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Generic Bifurcation in Manifolds with Boundary*

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INTRODUCTION

Let M be a C^∞ two-dimensional orientable compact manifold, *with boundary* ∂M. χ^r will denote the space of the C^r vector fields on M, with the C^r topology (it is a C^∞ Banach manifold).

We are concerned in this study with certain types of vector fields which are not structurally stable in χ^r; namely, in generic vector fields in $\chi_1{}^r = \chi^r - \Sigma_0$, where Σ_0 is the set of structurally stable vector fields of χ^r.

The main result is the following.

THEOREM A. *For $r > 3$, there exists a C^{r-1} submanifold Σ_1, having codimension one which is immersed in χ^r, and satisfies:*

(a) *Σ_1 is dense in $\chi_1{}^r$ (both with the relative topology);*

(b) *for any X in Σ_1, there exists a neighborhood B_1 in the intrinsic topology of Σ_1, such that any Y in B_1 is topologically equivalent to X".*

The part of Σ_1 imbedded in χ^r coincides with elements of χ^r which are first-order structurally stable.

In Section 0 we give definitions, recall standard facts, and establish our notation.

Section 1 is devoted to the construction of Σ_1 and the proof of Theorem A. It is divided into three parts. In the first part we adapt the quasi-generic fields studied by Sotomayor [22] to manifolds with a boundary. In part 2 we study fields which are nongeneric due to the contact between the field and ∂M. Finally in part 3 we prove Theorem A. At the end of certain paragraphs we include some remarks which prepare the way for the study of first-order structurally stable fields.

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0. Preliminaries

We will consider dynamical systems generated by tangent vector fields (differential equations) on manifolds with a boundary. For simplicity M will be imbedded on a two-dimensional C^∞ manifold N, without a boundary.

Two vector fields $\tilde{X}_1$, $\tilde{X}_2$ on N are said to be germ equivalent on M if they coincide on a neighborhood of M. A vector field X on M is, by definition, a class of germ equivalent (on M) tangent vector fields defined on N. It is said to be of class C^r if it has a representative $\tilde{X}$ of class C^r on N.

Let $\tilde{\Phi}$ be the flow of a representative $\tilde{X}$ of X; $\tilde{\Phi}$ is defined on a set $D(\tilde{X}) = \{(x, t) \in N \times R, t \in \tilde{I}(x)\}$, where $\tilde{I}(x)$ is an open interval with extremes $\tilde{\alpha}(x)$, $\tilde{\omega}(x)$. The flow Φ of X is defined by $\Phi(x, t) = \tilde{\Phi}(x, t)$ for $x \in M$ and $t \in I(x)$, where $I(x)$ is the maximal interval containing $t = 0$ ($\Phi(x, 0) = x$) for which $\tilde{\Phi}(x, t) \in M$. We denote by $\alpha(x)$ (resp. $\omega(x)$) the lower (resp. upper) extreme of this interval; it may be that one, both, or none of the extremes of $I(x)$ are infinite, finite, or even zero. Clearly Φ and its domain $D(X)$ do not depend on the representative $\tilde{X}$ of X. Furthermore, any two representatives of X define flows on N which coincide on a neighborhood of $D(X)$. We call $\tilde{\Phi}$ their germ on $D(X)$; $\Phi = \tilde{\Phi}\,|_{D(X)}$.

The orbit $\gamma(x)$ of X, passing through $x \in M$ is by definition the image of $I(x)$ by the integral curve map $\Phi_X(x, \ ): t \to \Phi_X(x, t)$. Orbits are oriented by the orientation induced by this map from the positive orientation of $I(x)$; an orbit of X, with no distinguished parametrization, is a *trajectory* of X.

Germ orbits and germ trajectories are defined similarly.

0.1. Definition. Two vector fields X, Y on M are said to be conjugate if there exists a homeomorphism $h: M \to M$ mapping trajectories of X onto trajectories of Y.

We denote by $\tilde{\chi}^r = \chi^r(N)$ the equivalent space of χ^r.

0.2. Definition. $X \in \chi^r$ is structurally stable in χ^r, if it has a neighborhood B in χ^r such that X is conjugate to every $Y \in B$.

It has been shown in [5, 6] that Σ_0 is open dense in χ^r ($r > 1$) and coincides with the collection of vector fields X such that:

Ω_1: X has all its singular points generic (or hyperbolic);

Ω_2: X has all its periodic trajectories generic (or hyperbolic);

Ω_3: X does not have saddle connections;

Ω_4: X does not have nontrivial recurrent trajectories;

B_1: X has all its singular points in the interior of M;

B_2: X has all its periodic trajectories in the interior of M;

B_3: any trajectory of X has at most one point of tangency with ∂M;

B_4: any saddle separatrix of X is transverse to ∂M;

B_5: if a trajectory of X is tangent to ∂M in p, then the contact between the two curves in p is of the 2nd order;

B_6: there exist only a finite number of points of tangency of X and ∂M.

It is proved in [11] that the conditions B_1, B_2, B_3, B_4, B_5 imply B_6.

For the sake of reference, the concepts of generic singular point, generic periodic trajectory, saddle connection, quasi-generic singular point, quasi-generic periodic trajectory, quasi-generic saddle connection are contained in [10].

We denote by $\Delta(X, p)$ and $\sigma(X, p)$ the determinant and the trace of DX_p (derivative of X at p), respectively.

The definitions of *imbedded* and *immersed* Banach submanifolds of class C^s and codimension K of a Banach manifold of class C^∞ are given in [10, p. 7].

0.3. *Observations and notations.* (a) We will fix on N a Riemannian metric of differentiability class large enough for our purposes.

(b) The positive limit set of an orbit $\gamma(p)$ of X is the set of points $y \in M$ which are limit points of sequence of the form $\Phi(p, t_n)$ with t_n tending to $\omega(p)$; we denote this set by $L^+(p)$ and the negative limit set $L^-(p)$ has a similar definition. These definitions do not depend on $q \in \gamma(p)$. If

$$\omega(p) < +\infty \qquad (\text{resp. } \alpha(p) > -\infty),$$

then $L^+(p)$ (resp. $L^-(p)$) is the single point $\Phi(p, \omega(p))$ (resp. $\Phi(p, \alpha(p))$) and belongs to ∂M.

(c) The following notations will be used in the text.

(i) $M - F$ is the set of points $q \in M$, such that $q \notin F$;

(ii) $\text{int}(M)$ is the interior (topologic) of M;

(iii) if $u, v \in T(M)$ ($T(M)$ is the tangent space of M), then $u \wedge v$ will denote the exterior product of u and v;

(iv) (F, p) is to be regarded as a flow box around p of some vector field.

For $Q \in \chi^r$ we have the definitions:

0.4. We say that Q satisfies the I condition (resp. M condition) if it is an immersed (resp. imbedded) Banach submanifold of class C^{r-1} and codimension one of χ^r.

0.5. We say that $\underset{\sim}{Q}$ satisfies the E condition if every $X \in \underset{\sim}{Q}$ has a neighborhood B in $\underset{\sim}{Q}$ such that every $Y \in B$ is conjugate to X.

0.6. We say that $\underset{\sim}{Q}$ satisfies the A condition if $\underset{\sim}{Q}$ is an open set of χ^r.

I. The Submanifold Σ_1

Part 1

We will consider in this part the quasi-generic elements of a vector field which belong to the interior of M; basically the demonstrations of 1.1, 1.2, 1.4, 2.1, and 3.1 are due to Sotomayor [10].

1

1.1. Proposition. *Denote by $\underset{\sim}{Q}_2$ the set of vector fields $X \in \chi^r$, $r > 2$, such that:*

(1) *X has one quasi-generic trajectory as unique nongeneric periodic trajectory;*

(2) *X satisfies Ω_2, Ω_3, Ω_4, B_2, B_3, B_4, B_5, and B_6. Then $\underset{\sim}{Q}_2$ satisfies the I and E conditions.*

See the proof of 1.1 in [10, p. 9].
It is convenient to give the following.

1.2. Lemma. *Call $\underset{\sim}{Q}_2(n)$ the set of $X \in \underset{\sim}{Q}_2$ such that its quasi-generic periodic trajectory has length less than n. Then $\underset{\sim}{Q}_2(n)$ satisfies the A, M, and E conditions.*

1.3. *Remark.* Call $\tilde{\underset{\sim}{Q}}_2$ the subset of $\underset{\sim}{Q}_2$ of vector fields X, which satisfy:

(a) There exists no $q \in M - \gamma_X$, such that $L^{+}(q) = L^{-}(q) = \gamma_X$.

(b) There exist no saddle points s_i of X in M, $i = 1, 2$, such that $L^{-}(W^{u}(s_1)) = L^{+}(W^{s}(s_2)) = \gamma_X$, where W^{s} (resp. W^{u}) is the stable (resp. unstable) submanifold associated to the critical point.

(c) Associated to X there exist no $(s, q) \in M \times M$, where s is a saddle point of X, $q \in \partial M$ and $X(q)$ is tangent to ∂M at this point, such that $L^{+}(q) = L^{-}(W^{s}(s)) = \gamma_X$.

(d) There exists no $p_i \in \partial M$, $i = 1, 2$, such that $X(p_i)$ is tangent to ∂M at p_i, with $L^{-}(p_1) = L^{-}(p_2) = \gamma_X$ (the case $p_1 = p_2$ is not excluded).

1.4. Proposition. *$\tilde{\underset{\sim}{Q}}_2$ satisfies the M, E, and A conditions.*

1.5. *Remarks.* (i) If γ_X is the α and ω limit of saddle separatrices, then it can be shown that there is Y, arbitrarily close to X, which has saddle connections having arbitrarily large length.

