ of [8] will be proved in § 2.1. The $\mathrm{M}_{\aleph^+}$ of [22] has an extra restriction on $\mathcal{P}$: that $\mathcal{P}$ has cardinality $\leq \aleph$. This restriction has no importance:

Lemma. *Let $\mathcal{P}$ be a partial ordering and let $\mathcal{F}$ be a collection of dense open subsets of $\mathcal{P}$. There is a $\mathcal{P}' \subseteq \mathcal{P}$ of cardinality $\leq \max(\aleph_0, (\mathrm{card}\,(\mathcal{F}))$ such that, if $\mathcal{F}'$ is the collection of $X \cap \mathcal{P}'$ for $X \in \mathcal{F}$, then $\mathcal{F}'$ consists of dense open subsets of $\mathcal{P}'$ and any $\mathcal{F}'$-generic filter on $\mathcal{P}'$ can be extended to an $\mathcal{F}$-generic filter on $\mathcal{P}$.*

Proof. For each $X \in \mathcal{F}$, let $f_x : \mathcal{P} \to \mathcal{P}$ be a function such that $p \preceq f_x(p) \in X$. Let p_0 be some element of $\mathcal{P}$. Let $\mathcal{P}'$ be the closure of $\{p_0\}$ under the f_x for $X \in \mathcal{F}$. If $\mathcal{F}$ is infinite, clearly the cardinality of $\mathcal{P}'$ is no greater than that of $\mathcal{F}$. If $X \in \mathcal{F}$, $X \cap \mathcal{P}'$ is a dense open subset of $\mathcal{P}'$. Let G' be an $\mathcal{F}'$-generic filter on $\mathcal{P}'$. Let $G = \{p \in \mathcal{P} : (\exists p')(p' \in \mathcal{P}'$ and $p \preceq p')\}$. G is an $\mathcal{F}$-generic filter on $\mathcal{P}$: G clearly satisfies a) and c'). If $p_1, p_2 \in G$, let $p'_1, p'_2 \in G'$ with $p_1 \preceq p'_1$ and $p_2 \preceq p'_2$. Since p'_1 and p'_2 have an extension in G' so do p_1 and p_2.

As we have essentially remarked already, $\mathsf{A}_{\aleph_0}$ is a theorem of ZF.

Theorem 1. *If* $\mathsf{A}_\aleph$ *then* $\aleph < 2^{\aleph_0}$.

Proof. Let $\mathcal{P}$ be as in Examples 1 and 2. For each subset s of ω let X_s be the set of $p \in \mathcal{P}$ such that p is not a subset of the characteristic function of s. Each X_s is dense open. Let $\mathcal{F} = \{X_s : s \subseteq \omega\}$. If $\mathsf{A}_\aleph$ for some $\aleph \geq 2^{\aleph_0}$, then there is an $\mathcal{F}$-generic filter G on $\mathcal{P}$. But then $\bigcup G$ is a subset of the characteristic function of a subset of ω differing from every subset of ω.

Let A be the proposition.

If $\aleph < 2^{\aleph_0}$ *then* $\mathsf{A}_\aleph$.

A is the axiom we wish to study (though many of our results will concern the $\mathsf{A}_\aleph$*'s*).

Clearly A is consistent with ZF, for CH → A. In fact we have the following much stronger consistency result (the "forcing" version of the "Boolean" theorem 7.11 of [22]):

Theorem 2. *Let* $\mathfrak{M}$ *be a standard model of* ZF. *Let* θ *be an ordinal such that in* $\mathfrak{M}$ *the statement "*θ *is an uncountable regular cardinal and* $\theta' < \theta \to 2^{\theta'} \leq \theta$*" is true. There is a partially ordered set* $\mathcal{P} \in \mathfrak{M}$ *such that "*$\mathcal{P}$ *has cardinality* θ *and* $\mathcal{P}$ *satisfies the cac" is true in* $\mathfrak{M}$ *and such that, if* G *is any* $\mathfrak{M}$*-generic filter on* $\mathcal{P}$, $\mathfrak{M}[G]$ *satisfies* $2^{\aleph_0} = \theta$ and A.

We shall see in § 3 that the conditions on θ cannot be dropped: A implies that $2^{\aleph_0}$ is regular and in fact that $\aleph < 2^{\aleph_0} \to 2^\aleph = 2^{\aleph_0}$.

§ 2. Propositions equivalent to A

In § 2.1 we prove the equivalence of A and its Boolean version. The rest of § 2 will be devoted in one way or another to a proposition A* which is also equivalent to A. The equivalence of A* and A is proved in § 2.4. In § 2.3 we prove a theorem about Boolean algebras which is the key fact in showing A* ⟷ A. In § 2.2 we introduce the main ideas of § 2.3 and use them to prove a consequence of A which will be used several times in this paper. In § 2.5 we use § 2.2 to study two propositions related to A*.

2.1. *The Boolean version.* The axiom A is stated in terms of forcing. In view of the general correspondence between Boolean algebras and forcing [13], there should be a translation of A into the Boolean language. We now give such a translation and recall enough of [13] to prove its equivalence to A. By $A'_{\aleph}$ we mean the following proposition:

Let $\mathcal{B}$ be a complete Boolean algebra satisfying the countable chain condition (ccc) and let $b_{i\alpha}$ be elements of $\mathcal{B}$ for all $i < \omega$ and all ordinals $\alpha < \aleph$. There is a homomorphism $h: \mathcal{B} \to \{\mathbf{0}, \mathbf{1}\}$ (the two element Boolean algebra) such that, for each $\alpha < \aleph$,

$$h\left(\sum_i b_{i\alpha}\right) = \sum_i h(b_{i\alpha}) .$$

A homomorphism preserving all infinite sums is impossible if $\mathcal{B}$ is atomless, but $A'_{\aleph}$ says that an h preserving any given $\aleph$ sums can be found. $A'_{\aleph}$ is $\mathrm{MA}(\aleph^+)$ of [8].

Theorem. $A_{\aleph}$ *and* $A'_{\aleph}$ *are equivalent.*

Proof. $A_{\aleph} \to A'_{\aleph}$. Let $\mathcal{B}$ and $b_{i\alpha}$ be as in the statement of $A'_{\aleph}$. With no loss of generality we may assume $\sum\limits_i b_{i\alpha} = \mathbf{1}$. For, if not, let $c_{0\alpha} = \mathbf{1} - \sum\limits_i b_{i\alpha}$ and $c_{i+1\alpha} = b_{i\alpha}$. If $h : \mathcal{B} \to \{\mathbf{0}, \mathbf{1}\}$ is a homomorphism with $\sum\limits_i h(c_{i\alpha}) = h(\sum\limits_i c_{i\alpha}) = h(\mathbf{1}) = \mathbf{1}$, then either $h(\mathbf{1} - \sum\limits_i b_{i\alpha}) = \mathbf{1}$ and so $h(\sum\limits_i b_{i\alpha}) = \mathbf{0} = \sum\limits_i h(b_{i\alpha})$ or else $h(b_{i\alpha}) = \mathbf{1}$ for some i and so $h(\sum\limits_i b_{i\alpha}) = \sum\limits_i h(b_{i\alpha}) = \mathbf{1}$.

Let $\mathcal{P} = \mathcal{B} - \{\mathbf{0}\}$. If $b_1, b_2 \in \mathcal{P}$, let $b_1 \preceq b_2$ if $b_2 \leq b_1$ where $\leq$ is the Boolean algebraic relation. If $b_1 \cdot b_2 \neq \mathbf{0}$, then $b_1 \cdot b_2 \succeq b_1$ and $b_1 \cdot b_2 \succeq b_2$. In other words, b_1 and b_2 are compatible if they are not disjoint. Since $\mathcal{B}$ satisfies the *ccc*, $\mathcal{P}$ satisfies the *cac*.

For $\alpha < \aleph$ let $X_\alpha = \{b \in \mathcal{P} : (\exists i)(b \leq b_{i\alpha})\}$. Since $\sum\limits_i b_{i\alpha} = \mathbf{1}$, X_α is dense open. Let $\mathcal{F} = \{X_\alpha : \alpha < \aleph\}$.

By $A_{\aleph}$ let G be an $\mathcal{F}$-generic filter on $\mathcal{P}$. Let $h : \mathcal{B} \to \{\mathbf{0}, \mathbf{1}\}$ be defined by $h(b) = \mathbf{1} \leftrightarrow b \in G$. By a) and b) of 1.1, G is a Boolean filter in $\mathcal{B}$, so that h is a homomorphism. Let $\alpha < \aleph$. G is $\mathcal{F}$-generic, so let

$b \in X_\alpha \cap G$. There is an i such that $b \leq b_{i\alpha}$. $h(b_{i\alpha}) \geq h(b) = \mathbf{1}$. Hence $\sum_i h(b_{i\alpha}) = \mathbf{1}$.

$\mathsf{A}'_{\aleph} \to \mathsf{A}_{\aleph}$. Let $\mathcal{P}$ be a partially ordered set. We define the *complete Boolean algebra* $\mathcal{B}_{\mathcal{P}}$ *associated with* $\mathcal{P}$.

For $p \in \mathcal{P}$, let $O_p = \{q \in \mathcal{P} : p \preceq q\}$. We can make $\mathcal{P}$ into a topological space by taking the O_p as a base for the open sets, for

$$O_{p_1} \cap O_{p_2} = \bigcup \{O_{p_3} : p_1 \preceq p_3 \ \&\ p_2 \preceq p_3\} \ .$$

Note that the term "dense open" is unambiguous. Let $\mathcal{B}_1$ be the Boolean algebra generated by the open sets. Let I be the ideal of sets whose complements are dense open. Let $\mathcal{B}_{\mathcal{P}} = \mathcal{B}_1/I$.

If $X \in \mathcal{B}_1$ let $[X]$ be the image of X in $\mathcal{B}_{\mathcal{P}}$. Every element of $\mathcal{B}_{\mathcal{P}}$ is of the form $[U]$ for some open U: Since this property is obviously preserved under sums, it is enough to show that it is preserved under complements. If $[U] \in \mathcal{B}_{\mathcal{P}}$, U open, let U' be the interior of $\mathcal{P} - U$. $U \cup U'$ is dense open, and U' and $\mathcal{P} - U$ are equal off the complement of $U \cup U'$. Therefore $[\mathcal{P} - U] = [U']$.

In [22, § 7.5] it is shown that $\mathcal{B}_{\mathcal{P}}$ is complete and that satisfies the *ccc* if $\mathcal{P}$ satisfies the *cac*.

