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Spin Models, Association Schemes and the Nakanishi–Montesinos Conjecture

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A 3-transformation of a link is a local change which replaces two strings that are three times half twisted around each other by two untwisted strings (and vice versa). The Nakanishi–Montesinos (NM) conjecture asserts that this 3-transformation can unknot any link. We introduce the notion of the NM-spin model, which gives a link invariant preserved by 3-transformation. We try to classify such spin models and determine the corresponding link invariant. It is proved that the dimension of the Bose–Mesner algebra generated by the spin model is ≤ 4. For dimension 1 and 2, there is no such spin model, but for dimension 3, there exists a unique one. Its link invariant is a non-trivial specialization of the Kauffman polynomial, but does not distinguish trivial links from the others, and hence cannot disprove the NM conjecture. For dimension 4, we give a family of NM-spin models. The corresponding link invariant is identified and does not distinguish trivial links from the others. Strong regularity and triple regularity of the Bose–Mesner algebra and its fusions are studied.

1. INTRODUCTION

This paper is intended to study spin models in relation with the Nakanishi–Montesinos (NM) conjecture. Spin models [2, 6, 9, 12] considered here consist of a pair (W_+, W_-) of square matrices satisfying some constraints called invariance equations. To every diagram, we associate a weighted graph with edge weight in $\{W_+, W_-\}$. Viewing vertices as atoms and weighted edges as interaction between atoms, we compute the so-called partition function of the system in the vein of statistical mechanics. Invariance equations insure that the partition function is an invariant of link. It turns out that $\{W_+, W_-\}$ generate a self-dual Bose–Mesner (BM) algebra [1] of an association scheme, which gives us algebraic and combinatorial tools to study such objects.

A 3-transformation is a local change in a link that exchanges two untwisted strings and two strings that twist around each other with three crossings (see Figure 1).

The NM conjecture [16] asserts that if we are allowed to perform 3-transformations, we can reduce any link to an unknotted collection of unknots. In terms of spin models, 3-transformations give four additional very simple invariance equations and conditions on the BM algebra; in particular we prove that its dimension is at most 4. The game is then to find solutions to these equations and to study the corresponding invariants. There is a unique solution when the associated BM algebra is two- or three-dimensional and then the corresponding invariant is a specialization of the Kauffman polynomial which has a cohomological interpretation [15], but it turns out not to distinguish trivial links from non-trivial ones. Concerning dimension 4, the situation is far more complicated. We exhibit a class of solutions whose corresponding invariant is the same as in dimension 2 or 3. We prove that the graphs of the association scheme related to any solution are all strongly regular, and that the scheme is not triply regular (except when its cardinality is 9). Long proofs have been omitted, they can be found in [5].

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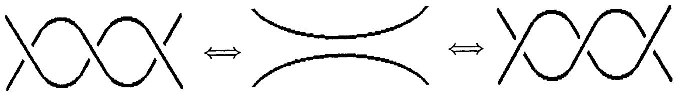

FIGURE 1. 3-Transformations.

$+1$ -1

FIGURE 2. Tait value of a crossing.

2. BACKGROUND: SPIN MODELS FOR LINK INVARIANTS

2.1. Links. A detailed introduction to links can be found in [12]. A *link* is a finite collection of mutually disjoint simple closed curves (the *components* of the link) in 3-space. An *oriented* link is a link with an orientation assigned to each component. Two links are said to be *ambient isotopic* if there exists an isotopic deformation of the ambient 3-space which carries one onto the other. For oriented links, it is required in addition that the isotopic deformation respects the orientation of each component. All the links from now on will be assumed to be tame, i.e., ambient isotopic to a link whose components are simple closed polygons, and will be denoted L with various indices.

2.2. Link diagrams, Tait number and Reidemeister's theorem. Every link can be represented by a *diagram*. This is a projection of the link on a plane which has a finite number of multiple points, each of which is a simple crossing. Near each crossing an obvious pictorial convention specifies which segment of the link goes under the other. Moreover, for oriented links the orientation of the components is indicated by arrows in the natural way. Throughout this article, D will denote a diagram.

The *Tait number* $T(D)$ of the oriented diagram D is the sum of the values of its crossings, where the value of a crossing is defined in Figure 2.

The following famous theorem gives a combinatorial reformulation of ambient isotopy. A proof can be found in [6].

THEOREM 1 (REIDEMEISTER). *Two non-oriented diagrams represent ambient isotopic links if and only if one can be obtained from the other by a finite sequence of moves represented on Figure* 3.

A move is performed by replacing a part of a diagram which is one of the configurations of Figure 3 by an equivalent configuration without modifying the remaining part of the diagram. The same result holds for oriented links if each move is replaced by a corresponding set of oriented moves defined in the obvious way.