(ii) If there exists a trajectory η of X, which has γ_X as the α and ω limits, then it can be shown that there is Y, arbitrarily close to X, which has a nongeneric periodic trajectory meeting F and arbitrarily large length.

(iii) If there exists a trajectory η_1 of X which has γ_X as the α limit and a saddle separatrix η_2 of X having γ_X as the ω limit then it can be shown there is Y, arbitrarily close to X, having a saddle separatrix tangent to ∂M; furthermore, its length is arbitrarily large.

(iv) If there exist two distinct trajectories of X, both having tangency points of ∂M and having its α and ω limits coinciding with a quasi-generic periodic trajectory, then there exists Y close to X, such that it has a trajectory which is tangent to ∂M at two distinct points; furthermore its length is arbitrarily large.

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2.1. PROPOSITION. *Denote by* $\underset{\sim}{Q}_1$ *the set of vector fields* $X \in \chi^r$, $R > 1$, *such that:*

(1) *X has a quasi-generic critical point as its unique nongeneric critical point;*

(2) *X satisfies* Ω_2, Ω_3, Ω_4, B_1, B_2, B_3, B_4, B_5, *and* B_6.

Then $\underset{\sim}{Q}_1$ *satisfies the M, E, and A conditions.*

3

3.1. PROPOSITION. *Denote by* $\underset{\sim}{Q}_3$ *the set of vector fields* $X \in \chi^r$, $r > 1$, *such that:*

(1) *X has one quasi-generic saddle connection as its unique saddle connection;*

(2) *X satisfies* Ω_1, Ω_2, Ω_4, B_1, B_2, B_3, B_4, B_5, *and* B_6.

Then $\underset{\sim}{Q}_3$ *satisfies the I and E conditions.*

3.2. *Remark.* Note that in 1.1, 2.1, and 3.1 the quasi-generic periodic trajectory, the quasi-generic critical point, and the quasi-generic saddle connection, respectively, are "away from" ∂M; then the B_i conditions, $i = 1, 2, \ldots, 6$ hold for small perturbation of $X \in \underset{\sim}{Q}_1 \cup \underset{\sim}{Q}_2 \cup \underset{\sim}{Q}_3$.

3.3. *Remark.* If the saddle connection of $X \in \underset{\sim}{Q}_3$ is an autoconnection at a saddle p, then a closed curve C is constructed in [10] which is arbitrarily close to $\gamma_X \cup \{P\}$ and such that any Y close to X is transverse to C. Denote by $\tilde{\underset{\sim}{Q}}_3$ the subset of $\underset{\sim}{Q}_3$ consisting of fields X which have the following properties:

No trajectory of X which is tangent to ∂M meets C and no saddle separatrix of X meets C. Then $\tilde{Q}_3$ satisfies the M, E, and A conditions.

3.4. *Remark.* Call $Q_3(n)$ the set of $X \in Q_3$, such that the saddle connection has length less than n. Then $\tilde{Q}_3$ satisfies the M, E, and A conditions.

Part 2

In this section we are going to study the families of nonstable fields, whose instability arises from the contact of the trajectories with ∂M. We will be using frequently techniques and results of Peixoto [6, 7] and Sotomayor [10].

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4.1. DEFINITION. $p \in \partial M$ is a generic critical element of $X \in \chi^r$ if it satisfies the conditions:

(b_1) no periodic trajectory of X is tangent to ∂M at p;

(b_2) $X(p) \neq 0$;

(b_3) if a trajectory γ of X is tangent to ∂M at p then γ is transversal to ∂M at any point $q \in \gamma$, $q \neq p$;

(b_4) no saddle separatrix of X is tangent to ∂M at p;

(b_5) if a trajectory γ of X is tangent to ∂M at p, then the contact between γ and ∂M at p is of 2nd order (we will say that the contact between X and ∂M at p is generic; see construction 4.2).

4.2. *A construction.* Let $p \in \partial M$, let $\gamma_X(p)$ be a trajectory of $X \in \chi^r$ passing through p, and let $\tilde{X} \in \tilde{\chi}^r$ be a representative of X. Let $u: (R, 0) \to (N, p)$ be a C^∞ germ of an imbedding, transverse to ∂M at p. Also, let $s: (R, 0) \to (\partial M, p)$ be a C^∞ germ of an imbedding. By the Implicit Function Theorem $\sigma = (s, u)$ is a C^∞ germ of a diffeomorphism $\sigma: (R^2, 0) \to (N, p)$. Denote by π the second component of the inverse function $\sigma^{-1}: (N, p) \to (R^2, 0)$. Finally we consider the germ $\pi_{\tilde{X}}: (R, 0) \to (R, 0)$ defined by $\pi_{\tilde{X}}(t) = \pi(\Phi_{\tilde{Y}}(p, t))$. By continuity, $\pi(\Phi_{\tilde{Y}}(q, t))$ is defined in a neighborhood $\tilde{B} \times \tilde{F}_1$ of $(\tilde{X}, p)$ in $\tilde{\chi}^r \times N$: For each $\tilde{Y} \in \tilde{B}$, consider the C^r germ $\pi_{\tilde{Y}}: (R, 0) \to (R, 0)$ defined by $\pi_{\tilde{Y}}(t) = \pi(\Phi_{\tilde{Y}}(q, t))$ for all $q \in \tilde{F}_1$. It is clear that γ_X is tangent to ∂M at p if and only if $\pi_X'(0) = 0$ for every representative $\tilde{X}$ of X. Observe that $\pi_{\tilde{X}}: (R, 0) \to (R, 0)$ can be defined without difficulties for $t_0 \neq 0$.

4.2.1. DEFINITION. We say that $p \in \partial M$ satisfies the *G condition* with respect to X, or the *contact* between X and ∂M at p is *generic*, if $\pi_{\tilde{X}}'(0) = 0$ and $\pi_{\tilde{X}}''(0) \neq 0$.

4.2.2. Definition. We say that $p \in \partial M$ satisfies the *QG condition* with respect to X or the *contact* between X and ∂M at p is *quasi-generic*, if $\pi_{\tilde{X}}'(0) = \pi_{\tilde{X}}''(0) = 0$, and $\pi_{\tilde{X}}'''(0) \neq 0$.

Obviously, these definitions depend neither on the transversal germ u, nor on the particular representative $\tilde{X}$ of X.

4.3. *Remarks.* (a) Condition b_5 is equivalent to the G condition.

(b) For future reference consider the coordinates $x = (x_1, x_2)$ (defined in a neighborhood $\tilde{F}_1$ of p in N) where

$$x_1(p) = x_2(p) = 0, \qquad x_1 \circ s = id, \qquad x_2 \circ u = id, \qquad x_1 \circ u = x_2 \circ s = 0.$$

It is convenient to observe $\pi_{\tilde{X}}(t) = x_2(\Phi_{\tilde{X}}(p, t))$.

(c) Denote by U and S arbitrarily small closed neighborhoods of p in $u(R)$ and $s(R)$, respectively. We will assume the positive orientation of U given by the outward sense from M.

4.4. Lemma. *Assume the notations of* 4.2. *If the contact between* $X \in \chi^r$ *and* ∂M *at* $p \in \partial M(X(p) \neq 0)$ *is generic, then there exist a neighborhood* B_0 *of* X *in* χ^r *and a* C^r *function* $\alpha\colon B_0 \to R$ *such that* $Y(s(\alpha(Y))$ *is tangent to* ∂M *at* $s(\alpha(Y))$; *furthermore, the contact between* Y *and* ∂M *at* $s(\alpha(Y))$ *is generic.*

Proof. Consider the germ $G\colon (\chi^r \times R, (X, 0)) \to (R, 0)$ of class C^r, defined by $G(Y, \alpha) = Y(s(\alpha)) \wedge s'(\alpha)$.

Let $x = (x_1, x_2)$ be a system of coordinates around p; assume $x_1(p) = x_2(p) = 0$, $\partial/\partial x_1 = X(p)$, and that $s = (s_1, s_2)$ are the components of s in this system with $s(0) = p$.

By a direct calculation we obtain $(\partial G/\partial\alpha)(X, 0) = s_2''(0) \neq 0$; this follows since the contact between X and ∂M at p is generic. By the Implicit Function Theorem, there are a neighborhood B_0 of X in χ^r and a unique C^r function $\alpha\colon B_0 \to R$ such that $\alpha(X) = 0$ and $G(Y, \alpha) = 0$ if and only if $\alpha = \alpha(Y)$. Furthermore, by continuity B_0 can be determined such that the contact between Y and ∂M at $s(\alpha(Y))$ is generic. This finishes the proof.

4.5. Definition. $p \in \partial M$ is a *quasi-generic critical element* of $X \in \chi^r$ of the type:

β_1 if $X(p) = 0$ and

(a) p is hyperbolic;

(b) the eigenspaces of DX_p are transverse to ∂M at p;

(c) the eigenvalues of DX_p are not equals;

(d) if p is a node (see [2]) then the trajectory of X that is tangent to the eigenspace of DX_p associated to the eigenvalue of larger absolute value, is not tangent to ∂M and is not a saddle separatrix;

β_2 if there exists a generic periodic trajectory of X tangent to ∂M only at p, where the G condition is true;

β_3 if the trajectory of X passing through p is neither periodic nor saddle separatrix and it has only one point of tangency q with ∂M besides p; furthermore, $p \neq q$ and both satisfy the G condition with respect to the field;

β_4 if there exists a saddle separatrix of X, tangent to ∂M only at p, satisfying the G condition with respect to the field;

β_5 if there exists a trajectory of X that is neither saddle separatrix nor periodic, is tangent to ∂M at p, and satisfies the QG condition with respect to the field.