Now suppose $\mathcal{P}$ is a partial ordering satisfying the *cac*, card $(\mathcal{P}) \leq \aleph$, $\mathcal{F} = \{X_\alpha ; \alpha < \aleph\}$ is a collection of dense open subsets of $\mathcal{P}$, and $\mathcal{B}_{\mathcal{P}}$ is the complete Boolean algebra associated with $\mathcal{P}$. For each $\alpha < \aleph$, let $\{p_{i\alpha}; i < \omega\}$ be a maximal antichain in X_α. Let $b_{i\alpha} = [O_{p_{i\alpha}}]$.

Let us compute $\sum_i b_{i\alpha}$. Suppose $U \subseteq \mathcal{P}$ is open and $[U] \geq \sum_i b_{i\alpha}$. Let $p \in \mathcal{P}$. There is an $i < \omega$ such that p and $p_{i\alpha}$ are compatible, since $\{p_{i\alpha}; i < \omega\}$ is a maximal antichain. Let $q \succeq p$ and $q \succeq p_{i\alpha}$. Then $O_q \subseteq O_p \cap O_{p_{i\alpha}}$. Hence $[O_q] \leq [O_{p_{i\alpha}}] = b_{i\alpha} \leq [U]$. Hence $U \cap O_q \neq \phi$ and so $U \cap O_p \neq \phi$. Since p was arbitrary, U is dense, i.e., $[U] = \mathbf{1}$. Thus $\sum_i b_{i\alpha} = \mathbf{1}$.

By $\mathsf{A}'_{\aleph}$, let $h : \mathcal{B}_{\mathcal{P}} \to \{\mathbf{0}, \mathbf{1}\}$ be a homomorphism such that $\sum_i h(b_{i\alpha}) = h(\sum_i b_{i\alpha}) = \mathbf{1}$ for each $\alpha < \aleph$ and $h([O_{p_1}] \cdot [O_{p_2}]) = \sum_{p_1, p_2 \preceq p_3} h([O_{p_3}])$ for $p_1, p_2 \in \mathcal{P}$. Let $G = \{p \in \mathcal{P} : h([O_p]) = \mathbf{1}\}$.

Since h is a homomorphism, it is readily seen that G satisfies a) and b) of 1.1. Let $\alpha < \aleph$. $\sum_i h(b_{i\alpha}) = \mathbf{1}$; so $h(b_{i\alpha}) = \mathbf{1}$ for some i. Hence $h([O_{p_{i\alpha}}]) = \mathbf{1}$ and $p_{i\alpha} \in G$. $p_{i\alpha} \in X_\alpha$ so $X_\alpha \cap G$ is nonempty. This means G satisfies c′) of § 1.2, and thus that G is $\mathcal{F}$-generic.

2.2. *Almost disjoint sets and the proposition* $\mathbf{S}_\aleph$. The method of this section was invented by Solovay in order to prove the consistency of "Every subset of $\aleph_1$ is constructible from a subset of ω and $2^{\aleph_0} > \aleph_1$" (§ 3.1). Among the theorems proved by this method are those of [6].

Let A be a collection of infinite subsets of ω. Let $\mathcal{P}_A$ be the set of all ordered pairs $\langle k, K\rangle$ with k a finite subset of ω and K a finite subset of A. We partially order $\mathcal{P}_A$ as follows:

$$\langle k_1, K_1\rangle \preceq \langle k_2, K_2\rangle \longleftrightarrow (k_1 \subseteq k_2 \;\&\; K_1 \subseteq K_2$$

$$\& \; k_2 \cap (\bigcup K_1) \subseteq k_1) .$$

Let $C_{\langle k,K\rangle}$ be the set of subsets t of ω such that $k \subseteq t$ and, for all $s \in K$, $s \cap t \subseteq k$. Then $\langle k_1, K_1\rangle \preceq \langle k_2, K_2\rangle$ if and only if $C_{\langle k_1,K_1\rangle} \supseteq C_{\langle k_2,K_2\rangle}$.

Lemma 1. $\mathcal{P}_A$ *satisfies the cac.*

Proof. $\langle k, K_1\rangle$ and $\langle k, K_2\rangle$ are always compatible, since $\langle k, K_1 \cup K_2\rangle$ extends them both. Since there are only countably many finite subsets of ω, the lemma is proved.

With each $x \subseteq \omega$ we now associate an $s_x \subseteq \omega$. Let $f_x : \omega \to \{0, 1\}$ be the chracteristic function of x. If $f : \omega \to \{0, 1\}$, $\bar{f}$ is defined by

$$\bar{f}(n) = \prod_{i=0}^{n-1} p_i^{f(i)+1},$$ where p_i is the $i + 1$st prime number. $\bar{f}(n)$ should be thought of as the finite sequence $f(0), f(1), \ldots, f(n-1)$. Now let $s_x = \{\bar{f}_x(n); n < \omega\}$. Note that s_x is always infinite.

Two subsets of ω are *almost disjoint* if their intersection is finite. Let $x, y \subseteq \omega$ and $x \neq y$. Then there is an $n \in \omega$ such that $n \in x \longleftrightarrow n \notin y$. If $m > n$, $\bar{f}_x(m) \notin s_y$ and $\bar{f}_y(m) \notin s_x$. Hence x and y are almost disjoint. In particular, we have shown:

Lemma 2. *There is a collection of infinite pairwise almost disjoint subsets of ω of cardinality $2^{\aleph_0}$.*

The following easy lemma is needed for the theorem of this section:

Lemma 3. *Let A be a set of subsets of ω. Let $t \subseteq \omega$ be such that for every finite subset K of A, $t - \bigcup K$ is infinite. For each $n \in \omega$, the set $X_{t,n}$ of $\langle k, K\rangle \in \mathcal{P}_A$ such that $k \cap t$ has cardinality $\geq n$ is dense open.*

Proof. Let $\langle k, K\rangle \in \mathcal{P}_A$. Since $t - (\bigcup K)$ is infinite, there is a subset k_1 of t of cardinality n disjoint from $\bigcup K$. Thus $\langle k, K\rangle \preceq \langle k \cup k_1, K\rangle$ and $\langle k \cup k_1, K\rangle \in X_{t,n}$.

By $\mathbf{S}_\aleph$ we mean the following proposition:

Let A and B be collections of subsets of ω, each of cardinality $\leq \aleph$, such that if $t \in B$ and K is a finite subset of A then $t - \bigcup K$ is infinite. There is a subset t_0 of ω such that $x \cap t_0$ is finite if $x \in A$ and infinite if $x \in B$.

Note that the hypothesis of $\mathbf{S}_\aleph$ is fulfilled if each member of B is infinite and almost disjoint from each member of A.

Theorem. $\mathbf{A}_\aleph \to \mathbf{S}_\aleph$.

Proof. Let A and B satisfy the hypothesis of $\mathbf{S}_\aleph$. Consider $\mathcal{P}_A$. For $s \subseteq \omega$, let Y_s be the set of $\langle k, K\rangle$ such that $s \in K$. Obviously Y_s is dense in $\mathcal{P}_A$ if $s \in A$. Define $X_{s,n}$ as in Lemma 3. Let

$$\mathcal{F} = \{ Y_s : s \in A \} \cup \{ X_{s,n} : s \in B \;\&\; n \in \omega \} .$$

By Lemma 3, $\mathcal{F}$ is a collection of dense open subsets of $\mathcal{P}_A$. By $\mathbf{A}_\aleph$ let G be an $\mathcal{F}$-generic filter on $\mathcal{P}_A$. Let

$$t_0 = \{ n : (\exists \langle k, K\rangle)(\langle k, K\rangle \in G \;\&\; n \in k)\} .$$

Let $s \in A$. Since $Y_s \in \mathcal{F}$, let $\langle k, K\rangle \in G$ with $s \in K$. Let $\langle k', K'\rangle \in G$. Then $\langle k, K\rangle$ and $\langle k', K'\rangle$ are compatible. Let $\langle k_1, K_1\rangle$ extend both. Since $\langle k, K\rangle \preceq \langle k_1, K_1\rangle$ we have by definition that $k_1 \cap s \subseteq k$. Hence $k' \cap s \subseteq k_1 \cap s \subseteq k$. Since k' was arbitrary, $t_0 \cap s \subseteq k$.

We have only to show that $s \in B \to t_0 \cap s$ is infinite. Let $s \in B$ and $n \in \omega$. We show that $t_0 \cap s$ has cardinality $\geq n$. Let $\langle k, K\rangle \in X_{s,n} \cap G$. $k \subseteq t_0$ and $k \cap s$ has cardinality $\geq n$, by the definition of $X_{s,n}$.

2.3. *An embedding theorem for Boolean algebras with the ccc.* Kripke [7] shows that every complete Boolean algebra can be embedded as a complete subalgebra in a countably generated complete Boolean algebra. In this paper we are concerned only with complete Boolean algebras satisfying the *ccc*. Can all such algebras be embedded as a complete subalgebra in a countably generated complete Boolean algebra satisfying the *ccc*? The answer is no, since it is readily seen that every countably generated complete Boolean algebra satisfying the *ccc* has cardinality $\leq 2^{\aleph_0}$.

Theorem. *Every complete Boolean algebra of cardinality $\leq 2^{\aleph_0}$ satisfying the ccc can be embedded as a complete subalgebra in a countably generated complete Boolean algebra satisfying the ccc.*

Proof. Our proof, like that of Kripke [7] and the proof of Solovay [18] on which it is based, is motivated by forcing. To indicate the motivation, suppose that $\mathcal{M}$ is a countable standard model of ZF, $\mathcal{P} \in \mathcal{M}$ is a partially ordered set of cardinality $\leq 2^{\aleph_0}$ in $\mathcal{M}$ satisfying the *cac*, and G is an $\mathcal{M}$-generic filter on $\mathcal{P}$. The Theorem of § 2.2 tells us how to find a *cac* Cohen extension $(\mathcal{M}[G])[t_0]$ of $\mathcal{M}[G]$ such that $G \in \mathcal{M}[t_0]$ (i.e., $\mathcal{M}[t_0] = (\mathcal{M}[G])[t_0]$) and $t_0 \subseteq \omega$. Results of [22] tell us that the composition of two *cac* Cohen extensions is a *cac* Cohen extension. Since $(\mathcal{M}[G])[t_0] = \mathcal{M}[t_0]$, we know that the Boolean algebra associated with this two stage extension is countably generated.

For the proof of the theorem, let $\mathcal{B}$ be a complete Boolean algebra of cardinality $\leq 2^{\aleph_0}$ satisfying the *ccc*. By Lemma 2 of § 2.2 let f map $\mathcal{B}$ one-one onto a collection of infinite pairwise almost disjoint subsets of ω. For $b \in \mathcal{B} - \{\mathbf{0}\}$, let $A(b) = \{f(b') : b' \geq b\}$. Let $\mathcal{P}$ be the set of all ordered triples $\langle b, k, K\rangle$, where $b \in \mathcal{B} - \{\mathbf{0}\}$ and $\langle k, K\rangle \in \mathcal{P}_{A(b)}$. Let

$$\langle b_1, k_1, K_1\rangle \preceq \langle b_2, k_2, K_2\rangle \longleftrightarrow$$

$$\longleftrightarrow b_1 \geq b_2 \;\&\; \langle k_1, K_1\rangle \preceq \langle k_2, K_2\rangle .$$

Lemma 1. *$\mathcal{P}$ satisfies the cac, and so $\mathcal{B}_{\mathcal{P}}$ (see § 2.1) satisfies the ccc.*

Proof. Suppose $b_1 \cdot b_2 \neq \mathbf{0}$. Then $\langle b_1, k, K_1\rangle$ and $\langle b_2, k, K_2\rangle$ are compatible, since $\langle b_1 \cdot b_2, k, K_1 \cup K_2\rangle$ extends both. If there were an uncountable antichain in $\mathcal{P}$, there would be one all of whose members had a fixed k. But this would give us an uncountable set of pairwise disjoint elements of $\mathcal{B}$.

Lemma 2. *$\mathcal{B}_{\mathcal{P}}$ is countably generated.*

Proof. For $n < \omega$, let $p_n = \langle \mathbf{1}_{B}, \{n\}, \phi\rangle$. Let $a_n = [O_{p_n}]$. It is enough to prove, for each $p \in \mathcal{P}$, that $[O_p]$ belongs to the complete subalgebra generated by the a_n, since the $[O_p]$ generate $\mathcal{B}_{\mathcal{P}}$.

We show that

$$[O_{\langle b,k,K\rangle}] = \Big(\prod_{n\in k} a_n\Big)\cdot\Big(\sum_{\substack{f(b)\supseteq t\\ \text{finite}}} \prod_{\substack{n\in t\cup(\bigcup K)\\ n\notin k}} (\mathbf{1} - a_n)\Big).$$

First we prove that $[O_{\langle b,k,K\rangle}] \leq \prod_{n\in k} a_n$. If $n \in k$, $\langle b, k, K\rangle$ is an extension of $\langle \mathbf{1}_{B}, \{n\}, \phi\rangle$ and so $O_{\langle b,k,K\rangle} \subseteq O_{\langle \mathbf{1}_{B}, \{n\}, \phi\rangle} = O_{p_n}$ and hence $[O_{\langle b,k,K\rangle}] \leq a_n$.

We next show that $[O_{\langle b,k,K\rangle}] \leq \sum_{\substack{f(b)\supseteq t\\ \text{finite}}} \prod_{\substack{n\in t\cup(\bigcup K)\\ n\notin k}} (\mathbf{1} - a_n)$. Call the right hand side of this inequality c. It is enough to show that $\{p \in \mathcal{P} : [O_p] \cdot [O_{\langle b,k,K\rangle}] = \mathbf{0} \text{ or } [O_p] \leq c\}$ is dense in $\mathcal{P}$, by the definition of $\mathcal{B}_{\mathcal{P}}$. Let $\langle b_1, k_1, K_1\rangle \in \mathcal{P}$. If $\langle b_1, k_1, K_1\rangle$ and $\langle b, k, K\rangle$ are incompatible, the $[O_{\langle b_1,k_1,K_1\rangle}] \cdot [O_{\langle b,k,K\rangle}] = \mathbf{0}$. Otherwise let $\langle b_2, k_2, K_2\rangle$ be an extension of both. $\langle b_2, k_2, K_2\rangle \preceq \langle b_2, k_2, K_2 \cup \{f(b)\}\rangle$ so we only need to show that $[O_{\langle b_2,k_2,K_2\cup\{f(b)\}\rangle}] \leq c$. Let $t = f(b) - k_2$. Let $n \in t \cup (\cup K)$ and $n \notin k$. We must prove that $[O_{\langle b_2,k_2,K_2\cup\{f(b)\}\rangle}] \cdot a_n = \mathbf{0}$, that is, that $\langle b_2, k_2, K_2 \cup \{f(b)\}\rangle$ and $\langle \mathbf{1}_{B}, \{n\}, \phi\rangle$ are incompatible. Suppose $\langle b_3, k_3, K_3\rangle$ extends both of them. If $n \in t$, then $n \in k_3 - k_2$ and $n \in f(b)$ which contradicts $\langle b_2, k_2, K_2 \cup \{f(b)\}\rangle \preceq \langle b_3, k_3, K_3\rangle$. Otherwise $n \in (\bigcup K) - k$, which contradicts $\langle b, k, K\rangle \preceq \langle b_3, k_3, K_3\rangle$.

Finally we show that $[O_{\langle b,k,K\rangle}] \geq (\prod_{n\in k} a_n)\cdot c$. To do this, we prove that, for each $t \subseteq \omega$ such that $f(b) - t$ is finite,

$$\{p \in \mathcal{P}: [O_p] \leq [O_{\langle b,k,K\rangle}] \quad \text{or}$$

$$[O_p] \cdot (\prod_{n \in k} a_n) \cdot (\prod_{\substack{n \in t \cup (\bigcup K) \\ n \notin k}} (\mathbf{1} - a_n)) = \mathbf{0}\}$$

is dense. Let $p \in \mathcal{P}$. If p and $\langle b, k, K\rangle$ are compatible, there is a $p' \geq p$ with $[O_{p'}] \leq [O_{\langle b,k,K\rangle}]$. Assume they are incompatible. Let $p = \langle b_1, k_1, K_1\rangle$.

Case 1. $b_1 \cdot b = \mathbf{0}_{\mathcal{B}}$. Then $(\bigcup K_1) \cap f(b)$ is finite by almost disjointness. Let $n \in t - (\bigcup K_1)$ and $n \notin k$. $\langle b_1, k_1 \cup \{n\}, K_1\rangle$ extends p and $\langle \mathbf{1}_{\mathcal{B}}, \{n\}, \phi\rangle$ so $[O_{\langle b_1, k_1 \cup \{n\}, K_1\rangle}] \cdot (\mathbf{1} - a_n) = \mathbf{0}$. Since $n \in t$ and $n \notin k$, we are done.

Case 2. $b_1 \cdot b \neq \mathbf{0}_{\mathcal{B}}$. Since $\langle b_1 \cdot b, k_1 \cup k, K_1 \cup K\rangle$ is not an extension of both p and $\langle b, k, K\rangle$, either there is an $n \in k_1 - k$ such that $n \in \bigcup K$ or there is an $n \in k - k_1$ such that $n \in \bigcup K_1$. In the first case, $[O_p] \leq a_n$ for an $n \in (\bigcup K) - k$. In the second case $[O_p] \cdot a_n = \mathbf{0}$ for an $n \in k$.

Lemma 3. *$\mathcal{B}$ can be embedded in $\mathcal{B}_{\mathcal{P}}$ as a complete subalgebra.*

Proof. Let $h : \mathcal{B} \to \mathcal{B}_{\mathcal{P}}$ be defined by $h(\mathbf{0}_{\mathcal{B}}) = \mathbf{0}$ and $h(b) = [O_{\langle b,\phi,\phi\rangle}]$ otherwise. The proof that h is a complete monomorphism is routine, so we omit it.

2.4. *The proposition* $\mathsf{A}^*_{\aleph}$. Recall that a σ-ideal I in a Boolean σ-algebra $\mathcal{B}$ is $\aleph_1$-*saturated* if every uncountable collection of disjoint elements of $\mathcal{B}$ meets I (in other words, if $\mathcal{B}/I$ satisfies the *ccc*). By $\mathsf{A}^*_{\aleph}$ we mean the following assertion.

If I is an $\aleph_1$-saturated σ-ideal in the Borel subsets of the real line $\mathcal{R}$ with $\mathcal{R} \notin I$, then $\mathcal{R}$ is not the union of $\aleph$ members of I.

Example. Let I be the set of Borel sets of Lebesgue measure zero. Then $\mathsf{A}^*_{\aleph}$ says that $\mathcal{R}$ is not the union of $\aleph$ sets of measure zero.

It is often convenient to consider a trivial variant of $\mathsf{A}^*_{\aleph}$. Give $\{0, 1\} = 2$ the discrete topology and 2^ω the product topology. Let $\mathcal{B}_0$

be the Borel subsets of 2^ω. Then $\mathsf{A}_\aleph^*$ is equivalent to the assertion:

If I is an $\aleph_1$-saturated σ-ideal in $\mathcal{B}_0$ with $2^\omega \notin I$, then 2^ω is not the union of $\aleph$ elements of I.

Theorem. $\mathsf{A}_\aleph^*$ *is equivalent to* $\mathsf{A}_\aleph$.

Proof. We show $\mathsf{A}_\aleph^*$ equivalent to $\mathsf{A}'_\aleph$.

Lemma 1. $\mathsf{A}'_\aleph \to \mathsf{A}_\aleph^*$.

Proof. Let $\mathcal{S}$ be a collection of (type ω) sequences of Borel subsets of 2^ω. We say that a subset of 2^ω is *$\mathcal{S}$-Borel* if it belongs to the smallest family $\mathcal{F}$ of subsets of 2^ω with the following three properties:
1) For each n, $\{g : g(n) = 1\} \in \mathcal{F}$.
2) If $A \in \mathcal{F}$, $2^\omega - A \in \mathcal{F}$.
3) If $\{A_n : n \in \omega\}$ is a sequence of sets in $\mathcal{F}$ and $\{A_n\} \in \mathcal{S}$, then $\bigcup_{n\in\omega} A_n$ belongs to $\mathcal{F}$. Clearly each Borel set is $\mathcal{S}$-Borel for some countable $\mathcal{S}$, since the family of sets with the latter property is a σ-algebra.