4.6. *Remark.* If p is a hyperbolic critical point of X and the eigenvalues of DX_p are complex conjugate, then we are allowing it to satisfy condition (b) of the definition of the quasi-generic critical element of β_1.

5

5.1. PROPOSITION. *Denote by H_2 the set of vector fields $X \in \chi^r$, $r > 2$, such that:*

(1) *there exists one point $p \in \partial M$ that is a quasi-generic critical element of X of the type β_2, a unique nongeneric critical element of X;*

(2) *X satisfies Ω_1, Ω_2, Ω_3, Ω_4. Then H_2 satisfies the M and E conditions.*

The proof of 5.1 depends on several lemmas.

5.2. LEMMA. *Let $\tilde{X} \in \tilde{\chi}^r$ have a generic periodic trajectory $\gamma_{\tilde{X}}$ of period τ_0. Given ϵ and T_0, positive integers, there exist neighborhoods $\tilde{B}$ of $\tilde{X}$ in $\tilde{\chi}^r$ and $\tilde{V}$ of $\gamma_{\tilde{X}}$ in $\tilde{N}$, such that:*

(a) *to each field $\tilde{Y} \in \tilde{B}$ corresponds a unique generic periodic trajectory $\gamma_{\tilde{Y}}$ contained in $\tilde{V}$ with period smaller than $|\tau_0 - \epsilon|$;*

(b) *every trajectory of $\tilde{X}$ meeting $\partial\tilde{V}$ is transverse to it and spends a time greater than T_0 in N. Furthermore, $\partial\tilde{V}$ is the union of two closed curves.*

See the proof of 5.2 in [4, part VIII].

5.3. LEMMA. *If $X \in H_2$ then there exists a neighborhood B of X in χ^r, such that:*

(a) *every* $Y \in B$ *satisfies* Ω_1, Ω_2, Ω_3, Ω_4, B_1, B_5, B_6;

(b) *if* $Y \in B \cap H_2$ *then* Y *satisfies* B_3 *and* B_4.

The proof of 5.3 follows immediately from [5].

5.4. LEMMA. *Let* $X \in \chi^r$, $r > 2$, *have one point* $p \in \partial M$ *as a quasi-generic critical element of the type* β_2. *Then there exist neighborhoods* B_2 *of* X *in* χ^r, F *of* p *in* M, *and a* C^{r-1} *function* $f: B_2 \to R$, *satisfying*:

(a) $f(Y) = 0$ *if and only if* Y *has one quasi-generic periodic trajectory that is tangent to* ∂M *only at the point* $p_Y \in F$ *and satisfies the* G *condition*; *if* $f(Y) \neq 0$ *then* Y *does not have a periodic trajectory meeting* F *and tangent to* ∂M;

(b) $df_X \neq 0$ (*see Fig.* 4).

Proof. Denote by γ_X the generic periodic trajectory of X tangent to ∂M at p, by τ_0 its period, and by $\Phi_X(p, t)$ its corresponding flow. Let $\tilde{X} \in \tilde{\chi}^r$ be a representative of X; obviously $\tau_X = \tau_{\tilde{X}}(p)$.

Take the neighborhoods $\tilde{F}_0$ of p in N, $\tilde{F}_0 \subset V$ and $\tilde{B}_0$ of $\tilde{X}$ in $\tilde{\chi}^r$ ($\tilde{V}$ and $\tilde{B}_0$ were given in 5.2); assume $\tilde{F}_0$ and $\tilde{B}_0$ are contained in $\tilde{F}_1$ and $\tilde{B}_1$ (given in 4.2), respectively; furthermore, if $\tilde{Y} \in \tilde{B}_0$ then its generic periodic trajectory contained in $\tilde{F}_1$ meets U transversely at a unique point $u_{\tilde{Y}}$; it is clear that the correspondence $\tilde{Y} \to u_{\tilde{Y}}$ is C^r.

Let $\tilde{G}: (\tilde{B}_0 \times J_0, (\tilde{X}, 0)) \to (R, 0)$ be a germ (C^r) defined by $\tilde{G}(\tilde{Y}, \tau) = \pi(\psi_{\tilde{Y}}(u_{\tilde{Y}}, \tau))$, where J_0 is an interval containing the origin and π was given in 4.2.

We have $\tilde{G}(\tilde{X}, 0) = (\partial\tilde{G}/\partial\tau)(\tilde{X}, 0) = 0$ and $(\partial^2\tilde{G}/\partial\tau^2)(\tilde{X}, 0) \neq 0$; this follows from the generic property of the contact between X and ∂M at p. By the Implicit Function Theorem, there are neighborhoods $\tilde{B}_2 \subset \tilde{B}_0$ of $\tilde{X}$, J of $\tau = 0$, and a unique C^{r-1} function $\tau: (\tilde{B}_2, \tilde{X}) \to (J, 0)$, such that $\tau(\tilde{Y}) = 0$, $(\partial\tilde{G}/\partial\tau)(\tilde{Y}, \tau) = 0$ if and only if $\tau = \tau(\tilde{Y})$; assume by continuity $(\partial^2\tilde{G}/\partial\tau^2)(\tilde{Y}, \tau(\tilde{Y})) < 0$ for $\tilde{Y} \in \tilde{B}_2$. Hence $\tau(\tilde{Y})$ is the maximum (nondegenerate critical point) of the mapping $\tau \to \tilde{G}(\tilde{Y}, \tau)$ for each $\tilde{Y} \in \tilde{B}_2$.

The function $\tilde{f}: (\tilde{B}_2, \tilde{X}) \to (R, 0)$ defined by $\tilde{f}(\tilde{Y}) = \tilde{G}(\tilde{Y}, \tau(\tilde{Y}))$ is C^{r-1} and $\tilde{Y} \in \tilde{f}^{-1}(0)$ if and only if $\tilde{Y}$ is tangent to ∂M at $P_{\tilde{Y}} = \Phi_{\tilde{Y}}(u_{\tilde{Y}}, \tau(\tilde{Y}))$.

Now, we will prove $d\tilde{f}_{\tilde{X}} \neq 0$.

First, consider the system of coordinates $y = (y_1, y_2)$ in a neighborhood $\tilde{F} \subset \tilde{F}_0$ of p, with $y_1(p) = y_2(p) = 0$, $(\partial/\partial y_1) = \tilde{X}$, $y_2 \circ u = id$, and $y_1 \circ u = 0$. If δ is a positive small number, let $\psi_1: \gamma_{\tilde{X}} \cap \tilde{F} \to R$ and $\psi_2: U \cap \tilde{F} \to R$ be C^∞ bump functions, having supports in $|y_1| < \delta$ and $|y_2| < \delta$, respectively.

We easily obtain $d\tilde{f}_{\tilde{X}}(Y) = (\partial\tilde{G}/\partial\tilde{Y})(\tilde{X}, 0)$.

Given the field $\tilde{Y} = \Psi_1\Psi_2(\partial/\partial y_2)$ in $\tilde{\chi}^r$, consider the C^r curve $h: (-\eta, \eta) \to \tilde{\chi}^r$ defined by $h(\lambda) = \tilde{Y}_\lambda = \tilde{X} + \lambda\tilde{Y}$. Clearly $\tilde{Y}_0 = \tilde{X}$ and $\tau(\tilde{Y}_\lambda) = 0$. By a known formula for the derivative of solutions of differential equations depending on parameters [3, p. 94] we have $(\partial\tilde{G}/\partial\tilde{Y})(\tilde{X}, 0) = (dG/d\lambda)(\tilde{X} + \lambda\tilde{Y})_{\lambda=0} \neq 0$. Therefore $d\tilde{f}_{\tilde{X}} \neq 0$.

Now, consider the neighborhood of X in χ^r, $B_2 = \{Y \in \chi^r$; there exists $\tilde{Y} \in \tilde{B}$ with $\tilde{Y}_{|M} = Y\}$ and the C^{r-1} function $f\colon B_2 \to R$ defined by $f(Y) = \tilde{f}(\tilde{Y})$ where $\tilde{Y} \in \tilde{B}_2$ is a continuous extension (at X) of $Y \in B_2$ [12, p. 67].

As $d\tilde{f}_{\tilde{X}} \neq 0$ we get $df_X \neq 0$ and the proof of part (a) of the lemma follows from the definition of the function $\tilde{f}$.

Proof of 5.1. Part (a) follows from 5.2, 5.3, and 5.4. It remains only to demonstrate part (b).

By an elementary technique, determine a neighborhood V of τ_X in M, satisfying:

(i) no periodic trajectory (except τ_X) and critical point of X meet V;

(ii) $M_2 = M - \mathrm{int}(V)$ is a C^∞ submanifold of M;

(iii) $X_{|M_2}$ is generic;

(iv) there is a unique point c_0 of tangency between ∂V and X, besides p (the trajectory passing through c_0 is contained in V and is different from τ_X);

(v) $\partial V = C_1 \cup C_2 \cup S_1$ (see Fig. 1), where $C_1 \cap \partial M = \Phi$, $C_2 \cup \partial M = \{v_1\} \cup \{v_2\}$, and $S_1 \subset S \subset \partial M$ (see 4.2); furthermore $c_0 \in C_2$.