Now let I be an $\aleph_1$-saturated σ-ideal in $\mathcal{B}_0$. $\mathcal{B}_0/I$ is a complete Boolean algebra [5]. Let $A_\alpha \in I$ for $\alpha < \aleph$. For each $\alpha < \aleph$ let $\mathcal{S}_\alpha$ be a countable set of sequences such that A_α is $\mathcal{S}_\alpha$-Borel. Let

$$\mathcal{S} = \bigcup_{\alpha<\aleph} \mathcal{S}_\alpha .$$

By $\mathsf{A}'_\aleph$, let $h : \mathcal{B}_0/I \to \{\mathbf{0}, \mathbf{1}\}$ be a homomorphism such that, for each sequence $\{C_n\}$ in $\mathcal{S}$,

$$h(\sum_n [C_n]) = \sum_n ([C_n]),$$

where $[C_n]$ is the image of C_n in $\mathcal{B}_0/I$. Let $f \in 2^\omega$ be defined by

$$f(n) = 1 \leftrightarrow h([\{g : g(n) = 1\}]) = \mathbf{1} .$$

We shall prove that, for every $\mathcal{S}$-Borel set C,

$$f \in C \longleftrightarrow h([C]) = \mathbf{1} .$$

Since each A_α is $\mathcal{S}$-Borel and $\in I$, we will be done.

By induction, $f \in 2^\omega - C \leftrightarrow f \notin C \leftrightarrow h([C]) \neq \mathbf{1} \leftrightarrow h([2^\omega - C]) = \mathbf{1}$, and $f \in \bigcup_i C_i \longleftrightarrow (\exists i)(f \in C_i) \longleftrightarrow (\exists i)(h([C_i]) = \mathbf{1}) \longleftrightarrow h(\sum_i [C_i]) = \mathbf{1}$.

Lemma 2. $\mathsf{A}^*_\aleph \to \mathsf{A}'_\aleph$.

Proof. Let $\mathcal{B}$ be a complete Boolean algebra satisfying the *ccc* and let $b_{i\alpha}$, $i < \omega$, $\alpha < \aleph$ be elements of $\mathcal{B}$ with $\sum_i b_{i\alpha} = \mathbf{1}$ for all α. We find a homomorphism $h : \mathcal{B} \to \{ \mathbf{0}, \mathbf{1} \}$ with $\sum_i h(b_{i\alpha}) = \mathbf{1}$ for each $\alpha < \aleph$. Let $\mathcal{B}'$ be the complete subalgebra of $\mathcal{B}$ generated by the $b_{i\alpha}$. It is enough to find a homomorphism $h : \mathcal{B}' \to \{ \mathbf{0}, \mathbf{1} \}$ with the required properties, since such a homomorphism can always be extended to a homomorphism of $\mathcal{B}$ into $\{ \mathbf{0}, \mathbf{1} \}$.

We next observe that $\mathsf{A}^*_\aleph$ implies that $\aleph < 2^{\aleph_0}$. Otherwise $\mathcal{R}$ is the union of $\aleph$ points, i.e., of $\aleph$ members of the ideal of sets of Lebesgue measure zero.

Since $\mathcal{B}$ satisfies the *ccc*, $\mathcal{B}'$ is the σ-subalgebra of generated by the $b_{i\alpha}$. It follows that the cardinality of $\mathcal{B}'$ is $\leq \aleph^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. By the Theorem of 2.3 $\mathcal{B}'$ is a complete subalgebra of a countably generated Boolean algebra satisfying the *ccc*. By [16, p. 108] every countably generated complete Boolean algebra satisfying the *ccc* is isomorphic to $\mathcal{B}_0/I$ for some σ-ideal I in $\mathcal{B}_0$. It is thus clear that with no loss of generality we may assume that our original algebra $\mathcal{B}$ was of the form $\mathcal{B}_0/I$ for I an $\aleph_1$-saturated σ-ideal in $\mathcal{B}_0$.

For each $i < \omega$ and $\alpha < \aleph$ pick $C_{i\alpha} \in \mathcal{B}_0$ such that $[C_{i\alpha}]$, the image of $C_{i\alpha}$ in $\mathcal{B}_0/I$, is $b_{i\alpha}$. Make sure that $\bigcup_i C_{i\alpha} = 2^\omega$ for each α (this can be done, since $\sum_i b_{i\alpha} = \mathbf{1}$). Let $\mathcal{B}^*$ be the Boolean subalgebra of $\mathcal{B}_0$ generated by the $C_{i\alpha}$. Let $J = \mathcal{B}^* \cap I$. Since J has cardinality $\leq \aleph$, by $\mathsf{A}^*_\aleph$ let $f \in 2^\omega - \bigcup J$. Define $h_0 : \mathcal{B}^* \to \{ \mathbf{0}, \mathbf{1} \}$ by $h_0(C) = \mathbf{1}$ if and only if $f \in C$. h_0 is a homomorphism. Since $h_0(C) = \mathbf{0}$ for C in $\mathcal{B}^* \cap I$, we can extend

h_0 to a homomorphism $h: \mathcal{B}_0 \to \{\mathbf{0}, \mathbf{1}\}$ which induces a homomorphism h^*: $\mathcal{B}_0/I \to \{\mathbf{0}, \mathbf{1}\}$. Since for each $\alpha < \aleph$ $\bigcup_i C_{i\alpha} = 2^\omega$, there is for each α an i such that $h_0(C_{i\alpha}) = \mathbf{1}$. Thus $h^*([b_{i\alpha}]) = \mathbf{1}$ and $\sum_i h^*([b_{i\alpha}]) = \mathbf{1}$.

2.5. *Questions related to* $\mathrm{A}^*_\aleph$. We first note that we cannot drop from $\mathrm{A}^*_\aleph$ the restriction that I be $\aleph_1$-saturated. This is related to the fact that we cannot drop from $\mathrm{A}_\aleph$ the restriction that $\mathcal{P}$ satisfy the *cac*. Let $A_\alpha : \alpha < \aleph_1$ be disjoint non-empty Borel subsets of $\mathcal{R}$ with $\bigcup_{\alpha<\aleph_1} A_\alpha = \mathcal{R}$. (The Lebesgue decomposition of $\mathcal{R}$ supplies such A_α.) Let I be the ideal of Borel sets disjoint from all but countably many A_α. I is a σ-ideal. But each $A_\alpha \in I$ and $\mathcal{R} = \bigcup_{\alpha<\aleph_1} A_\alpha$.

A more interesting question concerns the additivity of ideals in the Borel sets. An ideal I in an algebra of sets is $\aleph$-additive if every union of fewer than $\aleph$ members of I is a subset of a member of I. Can $\mathrm{A}^*_\aleph$ be strengthened to: If I is an $\aleph_1$-saturated σ-ideal in $\mathcal{B}_0$, then I is $\aleph^+$-additive (where $\aleph^+$ is the least cardinal greater than $\aleph$)? The answer is once again no, if $\aleph > \aleph_0$.

Theorem 1. *There is an $\aleph_1$-saturated σ-ideal in $\mathcal{B}_0$ (the Borel subsets of 2^ω) which is not $\aleph_2$-additive.*

Proof. Let A be an uncountable collection of infinite, almost disjoint subsets of ω. Each element $\langle k, K\rangle$ of $\mathcal{P}_A$ is associated with an element $C_{\langle k,K\rangle}$ of $\mathcal{B}_0$ (see § 2.2).

Let I be the σ-ideal in $\mathcal{B}_0$ generated by sets of the form

$$2^\omega - \bigcup_{p\in\mathcal{D}} C_p$$

where $\mathcal{D}$ is a maximal antichain in $\mathcal{P}_A$. It is fairly easy to show that $\mathcal{B}_0/I$ is isomorphic to the complete Boolean algebra associated with $\mathcal{P}_A$. I is then $\aleph_1$-saturated.

For each $a \in A$, let

$$B_a = \{ f \in 2^\omega : a \cap \{ n : f(n) = 1 \} \text{ is infinite} \} .$$

Since $\{ \langle k, K \rangle : a \in K \}$ is dense in $\mathcal{P}_A$, $B_a \in I$. Let A^* be any uncountable subset of A. We show that $\bigcup_{a \in A^*} B_a$ is not a subset of any member of I. Since A^* can have cardinality $\aleph_1$, this proves that I is not $\aleph_2$-additive.

Let $\mathcal{D}_i$, $i < \omega$ be maximal antichains in $\mathcal{P}_A$. We must show that $\bigcap_i \bigcup_{p \in \mathcal{D}_i} C_p$ meets $\bigcup_{a \in A^*} B_a$. Let

$$D = \{ a \in A : (\exists i)(\exists \langle k, K \rangle)(\langle k, K \rangle \in \mathcal{D}_i \ \& \ a \in K) \} .$$

D is countable, so let $a \in A^* - D$. We show that $\bigcap_i \bigcup_{p \in \mathcal{D}_i} C_p$ meets B_a.

We define a sequence

$$p_0 \prec p_1 \prec p_2 \preceq \ldots$$

of elements of $\mathcal{P}_A$. Suppose $p_i = \langle k_i, K_i \rangle$ is defined for $i < n$ and suppose $\bigcup_{i<n} K_i \subseteq D$. Since a is almost disjoint from each member of D, let $m \in a - (k_{n-1} \cup (\bigcup K_{n-1}))$ if $n > 0$ and $m \in a$ if $n = 0$. Let

$$q_n = \langle k_{n-1} \cup \{ m \} , K_{n-1} \rangle \text{ if } n > 0 \text{ and}$$

$$q_n = \langle \{ m \} , \phi \rangle \text{ if } n = 0 .$$

Since $\mathcal{D}_n$ is a maximal antichain in $\mathcal{P}_A$, let $\langle k, K \rangle \in \mathcal{D}_n$ be compatible with q_n. Let

$$p_n = \langle k_{n-1} \cup \{ m \} \cup k, K_{n-1} \cup K \rangle .$$

Note that $K_n = K_{n-1} \cup K \subseteq D$. Let $f(n) = 1$ if $n \in \bigcup_i k_i$ and $f(n) = 0$ otherwise. Clearly $a \cap \bigcup_i k_i$ is infinite so $f \in B_a$. Since $f \in \bigcap_n C_{p_n}$ and $C_{p_n} \subseteq C_p$ for some $p \in \mathcal{D}_n$, $f \in \bigcap_i \bigcup_{p \in \mathcal{D}_i} C_p$.

Despite Theorem 1, we shall see in § 4 that the two most important $\aleph_1$-saturated σ-ideals in $\mathcal{B}_0$ are, in the presence of $\mathsf{A}_{\aleph}$, $\aleph^+$-additive.

Let $\mathcal{B}$ be an algebra of subsets of a set X containing all points (unit subsets). Call an ideal J in $\mathcal{B}$ *non-trivial* if $X \notin J$ and every point belongs to J. A subset A of X is a *null set* of an ideal J in $\mathcal{B}$ if A is a subset of a member of J.

Let I be a non-trivial $\aleph_1$-saturated σ-ideal in $\mathcal{B}_0$. Does A imply that every $A \subseteq 2^\omega$ of cardinality $< 2^{\aleph_0}$ is a null set of I? To answer this question, we first prove the following theorem.

Theorem 2. *If* A *then the following two assertions are equivalent.*
1) *There is a non-trivial $\aleph_1$-saturated σ-ideal I in $\mathcal{B}_0$ such that not every set of cardinality $< 2^{\aleph_0}$ is a null set of I.*
2) *There is an uncountable cardinal $\kappa < 2^{\aleph_0}$ with a non-trivial $\aleph_1$-saturated κ-additive ideal in $P(\kappa)$, the set of all subsets of κ.*

Proof. 2) → 1). (A is not used in this half of the proof.) Let $\aleph_0 < \kappa < < 2^{\aleph_0}$ and let A be a subset of 2^ω of cardinality κ. If there is a non-trivial $\aleph_1$-saturated σ-ideal in $P(\kappa)$ then there is a non-trivial $\aleph_1$-saturated σ-ideal J in $P(A)$. Let

$$I = \{ C \in \mathcal{B}_0 : C \cap A \in J \} .$$

I is a non-trivial $\aleph_1$-saturated σ-ideal in $\mathcal{B}_0$ since J is such an ideal in $P(A)$. Also A is not a null set of I.

If A, then 1) → 2). Suppose I is a non-trivial $\aleph_1$-saturated σ-ideal in $\mathcal{B}_0$. Let κ be the least cardinal such that some $A \subseteq 2^\omega$ of cardinality κ is not a null set of I. Suppose $\kappa < 2^{\aleph_0}$ and let $A \subseteq 2^\omega$ be a set of cardinality κ which is not a null set of I. Let

$$J = \{ C \subseteq A : C \text{ is a null set of } I \} .$$

We need the following lemma, due to Silver, which also answers (assuming $\mathsf{A} + 2^{\aleph_0} > \aleph_1$) a question of Sierpinski [15, p. 90].

Lemma (Silver). *If* A *and if $C \subseteq A$, where A is a subset of 2^ω of cardinality less than $2^{\aleph_0}$, then there is a $C^* \subseteq 2^\omega$ such that $C^* \cap A = C$ and C^* is a G_δ.*

Proof. By $\mathbf{S}_\kappa$ let $f \in 2^\omega$ be such that $\{n : f(n) = 1\} \cap s_a$ is infinite if a is a subset of ω with $f_a \in C \cap A$ and $\{n : f(n) = 1\} \cap s_a$ is finite if $f_a \in A - C$. (s_a and f_a were defined in § 2.2.) Let

$$C^* = \{f_a : s_a \cap \{n : f(n) = 1\} \text{ is infinite }\}.$$

C^* clearly has the required properties.

We show that J is a non-trivial $\aleph_1$-saturated σ-ideal in $P(A)$. J is a non-trivial σ-ideal, since I is. To see that J is $\aleph_1$-saturated, suppose C_α, $\alpha < \aleph_1$, are disjoint subsets of A. By the lemma, let C^*_α be Borel sets such that $C^*_\alpha \cap A = C_\alpha$. Let $D_\alpha = C^*_\alpha - \bigcup_{\beta<\alpha} C^*_\beta$. The D_β are disjoint. Since I is $\aleph_1$-saturated, some $D_\beta \in I$. Since the C_α are disjoint, $C_\beta = D_\beta \cap A \in J$.

It is easily seen that the least κ^* such that J is not $(\kappa^*)^+$-additive is such that $P(\kappa^*)$ bears a non-trivial κ^*-additive $\aleph_1$-saturated ideal. Indeed let A_α, $\alpha < \kappa^*$ be disjoint subsets of A such that each $A_\alpha \in J$ but $\bigcup_{\alpha<\kappa^*} A_\alpha \notin J$. Let J^* be the collection of subsets X of κ^* such that $\bigcup_{\alpha\in X} A_\alpha \in J$. J^* is the desired ideal.

We note that the existence of an uncountable $\kappa < 2^{\aleph_0}$ such that $P(\kappa)$ bears a non-trivial κ-additive $\aleph_1$-saturated ideal is consistent with ZF + A if and only if the existence of a $\{0, 1\}$-measurable cardinal is consistent with ZF. If $P(\kappa)$ bears such an ideal J, then κ is $\{0, 1\}$-measurable in $L[J]$. (See [19].) If $\mathcal{M}$ is a countable standard model of ZF + "There is a $\{0, 1\}$-measurable cardinal" + the generalized continuum hypothesis, there is by Theorem 2 of § 1.2 a *cac* Cohen extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}$ satisfies ZF + A and $2^{\aleph_0} > \kappa$ in $\mathcal{N}$. By a theorem of Prikry [12], $P(\kappa)$ bears a non-trivial $\aleph_1$-saturated κ-additive ideal in $\mathcal{N}$ (namely, the ideal generated by the sets of measure 0 in $\mathcal{M}$).

§ 3. The cardinal of the continuum and a hypothesis of Lusin

3.1. *Subsets of cardinals* $< 2^{\aleph_0}$. Lusin [10] propounded a hypothesis which we call **L** which implies $2^{\aleph_0} = 2^{\aleph_1}$. (This latter equation is known as *Lusin's continuum hypothesis.*) In § 3.2 we shall see that **L** is consistent with and independent of $\mathsf{A}_{\aleph_1}$. We now show, using the proposition $\mathbf{S}_{\aleph_1}$ of § 2.2, that the consequence $2^{\aleph_0} = 2^{\aleph_1}$ of **L** does follow from $\mathsf{A}_{\aleph_1}$.

Theorem 1. *If* $\mathsf{A}_{\aleph}$ *then* $2^{\aleph} = 2^{\aleph_0}$.

Proof. Let $\{s_\alpha : \alpha < \aleph\}$ be a set of infinite pairwise almost disjoint subsets of ω. Let $G : P(\aleph_0) \to P(\aleph)$ be defined by $G(t) = \{\alpha < \aleph : t \cap s_\alpha$ is infinite$\}$. By $\mathsf{S}_{\aleph}$, G is surjective.

Corollary 1. *If* A *and* $\aleph < 2^{\aleph_0}$, *then* $2^{\aleph} = 2^{\aleph_0}$.

Corollary 2. *If* A, *then* $2^{\aleph_0}$ *is regular.*

Proof. Otherwise $2^{\aleph_0}$ is cofinal with some $\aleph$ less than $2^{\aleph_0}$. By Konig's Theorem, $2^{\aleph}$ is not cofinal with $\aleph$. Since $2^{\aleph} = 2^{\aleph_0}$, we have a contradiction.

Theorem 2. *If* $\mathsf{A}_{\aleph}$, *there is a fixed subset* Y *of* $\aleph$ *such that every subset of* $\aleph$ *is constructible from* Y *together with some subset of* $\aleph_0$.

Proof. Let G be the function defined in the proof of Theorem 1. $G(t)$ is constructible from t and the sequence $\{s_\alpha\}$. Let Y be a subset of $\aleph$ coding this sequence.

Corollary 3. *If* $\mathsf{A}_{\aleph_1}$, *every subset of* $\aleph_1$ *is constructible from a subset of* ω *if and only if* $\aleph_1 = \aleph_1^{\mathrm{L}[t]}$ *for some* $t \subseteq \omega$.

Proof. Let $Y \subseteq \aleph_1$ code a sequence $s_\alpha : \alpha < \aleph_1$ of distinct subsets of ω. If Y is constructible from $t \subseteq \omega$, then each s_α is constructible from t and so, since by Godel [3] the continuum hypothesis holds in $\mathrm{L}[t]$, $\aleph_1 = \aleph_1^{\mathrm{L}[t]}$.

On the other hand, if $\aleph_1 = \aleph_1^{\mathrm{L}[t]}$ then a sequence of $\aleph_1$ almost disjoint subsets of ω is constructible from t, so $G(s)$ is constructible from s and t, where G is the function defined in the proof of Theorem 1.

3.2. *The hypothesis* **L**. For information about projective sets, see [14]. **L** is the assertion that every subset of $P(\omega)$ of cardinality $\aleph_1$ is $\mathbf{\Pi}^1_1$(**CA**). (L = *I* on page 129 of [10].)

Theorem. *If* $\mathsf{A}_{\aleph_1}$, *then* **L** *if and only if there is a* $t \subseteq \omega$ *with* $\aleph_1^{\mathsf{L}[t]} = \aleph_1$

Proof. Assume $t \subseteq \omega$ and $\aleph_1 = \aleph_1^{\mathsf{L}[t]}$. By a theorem essentially due to Godel (see [20]), it follows that there is a $\mathbf{\Pi}^1_1$ set A of cardinality $\aleph_1$. Let $A = \{a_\alpha : \alpha < \aleph_1\}$ and let $C = \{c_\alpha : \alpha < \aleph_1\}$ be any set of cardinality $\aleph_1$. If $x \subseteq \omega$ and $n \in \omega$, let $s_{x,n} = \{\bar{f}_x(m) : m$ is a power of the $n+1$st prime number$\}$. By $\mathbf{S}_\aleph$ let $t_0 \subseteq \omega$ be such that

$$t_0 \cap s_{c_\alpha, 2n+1} \text{ is finite } \longleftrightarrow n \in a_\alpha ,$$

$$t_0 \cap s_{a_\alpha, 2n+2} \text{ is finite } \longleftrightarrow n \in c_\alpha .$$

For each $x \subseteq \omega$, let $y_x = \{n : s_{x,2n+1} \cap t_0$ is finite$\}$ and $z_x = \{n : s_{x,2n+2} \cap t_0\}$ is finite. Then

$$x \in C \longleftrightarrow y_x \in A \text{ and } z_{y_x} = x .$$

Since A is $\mathbf{\Pi}^1_1$ so is C.

On the other hand, if there is no $t \subseteq \omega$ such that $\aleph_1^{\mathsf{L}[t]} = \aleph_1$, then it is a result of Solovay [20] and Mansfield that *no* set of cardinality $< 2^{\aleph_0}$ is $\mathbf{\Pi}^1_1$ (or even $\mathbf{\Sigma}^1_2$). Since $\mathsf{A}_{\aleph_1} \to \aleph_1 < 2^{\aleph_0}$, no set of cardinality $\aleph_1$ is $\mathbf{\Pi}^1_1$.