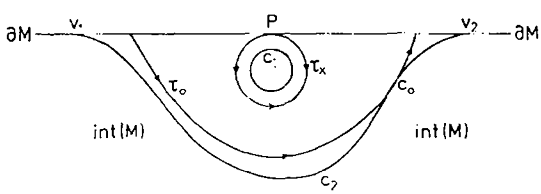

FIG. 1. The neighborhood V of τ_X.

V can be obtained such that any saddle separatrix and any trajectory (of X) that is tangent to ∂M meet C_2; if S is the arc of ∂M given in 4.2, we consider $s[-1, 1] = S$, $s(0) = p$, $S^- = s[1, 0)$, $S^+ = s(0, 1]$, $S_1^- = S_1 \cap S^-$, and $S_1^+ = S_1 \cap S^+$.

Let B_2 be the neighborhood of X in χ^r satisfying. If $Y \in B_2 \cap H_2$, then the generic periodic trajectory of Y, $\tau_y(\bar{p})$, which is tangent to ∂M at $\bar{p} \in S_1$, is contained in V and any trajectory of Y meeting V is transverse to C_1, C_2, and S, except at $\bar{p}$ and $\tilde{c}_0$, where $\tilde{c}_0 \in C_2$ is the corresponding point to c_0, associated to Y ($\tilde{c}_0$ is close to c_0).

As $X_{|M_2}$ is generic, there is a neighborhood B of X in χ^r, $B \subset B_2$, such that if $Y \in B \cap H_2$ then there exists a homeomorphism $h_2\colon M_2 \to M_2$ (close to the identity) mapping trajectories of $X_{|M_2}$ on to trajectories of $Y_{|M_2}$.

Necessarily $h_2(c_0) = \tilde{c}_0$ and we require $h_2(v_i) = v_i$, $i = 1, 2$; this is possible because each v_i is contained in a canonical region of $Y \in B$ (see definition of canonical region in [10, p. 8]).

Now we will construct a homeomorphism $h: M \to M$, which is a conjugacy of X with $Y \in B \cap H_2$; this homeomorphism will be an extension of h_2.

Consider the following subregions of V:

(a) V_1, bounded by C_1 and τ_X;

(b) V_2, bounded by τ_X and C_2 (see Fig. 2).

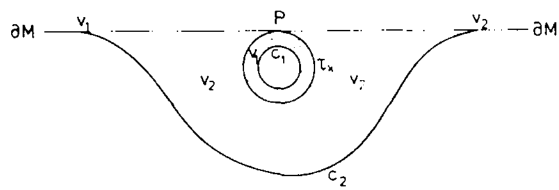

FIG. 2. The subregions V_1 and V_2 of V.

We begin by constructing h in V_1. Let Q be an arc in V_1 through to $q \in C_1$ and transverse to X; as h_2 is close to the identity, we determine an arc $\tilde{Q}$ (close to Q), joining $\tilde{p}$ to $h_2(q) = \bar{q}$, transverse to Y; necessarily $h(p) = \bar{p}$ and we define h for all the points of V_1 similarly to [22, p. 12] (note $h(V_1) = V_1$).

Let us construct h in V_2. We will determine three subregions (canonicals with respect to $X_{|V_2}$) in V_2 which will facilitate the above mentioned construction.

By the continuity of X, the trajectory of X passing through c_0 meets S_1^+ at c_2 and S_1^- at c_1. For $Y \in B \cap H_2$, there exist the correspondents $\tilde{c}_1$, $\tilde{c}_2$, and $\tilde{\gamma}_0$. We require $h(c_i) = \tilde{c}_i$, $i = 1, 2$. Thus γ_0 (resp. $\tilde{\gamma}_0$) determine in V_2 the following subregions (see Fig. 3):

(1) T_1 (resp. $\tilde{T}_1$): bounded by $(\overrightarrow{v_1 c_1})_{\partial M}$, $(\overrightarrow{v_1 c_0})_{\partial M_2}$, and $(\overrightarrow{c_0 c_1})_{\gamma_0}$ (resp. $(\overrightarrow{v_1 c_1})_{\partial M}$, $(\overrightarrow{v_1 c_0})_{\partial M_2}$, and $(\overrightarrow{\tilde{c}_0 \tilde{c}_1})_{\tilde{\gamma}_0}$);

(2) T_2 (resp. $\tilde{T}_2$): bounded by $(\overrightarrow{c_2 v_2})_{\partial M}$, $(\overrightarrow{c_0 v_2})_{\partial M}$, and $(\overrightarrow{c_2 c_0})_{\partial M_2}$ (resp. $(\overrightarrow{\tilde{c}_2 v_2})_{\partial M}$, $(\overrightarrow{\tilde{c}_0 v_2})_{\partial M_2}$, and $(\overrightarrow{\tilde{c}_2 \tilde{c}_0})_{\tilde{\gamma}_0}$);

(3) (resp. $\tilde{T}_3$): bounded by $(\overrightarrow{\tilde{c}_1 c_2})_{\partial M}$, $\gamma_X(p)$, and $(\overrightarrow{c_2 c_1})_{\gamma_0}$(resp. $(\overrightarrow{\tilde{c}_1 \tilde{c}_2})_{\partial M}$, $\gamma_Y(\tilde{p})$, and $(\overrightarrow{\tilde{c}_2 \tilde{c}_1})_{\tilde{\gamma}_0}$).

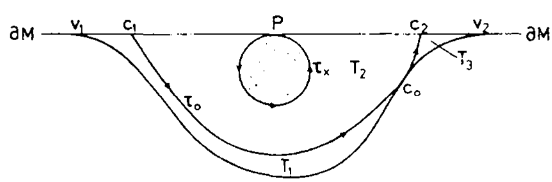

FIG. 3. The subregions T_i of V_2.

The critical region of $X_{|V_2}$ is formed by the union of $\gamma_X(p)$, γ_0, v_1, and v_2; we have similarly the critical region of $Y_{|V_2}$.

By the same techniques used at [10, p. 12] and [6, p. 153] we finally construct the homeomorphism h. By ratio of arc length we construct: $h[(\widehat{c_0c_1})_{\gamma_0}] = (\widehat{\tilde{c}_0\tilde{c}_1})_{\tilde{\gamma}_0}$; $h[(\widehat{c_2c_0})_{\gamma_0}] = (\widehat{\tilde{c}_2\tilde{c}_0})_{\tilde{\gamma}_0}$; $h[(\widehat{v_1c_1})] = (\widehat{v_1\tilde{c}_1})_{\partial M}$; and $h[(\widehat{c_2v_2})_{\partial M}] = (\widehat{\tilde{c}_2v_2})_{\partial M}$. We send T_i to its correspondent $\tilde{T}_i$; this is done in the following way:

On T_3: Let U be an arc in T_3, joining $q \in \gamma_X(p)$ to c_0, transverse to X and let K be an arc C^1, close to U joining $h_2(q) = q$ to $\tilde{c}_0$. By ratio of arc length, we construct $h[(\widehat{c_1, p})_{\partial M}] = (\widehat{\tilde{c}_1, \tilde{p}})_{\partial M}$. If $q_2 \in (\widehat{pc_2})_{\partial M}$ and $\gamma_X(q_2)$ meets $(\widehat{c_1p})_{\partial M}$ at q_1, we define $h(q_2) = \tilde{q}_2$, where $\tilde{q}_2$ is the intersection of $\gamma_Y(h(q_1))$ and $(\widehat{p\tilde{c}_2})_{\partial M}$. On U, h acts in the following manner: If $u \in U$, $\gamma_X(u)$ meets $u_1 \in (\widehat{c_1p})_{\partial M}$ and $u_2 \in (\widehat{pc_2})_{\partial M}$; assume $h(u) = \tilde{u}$, where $\tilde{u}$ is the intersection of $\gamma_Y(h(u_1))$ and $\bar{U}$. Now by a straightforward computation we construct h on T_3. Finally, by similar techniques, h is easily defined on T_1 and T_2.

Since every point of V_2 belongs to one trajectory, h is a one-to-one mapping of V_2 on to itself; it is continuous by the standard theorem on the continuous dependence of trajectories on initial data.

This ends the proof of 5.1.

5.5. *Remark.* Given any positive number $L > 0$, the neighborhood B of X may be taken, such that the length of every trajectory of $Y \in B$ is greater than L, in V_1; this is obvious by 5.2; furthermore any trajectory of $Y \in B$ meeting C_2 is transverse to ∂M in V.

5.6. *Remark.* Denote by $H_2(n)$ the set of $X \in H_2$ such that its periodic trajectory tangent to ∂M has length $L_0 < n$; by continuity arguments we verify that 5.1 holds for $H_2(n)$.

It is not difficult to prove the following.

5.7. PROPOSITION. *Denote by $\tilde{H}_2$ the subset of H_2, of fields X which satisfy the following additional axiom*: (3) *The periodic trajectory of X tangent to ∂M is neither the α nor the ω limit, of either the saddle separatrices or of the trajectory tangent to ∂M. Then*:

(a) *$\tilde{H}_2$ satisfies the M, E, and A conditions*;

(b) *$\tilde{H}_2' = H_2 - \tilde{H}_2$ is open in H_2*;

(c) *if $X \in \tilde{H}_2'$ then there exists a neighborhood B of X in χ^r, such that, if $Y \in B \cap H_2$ we have* (i) $Y \in \Sigma_0$, (ii) *Y has one unique saddle separatrix tangent to ∂M, or* (iii) *Y has one unique trajectory tangent to ∂M at two and only two points. Moreover, in* (ii) *and* (iii) *the contact between Y and ∂M is generic.*

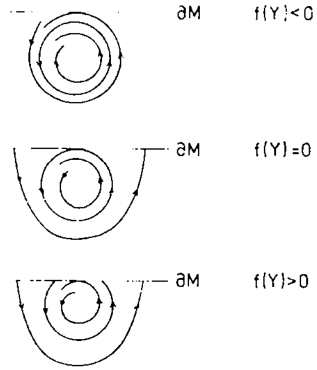

FIG. 4. The unfolding of $X \in H_2$.