Lusin proposed in [10] another hypothesis which he considered, unlike **L**, to be only probable: Every union of $\aleph_1$ Borel sets is a projective set of the second class. Let **L**$'$ be the assertion: Every union of $\aleph_1$ Borel sets is $\mathbf{\Sigma}^1_2$. We note that **L**$'$ follows easily from **L** (in ZF). It suffices to show from **L** that the union of $\aleph_1$ $\mathbf{\Sigma}^1_2$ sets is $\mathbf{\Sigma}^1_2$. For this, it is enough to prove that any set of the form

$$\{ t : (\exists a)(a \in A \text{ and } \langle a, t\rangle \in C)\} ,$$

where A has cardinality $\aleph_1$ and $C \subseteq P(\omega) \times P(\omega)$ is $\mathbf{\Sigma}^1_2$. But this is true if A is $\mathbf{\Pi}^1_1$.

Corollary. *If* $\mathsf{A}_{\aleph_1}$, *then* $\mathbf{L}'$ *if and only if there is a* $t \subseteq \omega$ *such that* $\aleph_1^{\mathbf{L}[t]} = \aleph_1$.

Still another proposition (II of [10]) is mentioned and described as "certain" by Lusin. Solovay will show elsewhere that II implies that $\aleph_1$ is a $\{0, 1\}$-measurable cardinal, so that II contradicts the axiom of choice.

§4. Measure and category

4.1. *Lebesgue measure.* Let $\aleph < 2^{\aleph_0}$. If a set of reals has cardinality $\aleph$ and is Lebesgue measurable, it must have measure zero. But is every such set measurable? If so, is every union of $\aleph$ sets of measure zero measurable? Does every such union have measure zero? If the continuum hypothesis holds, the answer to all these questions is yes. We shall see momentarily that the weaker proposition A also yields affirmative answers. On the other hand, there are models of ZF in which $2^{\aleph_0} > \aleph_1$ and
(a) There is a set of cardinality $\aleph_1$ (namely, the set of constructible reals) which is not Lebesgue measurable;
(b) Every set of cardinality $< 2^{\aleph_0}$ has measure zero, but $\mathcal{R}$ is the union of $\aleph_1$ sets of measure zero.

Briefly, let $\mathcal{M}$ be a countable transitive standard model of ZF + V = L. Let α be a regular cardinal $> 2^{\aleph_0}$ in $\mathcal{M}$. In $\mathcal{M}$, give $\{0, 1\}$ the discrete topology and give $\{0\}$ and $\{1\}$ each measure $\frac{1}{2}$; give 2^{α} the product topology and the product measure. Let $\mathcal{B}$ be the Borel subsets of 2^{α} and let I_a and I_b be the ideals of measure zero Borel sets and of meager Borel sets respectively. (A set is *meager* if it is disjoint from an intersection of countably many dense open sets.) Let $\mathcal{P}_a$ be the non-zero elements of the Boolean algebra $\mathcal{B}/I_a$ and let $\mathcal{P}_b$ be the non-zero elements of $\mathcal{B}/I_b$. In each case define

$$b_1 \preceq b_2 \longleftrightarrow b_2 - b_1 = \mathbf{0}.$$

Let G_a and G_b be $\mathcal{M}$-generic filters on $\mathcal{P}_a$ and $\mathcal{P}_b$ respectively. Then $\mathcal{M}[G_a]$ and $\mathcal{M}[G_b]$ are models of (a) and (b) respectively. The proofs of these facts, which are – like the analogous ones cited in §4.2 – due to Solovay, are omitted.

Let I be the σ-ideal of Borel subsets of $\mathcal{R}$ of measure 0. Suppose $\{A_\alpha : \alpha < \aleph < 2^{\aleph_0}\}$ are sets of measure 0 and suppose that $\bigcup_{\alpha<\aleph} A_\alpha$ has positive inner measure. Let $A \subseteq \bigcup_{\alpha<\aleph} A_\alpha$ be a Borel set of positive Lebesgue measure. Let I' be the set of Borel sets C such that $C \cap A \in I$. $(\mathcal{R} - A) \cup \bigcup_{\alpha<\aleph} A_\alpha = \mathcal{R}$, so $\mathcal{R}$ is the union of $\aleph < 2^{\aleph_0}$ members of the $\aleph_1$-saturated σ-ideal I', which contradicts A. We have then shown that, *if* A, *then the union of* $< 2^{\aleph_0}$ *sets of measure* 0 *has inner measure* 0. This is all we get from a direct application of A to the ideal of sets of measure 0. We now prove a much stronger theorem by applying A to a different ideal.

Theorem 1. *If* $\mathsf{A}_\aleph$ *then the union of* $\aleph$ *sets of Lebesgue measure* 0 *has Lebesgue measure* 0.

Proof. Let A_α, $\alpha < \aleph$ be sets of measure 0. Let ϵ be a real number > 0. We show that $\bigcup_{\alpha<\aleph} A_\alpha$ has outer measure $\leq \epsilon$. Let $\mathcal{P}$ be the set of open subsets of $\mathcal{R}$ of measure $< \epsilon$. Partially order $\mathcal{P}$ by inclusion. We denote Lebesgue measure by μ during the rest of this section. For $\alpha < \aleph$, let $X_\alpha = \{p \in \mathcal{P}: A_\alpha \subseteq p\}$. We show that each X_α is dense. Let $p \in \mathcal{P}$. Since $\mu(A_\alpha) = 0$, there is an open q with $\mu(q) < \epsilon - \mu(p)$ and $q \supseteq A_\alpha$. Then $p \cup q \in X_\alpha$. Let $\mathcal{F} = \{X_\alpha : \alpha < \aleph\}$. Since $\mathcal{F}$ consists of dense open subsets of $\mathcal{P}$, if we can show that $\mathcal{P}$ satisfies the *cac*, then by $A_\aleph$ there is an $\mathcal{F}$-generic filter G on $\mathcal{P}$. Evidently $\bigcup G$ is an open set of reals and $\bigcup_{\alpha<\aleph} A_\alpha \subseteq \bigcup G$. If $\mu(\bigcup G) > \epsilon$ then there are $A_1, A_2, \ldots, A_n \in G$

with $\mu(\bigcup_1^n A_i) > \epsilon$. By repeated application of condition b) on $\mathcal{F}$-generic filters (§ 1), $\bigcup_1^n A_i \in G$, a contradiction.

Suppose $\mathcal{D}$ is an uncountable antichain in $\mathcal{P}$. There is a $\delta > 0$ such that $\mathcal{E} = \{ p \in \mathcal{D} : \mu(p) < \epsilon - \delta \}$ is uncountable. Since $\mathcal{R}$ is separable, let $\{ b_n : n < \omega \}$ be a base for the open sets of $\mathcal{R}$. For each $p \in \mathcal{E}$ let q_p be a finite union of basic open subsets of p such that $\mu(p - q_p) \leq \leq \delta/2$. $\{q_p : p \in \mathcal{E}\}$ is countable, since there are only countably many finite unions of basic open sets. If $p_1, p_2 \in \mathcal{E}$ and $p_1 \neq p_2$, then p_1 and p_2 are incompatible, so $\mu(p_1 \cup p_2) \geq \epsilon$. But $\mu(q_{p_1} \cup q_{p_2}) \geq \mu(p_1 \cup p_2) - (\delta/2) - (\delta/2) \geq \epsilon - \delta$. Since $\mu(q_{p_1}) \leq \mu(p_1) < \epsilon - \delta$, $q_{p_1} \neq q_{p_2}$. Therefore the countability of $\{q_p ; p \in \mathcal{E}\}$ implies the countability of $\mathcal{E}$.

Corollary 1. *If* A, (1) *the ideal of sets of Lebesgue measure* 0 *is* $2^{\aleph_0}$-*additive*; (2) *the σ-algebra of Lebesgue measurable sets is* $\aleph$-*complete for every* $\aleph < 2^{\aleph_0}$; *and* (3) *Lebesgue measure is* $2^{\aleph_0}$-*additive*.

Proof. Assume A. (1) is evident. Let A_α, $\alpha < \aleph < 2^{\aleph_0}$, be Lesbesgue measurable. Let $C \subseteq \bigcup_{\alpha<\aleph} A_\alpha$ be a Borel set such that $\bigcup_{\alpha<\aleph} A_\alpha - C$ has inner measure 0. For each α, $\mu(A_\alpha - C) = 0$, so $\mu(\bigcup_{\alpha<\aleph} (A_\alpha - C)) = 0$. Since $\bigcup_{\alpha<\aleph} A_\alpha - C = \bigcup_{\alpha<\aleph} (A_\alpha - C)$, $\bigcup_{\alpha<\aleph} A_\alpha - C$ has measure 0; hence $\bigcup_{\alpha<\aleph} A_\alpha$ is measurable, and (2) is proved. If A_α, $\alpha < \aleph < 2^{\aleph_0}$ are pairwise disjoint measurable sets, then only countably many of them, say A_{α_i}, $i < \omega$, can have positive measure. Hence

$$\mu(\bigcup_{\alpha<\aleph} A_\alpha) = \mu(\bigcup_{i<\omega} A_{\alpha_i}) + \mu(\bigcup_{\alpha<\aleph} A_\alpha - \bigcup_{i<\omega} A_{\alpha_i}) =$$

$$= \sum_{i<\omega} \mu(A_{\alpha_i}) = \sum_{\alpha<\aleph} \mu(A_\alpha),$$

and so we have (3).

Corollary 2. *If* $\mathsf{A}_{\aleph_1}$, *every* $\mathbf{\Sigma}^1_2$ (**PCA**) *set is Lebesgue measurable.*

Proof. Every $\mathbf{\Sigma}^1_2$ set is the union of $\aleph_1$ Borel sets.

It is a theorem of Gödel that the measurability of $\mathbf{\Sigma}^1_2$ sets cannot be proved in ZF alone. On the other hand, that every $\mathbf{\Sigma}^1_2$ set is measurable also follows from the existence of a measurable cardinal (Solovay). We now indicate why two axioms are better than one. The proof of the following theorem of Martin, which uses the methods of [11], will appear elsewhere.

Theorem 2. *If a measurable cardinal exists, every* $\mathbf{\Sigma}^1_3$ *set is the union of* $\aleph_2$ *Borel sets.*

Corollary 3. *If* $\mathsf{A}_{\aleph_2}$ *and there exists a measurable cardinal, every* $\mathbf{\Sigma}^1_3$ *set is Lebesgue measurable.*

We do not know whether the hypothesis "There exists a measurable cardinal" can be dropped from Corollary 3. We conjecture that it cannot. We do not know whether $\mathsf{A}_{\aleph_2}$ can be weakened to $\mathsf{A}_{\aleph_1}$. We conjecture that it cannot. We do know that the measurability of $\mathbf{\Sigma}^1_3$ sets does not follow from the existence of a measurable cardinal. This fact is due to Silver [17].

We close §4.1 with two remarks: (1) Theorem 1 also shows that $\mathsf{A} \to 2^{\aleph_0}$ is regular (Corollary 1 of the Theorem of §3.1); 2) Theorem 1 readily generalizes to the completion of any regular Borel measure in a separable space.

4.2. *The Baire categories.* Recall that a subset of $\mathcal{R}$ is *meager* (*first category*) if its complement contains an intersection of countably many dense open sets. A set is *comeager* if its complement is meager. The *Baire Category Theorem* says that the intersection of $\aleph_0$ dense open sets is dense. If we apply A directly to the σ-ideal of meager Borel sets we see that A implies a *Strong Baire Category Theorem*: *The intersection of* $< 2^{\aleph_0}$ *dense open sets is dense.* To see this, let A_α, $\alpha < \aleph <$ $< 2^{\aleph_0}$, be dense open and let A be open. Let I be the σ-ideal of Borel sets whose intersection with A is meager. I is $\aleph_1$-saturated [5]. Since each $\mathcal{R} - A_\alpha \in I$, A implies that $(\mathcal{R} - A) \cup \bigcup_{\alpha < \aleph} (\mathcal{R} - A_\alpha) \neq \mathcal{R}$. Hence there is a real in $A \cap \bigcap_{\alpha < \aleph} A_\alpha$.

A set of reals A has *the Baire property* if there is an open set U and a meager set N such that, for all $x \notin N$,

$$x \in A \leftrightarrow x \in U$$

(i.e., A equals U outside the meager set N). Every Borel set has the Baire property [5]. Questions about the Baire property corresponding to those asked about measurability at the beginning of § 4.1 can be raised, and one gets the corresponding answers. There are models of ZF which satisfy $2^{\aleph_0} > \aleph_1$, and

(a′) There is a set of cardinality $\aleph_1$ which does not have the Baire property.

(b′) Every set of cardinality $< 2^{\aleph_0}$ is meager, but $\mathcal{R}$ is the union of $\aleph_1$ meager sets.

Recall the $\mathcal{M}[G_a]$ and $\mathcal{M}[G_b]$ mentioned in § 4.1. These are models of (b′) and (a′) respectively.

The following theorem was discovered independently by each of the authors. One of our proofs used an unpublished construction of R. Cotton.

Theorem. *If* $\mathrm{A}_\aleph$ *then the union of* $\aleph$ *meager sets is meager.*

Proof. Every union of $\aleph$ meager sets is meager if and only if the intersection of any $\aleph$ comeager sets is comeager. A comeager set contains the intersection of countably many dense open sets. What we have to prove then is that the intersection of $\aleph$ dense open sets is comeager.

Let D_α, $\alpha < \aleph$ be dense open sets. Let B_i, $i < \omega$, be a base for the open sets of $\mathcal{R}$. If W is a dense open set, let

$$s(W) = \{\, i \in \omega \, ; \; B_i \not\subseteq W \,\} \, .$$

For $j \in \omega$, let

$$t(j) = \{\, i \in \omega : B_i \subseteq B_j \,\} \, .$$

Let $A = \{ s(D_\alpha) : \alpha < \aleph \}$ and let $B = \{ t(j) : j \in \omega \}$. Let $n, j < \omega$ and $s(D_{\alpha_1}), \ldots, s(D_{\alpha_n}) \in A$. Since the intersection of finitely many dense open sets is dense open, there is a $B_i \subseteq B_j \cap D_{\alpha_1} \cap \ldots \cap D_{\alpha_n}$; $\{ k : B_k \subseteq B_i \}$ is then an infinite subset of $t(j) - (s(D_{\alpha_1}) \cup \ldots \cup s(D_{\alpha_n}))$. By $\mathbf{S}_\aleph$ and the Theorem of § 2.2, let t be a set of integers such that $t \cap t(j)$ is infinite for all $j \in \omega$ and $t \cap s$ is finite for $s \in A$. Let $W_n = \bigcup_{n < i \in t} B_i$. For each $j \in \omega$, since $t \cap t(j)$ is infinite, there is a $B_i \subseteq W_n \cap B_j$. Hence W_n is a dense open set. For each $\alpha < \aleph$, $t \cap s(D_\alpha)$ is finite, and so there is an n such that $W_n \subseteq D_\alpha$. Hence

$$\bigcap_{n \in \omega} W_n \subseteq \bigcap_{\alpha < \aleph} D_\alpha .$$

Since $\bigcap_{n \in \omega} W_n$ is comeager, so is $\bigcap_{\alpha < \aleph} D_\alpha$.

Corollary 1. *If* A, *then the ideal of meager sets is* $2^{\aleph_0}$*-additive and the* σ*-algebra of sets with the Baire property is* $\aleph$*-complete for every* $\aleph < 2^{\aleph_0}$.

Corollary 2. *If* $\mathsf{A}_{\aleph_1}$, *every* $\mathbf{\Sigma}^1_2$ *set of reals has the Baire property.*

Corollary 3 *If* $\mathsf{A}_{\aleph_2}$ *and there exists a measurable cardinal, then every* $\mathbf{\Sigma}^1_3$ *set of reals has the Baire property.*

The theorem, like Theorem 1 of 4.1, shows that $\mathsf{A} \to 2^{\aleph_0}$ is regular. The Theorem can readily be generalized to separable topological spaces. The construction needed to prove the Theorem antedates the almost-disjoint set technique; however, it was noticed only recently that it is essentially an example of that technique.

4.3. *The measure problem.* In [21] it is proved that if ZF plus "there exists an inaccessible cardinal" is consistent, then ZF minus the axiom of choice, together with the axiom of dependent choice and "every set of reals is Lebesgue measurable and has the Baire property", is consistent. The use of an inaccessible cardinal is a minor annoyance, since it appears unlikely that the inaccessible is really necessary. The methods of § 4.1 and § 4.2 suggest a method which might remove this annoyance.

Let $B_1, B_2, \ldots$ be the open intervals of $\mathcal{R}$ with rational endpoints. If A is a Borel subset of $\mathcal{R}$, a *code* for A is a real t which codes (in some standard way) a method for generating A from the B_i by taking countable unions, complements, etc. A real t is *random* (*Cohen generic*) over a class $\mathcal{M}$ satisfying the axioms of set theory if t belongs to no Borel set of measure zero (of the first category) some code for which belongs to $\mathcal{M}$.

It is a result of Solovay that if every set is Lebesgue measurable (has the Baire property), then

(*) If $\mathcal{M}$ satisfies the axiom of choice, the set of reals random (Cohen generic) over $\mathcal{M}$ has a measure zero (meager) complement.

The method of [21] is to find a model satisfying

(**) Any well-ordered sequence of reals is countable.

(*) follows easily from (**). However (**) plus dependent choice implies that $\aleph_1$ is inaccessible in L. That is why [21] requires an inaccessible cardinal.

The results of § 4.1 and § 4.2 can be restated as

If A *and if* $\mathcal{M}$ *has fewer than* $2^{\aleph_0}$ *reals, then the set of reals random (Cohen generic) over* $\mathcal{M}$ *has a measure zero (meager) complement.* (In the statement of A we here require that card $(\mathcal{P}) < \aleph$.)

Hence a model will satisfy (*) if it satisfies A and

(***) $\mathcal{R}$ cannot be well-ordered.

Suppose that $\mathcal{N}$ is a model of ZF + A + $2^{\aleph_0} > \aleph_1$. Let $\mathcal{N}'$ be the collection of members of $\mathcal{N}$ which are hereditarily ordinal definable from a real. $\mathcal{N}'$ satisfies A and $\mathcal{N}$ can be chosen so that $\mathcal{N}'$ satisfies (***) and hence (*). Perhaps then an $\mathcal{N}$ can be found such that $\mathcal{N}'$ satisfies "all sets are Lebesgue measurable and have the Baire property". If this can be done, however, it appears that the set of forcing conditions used to get $\mathcal{N}$ must be chosen with care. The proof in [21] depends not only on having a model of (*) but also on the fact that this model is a Cohen extension via a $\mathcal{P}$ whose associated Boolean algebra is quite homogeneous. It is not clear how to get our $\mathfrak{N}$ via a $\mathcal{P}$ with the analogous homogeneity property.

Another approach is not to get a model of A but simply to get a model of (*) using the special constructions of § 4.1 and § 4.2. This can be done without making $2^{\aleph_0} > \aleph_1$ in $\mathfrak{N}$.

§ 5. A and the continuum hypothesis

5.1. *The relative strength of* A *and* CH. Many of the most interesting applications of A occur only when $2^{\aleph_0} > \aleph_1$. Examples are Souslin's hypothesis and the measurability of $\mathbf{\Sigma}_2^1$ sets. Nevertheless, A is a consequence of CH and many of the consequences of CH follow also from A. We have seen several examples of this: Corollaries 1 and 2 of § 3.1, Corollary 1 of § 4.1 and Corollary 1 of § 4.2. For more examples, we turn to Sierpinski [15]. Of the consequences $C_1 - C_{82}$ of CH demonstrated there, we know that at least 48 follow from A; at least 23 are refuted by A + $2^{\aleph_0} > \aleph_1$; at least three (C_{52}, C_{78}, and C_{81}) are consistent with and independent of A + $2^{\aleph_0} > \aleph_1$ (provided that the existence of an inaccessible cardinal is consistent with ZF – for the consistency of C_{78} with A + $2^{\aleph_0} > \aleph_1$ and for the independence of C_{81} from A – and that the existence of a measurable cardinal is consistent with ZF – for the independence of C_{52} from A). There are only 8 of the C_n whose relation to A + $2^{\aleph_0} > \aleph_1$ we do not know about at present (C_8, C_{13}, C_{47}, C_{48}, C_{61}, C_{62}, C_{70}, and C_{80}).