6

6.1. PROPOSITION. *Denote by H_3 the set of fields $X \in \chi^r$, $r > 2$, such that:*

(1) *There is one point $p \in \partial M$, that is a quasi-generic critical element of X of type β_3, as the unique nongeneric critical element of X;*

(2) *X satisfies Ω_1, Ω_2, Ω_3, Ω_4.*

Then H_3 satisfies the M, E, and A conditions.

The proof of 6.1 depends on several lemmas.

6.2. LEMMA. *If $X \in H_3$, then there exists a neighborhood B of X in χ^r such that any $Y \in B$ satisfies Ω_1, Ω_2, Ω_3, Ω_4, B_1, B_2, B_4, B_5, and B_6.*

The proof of this lemma follows immediately from [5] and 4.4.
We can prove the next lemma in the same way as 5.4.

6.3. LEMMA. *Let $X \in \chi^r$, such that there exists one trajectory γ_X tangent to ∂M only at two points P_1 and P_2 ($P_1 \neq P_2$). Suppose the contact between X and ∂M at P_1 and P_2 is generic. Then, there exist neighborhoods B_3 of X in χ^r, F_i of P_i in N ($i = 1, 2$), and a C^{r-1} function $f\colon B_3 \to R$, such that:*

(a) *$f(Y) = 0$ if and only if the trajectory of Y is tangent to ∂M at two points $q_1 \in F_1$ and $q_2 \in F_2$, whose contact between the curve and the field is generic; if $f(Y) \neq 0$, then there exists a unique trajectory tangent to ∂M in F_1 (resp. F_2) at a unique point and it is not tangent to ∂M at any other point;*

(b) *$df_x \neq 0$ (see Fig. 5).*

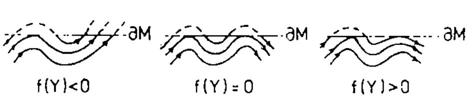

FIG. 5. The unfolding of $X \in H_3$.

Now, the proof of 6.1 is analogous to 5.1.

6.4. *Remark.* $Ad(H_3) \cap Q_2 \neq \Phi$ and $Ad(H_3) \cap H_2 \neq \Phi$.

6.5. *Remark.* Denote by $H_3(n)$ the subset of H_3 of fields X, such that γ_X has length $L < n$. Then 6.1 holds for $H_3(n)$.

7

The proof of the following proposition is similar to 5.1.

7.1. PROPOSITION. *Denote by H_4 the set of fields $X \in \chi^r$, $r > 2$, such that:*

(1) *there is a $p \in \partial M$, that is a quasi-generic critical element of X of type β_4 as a unique nongeneric critical element of X;*

(2) *X satisfies Ω_1, Ω_2, Ω_2, Ω_4. Then H_3 satisfies the M, E, and A conditions.*

It is convenient to state the following two lemmas.

7.2. LEMMA. *If $X \in H_4$, then there exists a neighborhood B of X in χ^r, such that every $Y \in B$ satisfies Ω_1, Ω_2, Ω_3, Ω_4, B_1, B_2, B_3, B_5, and B_6.*

We can prove the next lemma in the same way as 5.4.

7.3. LEMMA. *Let $X \in H_4$ have a saddle separatrix tangent to ∂M at only one point p. Then there exist neighborhoods B_4 of X in χ^r, F of p in M, and a C^{r-1} function F: $B_4 \to R$, such that:*

(a) *$f(Y) = 0$ if and only if there exists a saddle separatrix tangent to ∂M at only one point $P_Y \in F$, and satisfying the G condition with respect to the field; if $f(Y) \neq 0$ then there is no saddle separatrix of Y tangent to ∂M in F;*

(b) *$df_X \neq 0$ (see Fig. 6).*

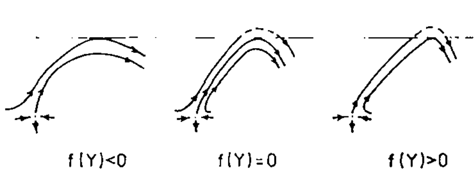

FIG. 6. The unfolding of $X \in H_4$.

7.5. *Remark.* $Ad(H_4) \cap Q_2 \neq \Phi$ and $Ad(H_4) \cap H_2 \neq \Phi$.

7.6. *Remark.* Denote by $H_4(n)$ the subset of H_4, of fields X such that the saddle separatrix tangent to ∂M has length $L < n$. Then 6.1 holds for $H_4(n)$.

8

8.1. Lemma. *Let $p \in \partial M$ be a simple critical point of $X \in \chi^r$. Then there exist neighborhoods B_0 of X in χ^r, F of p in M, and a C^r function $f: B_0 \to R$, such that:*

(a) *$f(Y) = 0$ if and only if Y has one unique critical point $p_Y \in \partial M \cap F$; furthermore p_Y is simple;*

(b) *if $f(Y) > 0$, Y has no critical point in F;*

(c) *if $f(Y) < 0$, Y has one unique simple critical point $p_Y \in F$, and $p_Y \in \text{int}(M)$.*

Proof. Choose $\tilde{X} \in \tilde{\chi}^r$ a representative of X, $\tilde{F}_1$ a neighborhood of p in N, and $\tilde{B}_0$ a neighborhood of $\tilde{X}$ in $\tilde{\chi}^r$ such that each $\tilde{Y} \in \tilde{B}_0$ has one unique critical point $p_{\tilde{Y}}$ in $\tilde{F}_1$, which is simple; it is clear that the correspondence $\tilde{Y} \to p_{\tilde{Y}}$ is C^r.

Define a C^r mapping $\tilde{f}: \tilde{B}_0 \to R$ by $\tilde{f}(\tilde{Y}) = \pi(p_{\tilde{Y}})$; it is obvious that $f(\tilde{X}) = 0$. Now we will prove that $d\tilde{f}_{\tilde{X}} \neq 0$.

Let $x = (x_1, x_2)$ be the system of coordinates around p given in 4.3. Let $\psi: N \to R$ be a C^∞ bump function with support in $F_\delta = \{q \in N \text{ with } |x(q)| < \delta\}$ $(\delta > 0)$ and $\psi(q) = 1$ if $|x(q)| < \delta$.

Since p is a simple critical point suppose, for simplicity, that $(\partial X^1/\partial x_1)(p) \neq 0$. The equality $d\tilde{f}_{\tilde{X}}(E) = \pi[(D\tilde{X}_p)^{-1}(Z(p))]$ (see [10, p. 24]) implies $d\tilde{f}_{\tilde{X}}(\tilde{V}) \neq 0$, where $\tilde{V} = \psi(\partial/\partial x_2) + (1 - \psi)\tilde{X}$.

Consider the neighborhood B_0 of X in χ^r given by $B_0 = \{Y \in \chi^r$ such that there exist $\tilde{Y} \in \tilde{B}_0$ and $\tilde{Y}_{|M} = Y\}$ and the C^r function $f: B_0 \to R$, defined by $f(Y) = \tilde{f}(\tilde{Y})$, where $\tilde{Y}$ is a continuous extension of Y (at X) in $\tilde{\chi}^r$. Now, the proposition follows immediately.

8.2. *Remark.* Denote by $H_1{}^1$ the set of fields $Y \in \chi^r$, such that:

(1) Y has one unique simple critical point $p_Y \in \partial M$ which is the unique nongeneric critical element of Y;

(2) Y satisfies Ω_2, Ω_3, Ω_4, and all the critical points of Y except p_Y are hyperbolic. Let D_1 be the subset of $H_1{}^1$ of fields Y satisfying the additional axiom: "The eigenvalues of DY_{p_Y} are real and are equals." Then $H_1{}^1 - D_1$ is open and dense $H_1{}^1$; this follows by considering the C^r function

$$g(Y) = \sigma^2(Y; p_y) - 4\Delta(Y; p_Y).$$

8.3. Proposition. *Denote by H_1 the set of fields $X \in \chi^r$, such that:*

(a) *there exists one point $p \in \partial M$, that is a quasi-generic critical element of type β_1 as the unique nongeneric critical element;*

(b) *X satisfies Ω_1, Ω_2, Ω_3, Ω_4. Then H_1 satisfies the M and E conditions.*

The proof of 8.3 depends on the following.

8.4. *Remark.* If $X \in H_1$, by condition (b) of the definition of the quasi-generic critical element of type β_1, there exists a neighborhood F of p in M, such that any trajectory of $X_{|F}$ meets ∂M transversally, does not meet ∂M, or if p is the α or ω limit of the trajectory, then "it tends transversally to ∂M at p."

8.5. *Remark.* Take $X \in H_1$, such that $p \in \partial M$, $X(p) = 0$ and $[\sigma^2(x, p) - 4\Delta(X, p)] < 0$. If s is the imbedding given in 4.3 ($s[-1, 1] = S$), the construction made in [10, pp. 24–25] implies that there exists a C^{r-1} diffeomorphism Θ_X of $S^- = s[-1, 0]$ on to $S^+ = s[0, 1]$ satisfying the conditions: $\Theta_X(s(-1)) = s(1)$, $\Theta_X(p) = p$, and, for each α, $s(\alpha)$ and $\Theta_X(s(\alpha))$ belong to the same trajectory; furthermore, every trajectory of X, except p, is transverse to $(S - \{p\})$.

8.6. Lemma. *If $X \in H_1$, then there exists a neighborhood B of X in χ^r, such that any $Y \in B$ satisfies Ω_1, Ω_2, Ω_3, and Ω_4.*

8.7. Lemma. *Every $X \in H$ has a neighborhood B_0 in H_1, such that if $Y \in B_0$, then*:

(a) *There exists a neighborhood F of p in M where given a trajectory of $Y_{|F}$, one of the following situations is possible*:

(i) *the given trajectory is the quasi-generic critical element $P_Y \in \partial M$*;

(ii) *the given trajectory meets ∂M transversally and*

(iii) *the given trajectory "tends to P_Y transversally to ∂M."*

(b) *If X has n and only n critical points* (*hyperbolics*) *in* int(M), *then Y has n and only n critical points* (*hyperbolics*) *in* int(M); *any $Y \in B_0$ satisfies Ω_1, Ω_2, Ω_3, Ω_4, B_2, B_3, B_4, B_5, and B_6.*

Part (a) of 8.7 follows by the transversality theory and parts (b) and (c) by [5] and 8.11.

Proof of 8.3. Proposition 8.3(a) is a direct consequence of 8.1, 8.2, 8.6, and 8.7. We will demonstrate part (b) of 8.3.

If $X \in H_1$, then we have the following possibilities: (0_1) p is a saddle point; (0_2) p is a nondegenerate node; (0_3) p is a generic focus. We will consider these cases separately.

(0_1) Let $\tilde{X} \in \tilde{\chi}^r$ be a representative of X. Denote by $\tilde{F}$ a neighborhood of p in N, such that $\tilde{X}_{\tilde{F}}$ is generic; hence the separatrices S_i, $i = 1, 2, 3, 4$ of p meet $\partial\tilde{F}$ transversally. These trajectories of $\tilde{X}$ determine four subregions $\tilde{T}_i$ of $\tilde{F}$ (see Fig. 7); let $T_i = \tilde{T}_i \cap M$ $(i = 1,..., 4)$, $F = \tilde{F} \cap M$, $L = \partial\tilde{F} \cap M$, and $L_2 = \tilde{F} \cap \partial M$. Assume there is no saddle separatrix different from S_i, no trajectory tangent to ∂M, and no periodic trajectory of X meeting F.

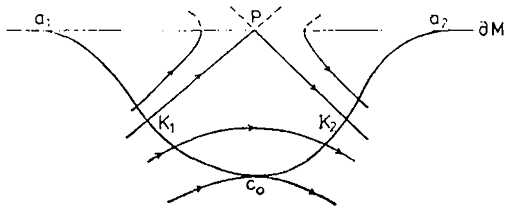

FIG. 7. The neighborhood F of a saddle point.

We know that $\tilde{F}$ can be chosen such that $\tilde{X}$ is tangent to $\partial\tilde{F}$ at only four points; assume without loss of generality that only one point c_0 belongs to M and $c_0 \in T_1$. Assume $L_2 \subsetneqq S$ (S, S^+, S^- given in 8.5) and X transversal to $S - \{p\}$ (see 8.9).

Consider $a_1 \in S^- \cap F^+$, $a_2 \in S^+ \cap F^+$ and assume for simplicity that if we go through $(\widehat{a_1a_2})_{L_1}$, we meet first one stable separatrix S_1 and then one unstable separatrix S_2, where $S_1 \cap L_1 = K_1$ and $S_2 \cap L_1 = K_2$.

We can assume that L_1 satisfies the conditions:

(i) there is a neighborhood B of X in χ^r, such that, if $Y \in B_0 \cap H_1$, then $p_Y \in F$;

(ii) the separatrices of p_Y, $\bar{S}_1$, and $\bar{S}_2$, corresponding to S_1 and S_2, respectively, meet L_1 in $\bar{K}_1$ and $\bar{K}_2$ transversally;

(iii) $M_2 = M - (\operatorname{int} F)$ is a C^∞ submanifold of M;

(iv) the contact between X and L_1 at c_0 is generic;

(v) X is transverse to L_2, except at p.

Since $X_{|M_2}$ is generic (by construction of M_2), there exists a neighborhood of X in χ^r, $\bar{B} \subset B_0$, such that, if $Y \in B \cap H$ then $Y_{|M_2}$ is conjugate to $X_{|M_2}$; so we have a homeomorphism (close to the identity) h: $M_2 \to M_2$ mapping trajectories of X_{M_2} onto those of $Y_{|M_2}$. In the process of the extension of h_2 to homeomorphism h: $M \to M$, conjugating X to Y, we note that the critical region of X_F is formed by the union of the following trajectories: a_1, a_2, p, c_0, S_1, and S_2 (see Fig. 7). Then we apply the technique of Peixoto [5] and we obtain without difficulties the homeomorphism h.

(0_2) Consider the following objects given below, and $\tilde{X}$, $\tilde{F}$, F, L_1, L_2, S, S^+, S^-, a_1, and a_2 given in (0_1). Call E_1 the eigenspace such that the trajectories of X, except one that we denote by γ_2, are tangent to it. F can be chosen such that X is transverse to L_1 except at one unique c_0 ($c_0 \notin \gamma_2$) and to L_2 except at p. Assume there is no saddle separatrix and no trajectory of X tangent to ∂M, meeting L_2; furthermore no periodic trajectory of X meets F. Finally we must observe that the critical region of $X_{|F}$ is formed by the union of a_1, a_2,

p, $\gamma_{X|_F}(c_0)$, and γ_2; for Y close to X, there exist the corresponding objects a_1, a_2, p_Y, $\gamma_{Y|_F}(\bar{c}_0)$, and $\tilde{\gamma}_2$, respectively. Then we use standard techniques to give the proof (see Fig. 8).

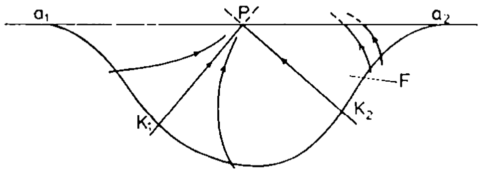

FIG. 8. The neighborhood F of a node.

(0_3) In the same same way as in (0_1) (or (0_2)) we can prove easily the case when $[(\sigma(X, p))^2 - 4\Delta(X, p)] < 0$ (see Fig. 9).

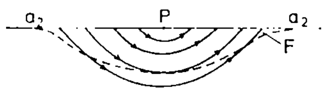

FIG. 9. The neighborhood F of a focus.

8.8. LEMMA. *H_1 is open in $\chi_1{}^r$.*

The proof of 8.8 depends on the lemmas given in 8.9.

8.9. *Remarks.* The following lemmas discuss the behavior of the trajectories of a field Y around a hyperbolic critical point, with respect to one given curve. Let V be a neighborhood of a point p of R^2 and let X be a field on R^2 of class C^r, $r > 2$, such that p is one unique singularity of $X|_V$; furthermore p is a hyperbolic critical point of X. Denote by λ_1, λ_2 the eigenvalues of DX_p and by T_1, T_2 their respectives eigenspaces. Consider $s: I = [-1, 1] \to R^2$ a C^∞ imbedding with $s(0) = p$ and $S = s(I)$.

8.9a. LEMMA. *Suppose $\lambda_1, \lambda_2 \in R$, $\lambda_1 \neq \lambda_2$ an S transversal to T_i, $i = i, 2$. Then there exist neighborhoods V_1 of p in R^2, $V_1 \subset V$ and B_1 of X in $\chi^r(R^2)$, such that:*

(i) *each $Y \in B_1$ has one unique hyperbolic singularity $P_Y \in V_1$ of the same kind as p;*

(ii) *there exists a C^r function $\alpha: B_1 \to R$, such that, if $P_Y \notin S \cap V_1$, then $Y(s(\alpha(Y)))$ is tangent to S at $s(\alpha(Y))$, for $Y \in B_1$;*

(iii) *the contact between Y and S in $s(\alpha(Y))$ is generic.*

Proof. It is known that there are neighborhoods B_0 of X in $\chi^r(R^2)$ and V_0

of p in R^2, such that each $Y \in B_0$ has one unique singularity P_Y in V_0. Consider the sets $S_0 = S \cap V_0$ and $I_0 = s^{-1}(S_0)$.

Define a C^r function $G: B_0 \times I_0 \to R$ by $G(Y, \alpha) = Y(s(\alpha)) \wedge s'(\alpha)$; it is obvious that $G(X, 0) = 0$.

Let $x = (x_1, x_2)$ be a system of coordinates around p (say in V_0) with $\partial/\partial x_i \in T_i$, $i = 1, 2$. In these coordinates the components of X, X_1, and X_2, satisfy

$$\frac{\partial X^1}{\partial x_2}(p) = \frac{\partial X^2}{\partial x_1}(p) = 0, \qquad \frac{\partial X^1}{\partial x_1}(p) = \lambda_1, \qquad \text{and} \qquad \frac{\partial X^2}{\partial x_2}(p) = \lambda_2.$$

If $s(\alpha) = (s_1(\alpha), s_2(\alpha))$, by hypothesis we have $s_1'(0) \neq 0$ and $s_2'(0) \neq 0$. Thus $G(Y, \alpha) = Y^1(s(\alpha))\, s_2'(\alpha) - Y^2(s(\alpha))\, s_1'(\alpha)$ and we get $(\partial G/\partial\alpha)(X, 0) = s_1'(0)\, s_2'(0)(\lambda_1 - \lambda_2)$.