Actually (as Kunen [8] remarks]) A is much closer to CH with respect to the C_n of [15] than our count makes it appear. Sierpinski often states his consequences of CH in terms of the denumerable/indenumerable dichotomy. Obviously, however, the effect of A is to say that all infinite cardinals $< 2^{\aleph_0}$ have many of the properties of $\aleph_0$, so that the important dichotomy in terms of A is the less than $2^{\aleph_0}/2^{\aleph_0}$ dichotomy. All the 23 consequences C_n of CH which we know to contradict A + $2^{\aleph_0} > \aleph_1$, and all of the 8 C_n about whose relation to A + $2^{\aleph_0} > \aleph_1$ we are ignorant, become – if we make the obvious replacements of "denumerable" by "of cardinality $< 2^{\aleph_0}$" and "indenumerable" by "of cardinality $2^{\aleph_0}$" – consequences of A. A few become in fact theorems of ZF, but most do not.

For instance (this example is also noted by Kunen [8]), C_1 is the assertion that there is a set of reals of cardinality $2^{\aleph_0}$ which has at most $\aleph_0$ members in common with each meager set. If $2^{\aleph_0} > \aleph_1$ and A, C_2 clearly contradicts the Theorem of § 4.2, which implies that every set of cardinality $< 2^{\aleph_0}$ is meager. On the other hand, the very same Theorem of § 4.2 allows us to repeat essentially Sierpinski's proof (which is due to Lusin) of CH $\to C_1$ to show that A $\to C_1^*$, where C_1^* is the proposition: There is a set of reals of cardinality $2^{\aleph_0}$ which has $< 2^{\aleph_0}$ members in common with each meager set. (Let A_α, $\alpha < 2^{\aleph_0}$ be all meager Borel sets. By A and the Theorem of 4.2 $\bigcup_{\alpha<\beta} A_\alpha$ is meager for each $\beta < 2^{\aleph_0}$. Let $t_\beta \in \mathcal{R} - \bigcup_{\alpha<\beta} A_\alpha$, be distinct from each t_α, $\alpha < \beta$. Then $\{ t_\beta : \beta < 2^{\aleph_0} \}$ is the required set.) C_1^*, together with the theorems of § 4 is enough to deduce many of the propositions which Sierpinski deduces from C_1.

For other consequences of CH, which are also consequences of A, see [8].

5.2. *Real valued measurable cardinals.* κ is a *real-valued measurable cardinal* if κ is an uncountable cardinal and there is a κ-additive real-valued measure ν defined on all subsets of κ such that $\nu(\{\alpha\}) = 0$ for each $\alpha < \kappa$ and $\nu(\kappa) = 1$ (i.e., if there is a non-trivial κ-additive ideal I in $P(\kappa)$ such that $P(\kappa)/I$ is a measure algebra). κ is a $\{0, 1\}$*-measurable cardinal* if there is a ν as above and in addition ν takes only the values 0 and 1.

The assumption that $2^{\aleph_0}$ is a real-valued measurable cardinal is, like A, an alternative to CH. It is known that $2^{\aleph_0}$ is very large if it is real-valued measurable (see [19]). Also many of the consequences of CH are decided – one way or other – by the assumption that $2^{\aleph_0}$ is real-valued measurable.

If I is an ideal in $P(\kappa)$ such that $P(\kappa)/I$ is a measure algebra, I is of course $\aleph_1$-saturated. We remarked in § 2.5 that, if the existence of a $\{0,1\}$-measurable cardinal is consistent, then so is A plus the existence of an uncountable $\kappa < 2^{\aleph_0}$ such that $P(\kappa)$ bears a non-trivial $\aleph_1$-

saturated κ-additive ideal. By the same argument, A is (on the same hypothesis) consistent with the assumption that $P(2^{\aleph_0})$ bears a non-trivial $\aleph_1$-saturated $2^{\aleph_0}$-additive ideal. Now Solovay [19] has shown that it is consistent that $2^{\aleph_0}$ is real-valued measurable if and only if it is consistent that a $\{0, 1\}$-measurable cardinal exists. Can one combine these consistency results to show that A is consistent with the assumption that $2^{\aleph_0}$ is real-valued measurable?

The answer is no. In fact, A implies that there is no real valued measurable cardinal $\leq 2^{\aleph_0}$. It is hard to decide to whom this fact is due, since almost any of the classical proofs that CH implies that there is no real-valued measurable cardinal $\leq 2^{\aleph_0}$ work just as well under the weaker hypothesis A. Kunen [8] gives three proofs and mentions still a fourth. Several recipes for proofs can be found in Sierpinski [15], though Sierpinski's *official* proof that CH $\to$ no $\kappa \leq 2^{\aleph_0}$ is real valued measurable unfortunately uses the fact that CH $\to P(2^{\aleph_0})$ does not bear an $\aleph_1$-saturated σ-ideal. Most of the proofs use only the consequence of A which we have in § 4.2 called the Strong Baire Category Theorem. One of Kunen's arguments uses only the even weaker assumption that every set of reals of cardinality $< 2^{\aleph_0}$ is Lebesgue measurable. We give a proof based on Theorem 6 of Sierpinski [15].

Theorem. *The Strong Baire Category Theorem* (SBCT) *implies (and so* A *implies) that there is no real valued measurable cardinal* $\leq 2^{\aleph_0}$.

Proof. Assume SBCT and that $\kappa < 2^{\aleph_0}$ is real-valued measurable. Let $C \subseteq \mathcal{R}$ have cardinality $2^{\aleph_0}$. It is easily seen (using the real-valued measurability of κ) that there is a countably additive real-valued measure ν defined on all subsets of $\mathcal{R}$ with $\nu(\{t\}) = 0$ for each real t and $\nu(C) = \nu(\mathcal{R}) = 1$.

We first use SBCT to show that every $A \subseteq \mathcal{R}$ of cardinality $< 2^{\aleph_0}$ has ν-measure 0. Let r_i, $i \in \omega$, be an enumeration of all rational numbers. Let B_{ij}, for $j \in \omega$, be the set of reals t such that $r_i < t$ and $\nu([r_i, t]) < \epsilon/2^{j+1}$. ($[r_i, t]$ is the closed interval from r_i to t.) For each i and j, $\nu(B_{ij}) \leq \epsilon/2^{j+1}$; and, for each j, $\bigcup_i B_{ij} = \mathcal{R}$, since otherwise there is a real t such that $\nu([r, t]) > \epsilon/2^{j+1}$ for each $r < t$, and so $\nu(\{t\}) >$ $> \epsilon/2^j$. Let C_j, $j \in \omega$, be a base for the open sets of $\mathcal{R}$. For each j, let

$A_{0j}, A_{1j}, \ldots$ be disjoint nonempty open subsets of C_j. For each real s let

$$D_s = \bigcup_{ij} \{ A_{ij} : s \in B_{ij} \} .$$

Clearly each D_s is open. Let $j \in \omega$. Let i be such that $s \in B_{ij}$. Then $A_{ij} \subseteq C_j$ and $A_{ij} \subseteq D_s$. Hence D_s is dense.

Let $A \subseteq \mathcal{R}$ have cardinality $< 2^{\aleph_0}$. By SBCT, there is a real

$$t \in \bigcap_{s \in A} D_s .$$

For each j there is a unique $i(j)$ such that $t \in A_{i(j),j}$. Consider

$$B = \bigcup_j B_{i(j),j} .$$

$\nu(B) \leq \sum_j \epsilon/2^{j+1} = \epsilon$. If $s \in A$, $t \in D_s$, so there is a j with $s \in B_{i(j),j}$. Hence $B \supseteq A$. Since ϵ was arbitrary, we have shown that $\nu(A) = 0$.

We next note that the proof of C_1^* of 5.1 uses only SBCT. We may then suppose that our set C has $< 2^{\aleph_0}$ members in common with every meager set. Let $\epsilon > 0$. For each rational r_i, let B_i be the interior of B_{ii}.

$$\nu(\bigcup_i B_i) \leq \sum_i \frac{\epsilon}{2^{i+1}} = \epsilon .$$

Now $\bigcup_i B_i$ is dense open, so $C - \bigcup_i B_i$ has cardinality $< 2^{\aleph_0}$ and so has ν-measure 0. But then $\nu(C) \leq \epsilon$. If we let $\epsilon < 1$ we have a contradiction.

5.3. *Is* A *true?* As we have indicated, many if not most of the interesting consequences of CH follow also from A. If CH is thought by some to be false – and if it is thought to be false because of its consequences – then may not the same consequences count against A?

Gödel [4] offers as indications of the falsity of CH six consequences of CH which he says are implausible. Three of these follow from A:

(a) There is a set of reals of cardinality $2^{\aleph_0}$ which is meager on every perfect set.

(b) There is a set of reals of cardinality $2^{\aleph_0}$ which is carried into a set of Lebesgue measure 0 by every continuous 1–1 mapping of $\mathcal{R}$ into $\mathcal{R}$.

(c) There is a set $A \subseteq \mathcal{R}$ of cardinality $2^{\aleph_0}$ which *has property* C: for any positive reals $a_0, a_1, \ldots$, there are intervals A_n, $n < \omega$, of length a_n such that $\bigcup_{n=0}^{\infty} A_n \supset A$.

The proofs given in [15] of (a), (b) and (c) from C_1 work just as well as proofs from C_1^* plus the assumption that every set of cardinality $< 2^{\aleph_0}$ is meager and has the property **C**. We might note that (b) and (c) follow from SBCT alone. The construction of the proof of the Theorem of § 5.2 goes through with no changes if we assume that ν is Lebesgue measure (since the only sets we assumed to be measurable were Borel sets). Suppose we replace $\epsilon/2^{i+1}$ by reals a_i in that construction. Then the first part of the proof shows that any set of cardinality $< 2^{\aleph_0}$ can be covered by the union of intervals $B_{i(j),j}$ of length a_i. In other words, it shows that every set of cardinality $< 2^{\aleph_0}$ has property **C**. The second part of the proof shows – with similar modifications – that any set C satisfying C_1^* has property **C**. It is easily seen that **C** is preserved under continuous mappings of $\mathcal{R}$ into $\mathcal{R}$, so our set C also satisfies the conditions of (b).

Another "implausible" consequence of CH mentioned by Gödel is C_1. Now A + $2^{\aleph_0} > \aleph_1$ implies that C_1 is false, but perhaps C_1^* is just as "implausible" as C_1. Of the other two consequences of CH cited by Gödel, one is equivalent to CH and the other is inconsistent with A + $2^{\aleph_0} > \aleph_1$. (This last fact is due to D.Booth.)

If one agrees with Gödel that (a), (b), and (c) are implausible, then one must consider A an unlikely proposition. The authors, however, have virtually no intuitions at all about (a), (b), and (c) – or about the other consequences of A discussed in this paper. We know of no very convincing evidence either of the truth of A or of its falsity, and we see no immediate hope for finding such evidence.

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