Since $\lambda_1 \neq \lambda_2$, then $(\partial G/\partial\alpha)(X, 0) \neq 0$.

By the Implicit Function Theorem, there exist neighborhoods B_1 of X in $\chi^r(R^2)$ $(B_1 \subset B_0)$, I_1 of $\alpha = 0$ in R $(I_1 \subset I_0)$, and a C^r function $\alpha: B_1 \to I_1$, such that $\alpha(X) = 0$ and $G(Y, \alpha) = 0$ if and only if $\alpha = \alpha(Y) = \alpha_Y$.

If $Y(s(\alpha_Y)) \neq 0$, then this vector and $s'(\alpha_Y)$ are linearly dependents.

The above assertions imply (i) and (ii); part (iii) follows immediately from $(\partial G/\partial\alpha)(Y, \alpha_Y) \neq 0$. This ends the proof of 8.9a.

8.9b. Lemma. *Suppose λ_i complex, $i = 1, 2$. Then there exist neighborhoods, V_1 of p in V, B_1 of X in $\chi^r(R^2)$, such that:*

(i) *each $Y \in B_1$ has one unique singularity p_Y in V that is hyperbolic and of the same kind as p;*

(ii) *there exists a C^r function $\alpha: B_1 \to R$, such that if $p_Y \notin S \cap V_1$ then $Y(s(\alpha(Y)))$ is tangent to S at $s(\alpha(Y))$, $Y \in B_1$;*

(iii) *the contact between Y and S at $s(\alpha(Y))$ is generic.*

Proof. Let S_0, I_0, V_0, and G be the objects given in the last demonstration.

Let $x = (x_1, x_2)$ be a system of coordinates around p (say in V_0), with $\partial/\partial x_1 = s'(0)$. Thus we have $\partial X^1/\partial x_1 = \partial X^2/\partial x_2 = \alpha$ and $\partial X^1/\partial x_2 = -\partial X^2/\partial x_1 = \beta$ $(\alpha \neq 0, \beta \neq 0)$. In the same way as 8.9a we prove this lemma without difficulties.

8.9c. Lemma. *Suppose p is a hyperbolic critical point of $X \in \chi^r(R^2)$, such that $\lambda_1 < \lambda_2 < 0$ (or $0 < \lambda_1 < \lambda_2$). Let $s: I \to R^2$ be the imbedding given in* 8.11a *having the following property. There exists one unique saddle (hyperbolic) separatrix γ_X, of length $L < \infty$, such that p is its ω limit and $\gamma_X \cap S = \Phi$. Then there exist neighborhoods V_1 of p in R^2 and B_1 of X in $\chi^r(R^2)$, such that:*

(i) *each $Y \in B_1$ is transverse to ∂V_1 (∂V_1 is a C^∞ curve);*

(ii) *each* $Y \in B_1$ *has a saddle separatrix* γ_X *meeting* ∂V_1 *at one unique point* ω_Y *and the correspondence* $Y \to \omega_Y$ *is* C^r;

(iii) $s(\alpha(Y)) \notin \gamma_Y$, *where* $s(\alpha(Y))$ *is the point of* S *obtained in* 8.11a.

Proof. Parts (i) and (ii) follow by [5], and its verification is similar to [9, Lemma 4.3, p. 27].

Consider V_1 and B_1 given in (i) and (ii) and satisfying Lemma 8.9a; assume S is transverse to ∂V_1 and $(V_1 - S)$ and has two connected components, S_1 and S_2 (see Fig. 10).

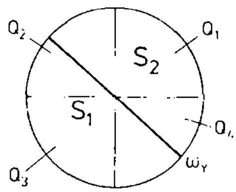

FIG. 10. The neighborhood V_1 .

Since $\omega_X = \gamma_X \cap \partial V_1 \notin S$, assume by continuity that $\omega_Y \notin S \cap \partial V_1$ for every $Y \in B_1$.

Fix in V_1 the coordinates $x = (x_1, x_2)$ around p, given in 8.9a; for $E > 0$, by [2, p. 90], V_1 can be chosen such that $| X^1(q)/X^2(q)| < E$ for $q \in V_1$ and $\gamma_X(q)$ is not tangent to T_1 ; so this inequality holds for $Y \in B_1$ and q does not belong to the trajectory of Y close to T_1 [2, p. 87]. Observe that

$$0 < K_1 < | s_2'(\alpha)/s_1'(\alpha) < \infty, \qquad \text{for} \quad \alpha \in I.$$

Assume T_1 and T_2 determine in V_1 four quadrants Q_i , $i = 1, 2, 3, 4$ (see Fig. 10).

Assume, for simplicity, that $S \cap Q_1 \cup Q_3 = p$ and $\omega_Y \in Q_4 \cap S_1$; we will analyze the cases:

(1) If $p_Y \in S$ then the demonstration is trivial.

(2) If $p_Y \in S_2 \cap Q_1$, since this point is the ω limit of γ_Y , then $\gamma_Y \cap S = A_y$ has (a) one unique point or (b) two points, at least. If (a) occurs then $s(\alpha(Y)) \in A_y$, since the contact between Y and S in this case is generic. If (b) occurs then the continuity of Y in S implies the existence in S of two points of tangency between the field and curve, and this is an absurdity.

(3) If $p_Y \in Q_3 \cap S_1$ then γ_Y does not meet S for Y close enough to X; so $s(\alpha(Y)) \notin \gamma_Y$.

The other cases are similar.

8.9d. LEMMA. *Lemma* 8.9c *holds if* γ_X *is the unique trajectory tangent to an imbedded curve in* R^2 (*distinct from* S) *at the unique point* q *where the contact is generic.*

Proof of 8.8. If p is a saddle point (case (0_1)), consider neighborhoods B of X in χ^r and F of p in M given in 8.6 and satisfying:

(i) no saddle separatrix of $Y \in B$, except the ones of p_Y, meets F;

(ii) no trajectory of $Y \in B$, tangent to ∂M, meets F. This is possible since the numbers of points of tangency between X and ∂M and critical points of the field are finite.

Lemma 8.9a permits us to choose B and F such that if $Y \in B$ and $f(Y) \neq 0$, then there exists one unique trajectory γ_Y of Y tangent to ∂M at $q_Y \in F$, generically: so $Y \in B$ satisfies B_5 and B_6.

Since the conditions Ω_1, Ω_2, Ω_3, Ω_4, B_1, B_2, B_3, and B_4 are trivially satisfied for Y close enough to X and $f(Y) \neq 0$, we have $Y \in \Sigma_0$ for $Y \in B$ and $f(Y) \neq 0$.

Using 8.9 we can finish this demonstration without difficulties; i.e., there exists a neighborhood of X in χ^r, such that every $Y \in B$ either belongs to H_1 or belongs to Σ_0.

9

9.1. PROPOSITION. *Denote by* H_5 *the set of fields* $X \in \chi^r$, *such that*:

(1) *there exists* $p \in \partial M$, *that is a quasi-generic critical element of* X *of type* β_5 *as a unique nongeneric critical element of* X;

(2) X *satisfies* Ω_1, Ω_2, Ω_3, *and* Ω_4. *Then* H_5 *satisfies the* M, E, *and* A *conditions.*

We have to state two preliminary lemmas.

9.2. LEMMA. *Every* $X \in H_5$ *has a neighborhood* B *in* χ^r, *such that every* $Y \in B$ *satisfies* Ω_1, Ω_2, Ω_3, Ω_4, B_2, B_3, *and* B_4.

9.3. *Remark.* Lemma 9.4 proves in particular that B can be chosen such that if $Y \in B$ then Y satisfies B_6.

9.4. LEMMA. *Let* $X \in \chi^r$, $r > 3$, *having a trajectory* γ_X *tangent to* ∂M *at the unique point* p *where the contact between the curve and the field is quasi-generic. Then there exist neighborhoods* B_5 *of* X *in* χ^r, F *of* p *in* M, *and a* C^{r-1} *function* $f: B_5 \to R$, *such that*:

(a) $f(Y) = 0$ *if and only if* Y *has a trajectory* γ_Y *tangent to* ∂M *at the unique point* $p_Y \in F$, *satisfying the Q.G. condition with respect to* Y; *if* $f(Y) > 0$, *then any trajectory of Y meeting F is transverse to ∂M in F; if $f(Y) < 0$ then there exist two and only two distinct trajectories of Y meeting F, each one tangent to ∂M at one point and both satisfying the G condition with respect to Y;*

(b) $df_X \neq 0$ (*see Fig.* 2).

Proof. Consider the neighborhoods B_0 of X in χ^r, and F of p in M, such that no $Y \in B_0$ has a critical point in F.

Define the C^r germ $G: (B_0 \times R, (X, 0)) \to (R, 0)$ by $G(Y, \alpha) = Y(s(\alpha)) \wedge s'(\alpha)$, where s is the imbedding given in 4.2. We have

$$\frac{\partial G}{\partial \alpha}(y, \alpha) = \frac{d}{d\alpha}[Y(s(\alpha))] \Lambda s'(\alpha) - Y(s(\alpha)) \Lambda s''(\alpha)',$$

$$\frac{\partial^2 G}{\partial \alpha^2}(Y, \alpha) = \frac{d^2}{d\alpha^2}[Y(s(\alpha))] \Lambda s'(\alpha) - 2\frac{d}{d\alpha} Y(s(\alpha)) \mid \Lambda s''(\alpha) + Y(s(\alpha)) \Lambda s'''(\alpha).$$

By a direct calculation, we obtain $(\partial G/\partial\alpha)(X, 0) = 0$ and $(\partial^2 G/\partial\alpha^2)(X, 0) \neq 0$ (we used here the quasi-generic property of the contact between X and ∂M at p).

Thus by the Implicit Function Theorem, there exist neighborhoods B_5 of X in χ^r ($B_5 \subset B_0$), J of $\alpha = 0$ in R, and a C^{r-1} function $\alpha: B_5 \to J$, satisfying $\alpha(X) = 0$ and $(\partial G/\partial\alpha)(Y, \alpha) = 0$ if and only if $\alpha = \alpha(Y) = \partial_Y$. Assume for simplicity that $(\partial^2 G/\partial\alpha^2)(X, 0) > 0$ (the other case is similar). Choose B_5 and J such that $(\partial^2 G/\partial\alpha^2)(Y, \alpha) > 0$ for $(Y, \alpha) \in B_5 \times J$.

So α_Y is the minimum of $g_Y(\alpha) = G(Y, \alpha)$ for each $Y \in B_5$, and:

(i) if $g_Y(\alpha_Y) > 0$ then $g_Y(\alpha) > 0$, $\alpha \in J$; this means that Y is transverse to ∂M around p in M;

(ii) if $g_Y(\alpha_Y) = 0$, then $g_Y(\alpha) = 0$ ($\alpha \in J$) only if $\alpha = \partial_Y$;

(iii) if $g_Y(\partial_Y) < 0$, by the Intermediate Value Theorem there exist $\alpha_1, \alpha_2 \in R$, $\alpha_1 < \alpha_Y < \alpha_2$, such that $g_Y(\alpha_1) = g_Y(\alpha_2) = 0$; however, $(\partial G/\partial\alpha)(Y, \alpha_i) \neq 0$, $i = 1, 2$.

If $g_Y(\alpha_Y) = 0$ and $(\partial G/\partial\alpha)(Y, \alpha_Y) = 0$ then the contact between Y and ∂M at $s(\alpha_Y)$ is nongeneric; $(\partial^2 G/\partial\alpha^2)(Y, \alpha_Y) \neq 0$ implies that the contact is quasi-generic.

If $g_Y(\bar\alpha) = 0$ and $(\partial G/\partial\alpha)(Y, \bar\alpha) \neq 0$ ($\bar\alpha \in J$), then the contact noted above at $s(\bar\alpha)$ is generic.

The application $f(Y) = G(Y, \alpha_Y)$ shows part (a) of 9.4. We will prove $df_X \neq 0$.

We have $f(X) = 0$ and $df_X(Y) = dG_{(X,0)}(Y, 0) + (\partial G/\partial\alpha)(X, 0)\, d\alpha_X(Y)$.

Since $(\partial G/\partial\alpha)(X, 0) = 0$, we need only show that

$$dG_{(X,0)}(Y, 0) \neq 0.$$

Consider coordinates around p in M, $y = (y_1, y_2)$ with $y(p) = 0$, $\partial/\partial y_1 = X$ and a bump function $\psi: M \to R$ with support in $|y(q)| \leqslant \delta$, $\delta > 0$ and small enough; furthermore $\psi(q) = 1$ for $|y(q)| < \delta_1$, with $0 < \delta_1 < \delta$.

If $Y = \psi(\partial/\partial y_2)$, consider the C^r curve $h: [-\eta, \eta] \to \chi^r$ defined by $h(\lambda) = X + \lambda Y$; call $h(\lambda) = Y_\lambda$. In coordinates $Y = (1, \lambda)$ and $G(Y_\lambda, 0) = \lambda$. This proves 9.4.

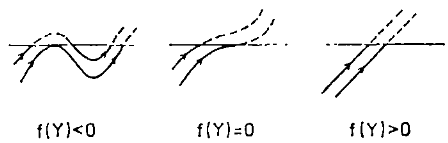

FIG. 11. The unfolding of $X \in H_5$.

Proof of 9.1. The M and A conditions follow from 9.2, 9.3, and 9.4, while the E condition is demonstrated by already known methods.

Part 3. *The Submanifold* Σ_1

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Consider the sets $S_i = Q_1 \cup Q_2(i) \cup Q_3(i) \cup H_1 \cup H_2(i) \cup H_3(i) \cup H_4(i) \cup H_5$ and $\Sigma_1 = \bigcup_{j=1}^{3} Q_j \bigcup_{k=1}^{5} H_k$. By 1.2, 2.1, 3.4, 5.6, 6.5, 7.6, 8.3, and 9.1 each S_i ($i = 0, 1, 2, \ldots$) satisfies the M condition; since $S_i \subset S_{i+1}$ and $\Sigma_1 = \bigcup_i S_i$, this subset of χ^r satisfies the I condition.

Proof of Theorem A. The above considerations guarantee us the existence of Σ_1. Part (b) follows from 1.1, 2.1, 3.1, 5.1, 7.1, 8.3, and 9.1. Part (a) follows from a sequence of approximations similar to those used in [5] (to get the density of Σ_0 in χ^r) and [22]. By a straightforward computation one proves the following lemmas.

LEMMA A. *Denote by Q_2^0 the set of fields $X \in \chi^r$ having nongeneric periodic trajectories contained in* int(M). *Then Q_2 is dense in Q_2^0.*

LEMMA B. *Denote by Q_1^0 the set of fields $X \in \chi_1^r$ having nongeneric critical points contained in* int(M). *Then Q_1 is dense in Q_1^0.*

LEMMA C. *Denote by Q_3^0 the set of fields $X \in \chi_1^r$ which have saddle connections (contained in* int(M)) *or nontrivial recurrent orbits, and all the field's critical points and periodic trajectories are in* int(M). *Then $Q_1 \cup Q_2 \cup Q_3$ is dense in Q_3^0.*

LEMMA D. *Denote by H_3^0 the set of fields $X \in \chi_1^r$ having periodic trajectories tangent to ∂M. Then $H_2^0 \subset Ad(H_2 \cup Q_2)$.*

LEMMA E. *Denote by* H_1^0 *the set of fields* $X \in \chi_1^r$ *having critical points in* ∂M. *Then* $H_1^0 \subset Ad(H_1 \cup Q_1)$.

LEMMA F. *Denote by* H_3^0 *the set of fields* $X \in \chi_1^r$ *having trajectories tangent to* ∂M *in more than one point, none of then being periodic or saddle separatrix. Then* $H_3^0 \subset Ad(H_3)$.

LEMMA G. *Denote by* H_4^0 *the set of fields* $X \in \chi_1^r$ *having saddle separatrices tangent to* ∂M. *Then* $H_4 \subset Ad(H_1 \cup H_4 \cup Q_1 \cup Q_3)$.

LEMMA H. *Denote by* H_5^0 *the set of fields* $X \in \chi_1^r$ *having one point* $p \in \partial M$ *such that it does not satisfy the G condition with respect to X. Then* $H_5^0 \subset Ad(\Sigma_1)$.

Since $\chi_1^r = Q_1^0 \cup Q_2^0 \cup Q_3^0 \cup H_1^0 \cup H_2^0 \cup H_3^0 \cup H_4^0 \cup H_5^0$, Lemmas A–H imply immediately that Σ_1 is dense in χ_1^r.

11. *Final Remarks*

11.1. *Remark.* Denote by $\tilde{\Sigma}_1$ the set of first-order structurally stable vector fields of χ^r (see the definition in [10, p. 35]). Then $\tilde{\Sigma}_1 = Q_1 \cup \tilde{Q}_2 \cup \tilde{Q}_3 \cup H_1 \cup \tilde{H}_2 \cup H_3 \cup H_4 \cup H_5$; furthermore $\tilde{\Sigma}_1$ satisfies the M and A conditions. This follows by 1.4, 2.1, 3.3, 5.7, 6.1, 7.1, 8.8, and 9.1.

11.2. *Remark.* Let $I = [a, b]$ be a closed interval. Denote by Φ^r the space of C^1 mappings $\xi: I \to \chi^r$, with the C^1 topology. We say $\lambda_0 \in J$ is an ordinary value of $\xi \in \Phi^r$ if there is a neighborhood N of λ_0 such that $\xi(\lambda)$ is topologically equivalent to $\xi(\lambda_0)$ for every $\lambda \in N$; if λ_0 is not an ordinary value of ξ, it is called a bifurcation value of ξ. Obviously, if $\xi(\lambda_0) \in \Sigma_0$, (resp. $\xi(\lambda_0) \in \chi_1^r$), λ_0 is an ordinary (resp. bifurcation) value of ξ. If ξ is transverse to Σ_1 then every $\lambda_0 \in \xi^{-1}(\Sigma_1)$ is a bifurcation value of ξ.

11.3. *Remark.* We say ξ_1 and ξ_2 of Φ^r are conjugate if there is a homeomorphism $h: I \to I$ and a map $H: I \to$ homeo.(M), such that $H(\lambda)$ is a conjugation between $\xi_1(\lambda)$ and $\xi_2(h(\lambda))$ (homeo.(M) denotes the group of homeomorphisms of M). With this concept of conjugacy, the structural stability in Φ^r is defined in an obvious way. Let us denote by A^r, the collection of the elements $\xi \in \Phi^r$ such that:

(1) $\xi(I) \subset \Sigma_0 \cup \tilde{\Sigma}_1$;

(2) ξ is transversal to $\tilde{\Sigma}_1$;

(3) $\xi(a)$ and $\xi(b)$ are in Σ_0. We have the result, "Any $\xi \in A^r$ is structuraly stable."

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