2.3. Graph associated to a diagram. With every diagram D one can associate a (unoriented, finite, plane, possibly with loops and multiple edges) graph $G(D)$ as follows (see Figure 4): colour the plane open regions delimited by the diagram with two colours, black and white, so that no two adjacent regions receive the same colour, and so that the unbounded region is white. It is easy to see that such a colouring always exists. We call $B(D)$ the set of

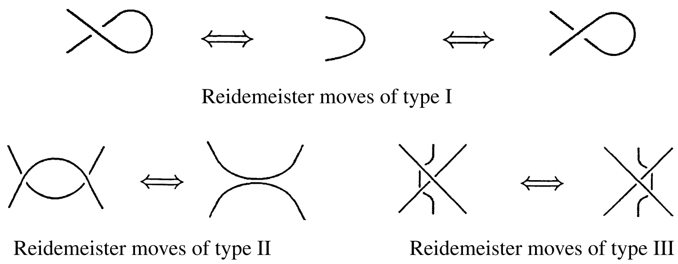

FIGURE 3. Reidemeister moves.

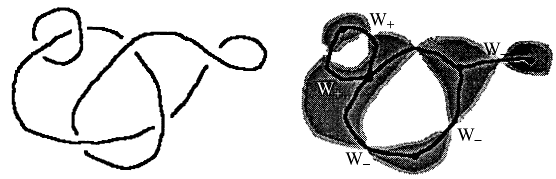

FIGURE 4. The graph associated to a link.

black regions. Take one point in each black region (the *capital* of the region), these are the vertices of $G(D)$. Then draw one edge through each crossing, adjacent to the two capitals of the concerned black regions. Draw the edges in such a way that you obtain a plane embedding of $G(D)$.

2.4. Matrix weighted graphs and partition functions. Let X be a set of size n. Let $\mathcal{S}_X(\mathbb{C})$ denote the set of $n \times n$ symmetric matrices indexed by X. Let $G = (V, E)$ be an unoriented multigraph. Itf w is a mapping from the multiset E to $\mathcal{S}_X(\mathbb{C})$, we say that (G, w) is a *matrix weighted graph*. A *state* of (G, w) is a mapping σ from the vertex set V to X. The *weight of an edge* $e = \{x, y\}$ with respect to σ is the $(\sigma(x), \sigma(y))$ entry of $w(e)$. The *weight of a state* σ is the product of the edge weights with respect to σ over all the edges of G (it will be set to 1 if G has no edge). Eventually, the *partition function* of the matrix weighted graph (G, w) is the sum of the state weights over all states. In other words, the partition function $Z(G, w)$ of the matrix weighted graph (G, w) is

$$Z(G, w) = \sum_{\sigma: E \mapsto X} \prod_{e=\{x,y\} \in E} w(e)[\sigma(x), \sigma(y)]. \tag{1}$$

EXAMPLES.

- Set $w(e) = J - I$ for all $e \in E$. Then $Z(G, w)$ counts the number of proper n-colourings of G.
- Set $w(e) = H$ for all $e \in E$, where H is the adjacency matrix of the graph H.

Then $Z(G, w)$ counts the number of morphisms from G to H.

2.5. Symmetric spin models. Jones [12] proposed the following construction to obtain link invariants. Let D be a link diagram, and $W_+ W_-$ two matrices of $\mathcal{S}_X(\mathbb{C})$.

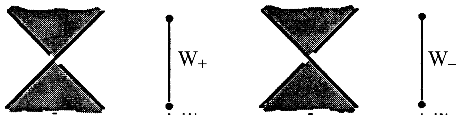

FIGURE 5. Edge assignment at a crossing.

We distinguish two situations that can occur at a crossing. For each of these, we choose $w(e)$ among W_+, W_- according to Figure 5. $(G(D), w)$ is thus a matrix weighted graph. Then we have the following.

THEOREM 2. *Let $a \in \mathbb{C}^*$ and q a square root of $|X|$.*
If w is defined as earlier, and $W_+, W_- \in \mathcal{S}_X(\mathbb{C})$ satisfy the following conditions

$$I \circ W_+ = aI, \qquad I \circ W_- = a^{-1} I, \tag{2}$$
$$JW_+ = W_+J = qa^{-1}J, \qquad JW_- = W_-J = qaJ, \tag{3}$$
$$W_+W_- = nI, \tag{4}$$
$$W_+ \circ W_- = J, \tag{5}$$

(Star-triangle equation) for every α, β, γ in X,

$$\sum_{x \in X} W_+[\alpha, x] W_+[\beta, x] W_-[\gamma, x] = q W_+[\alpha, \beta] W_-[\beta, \gamma] W_-[\gamma, \alpha], \tag{6}$$

then the so-called normalized partition function defined by

$$Z'(D, W_+, W_-, a, q) = a^{-T(D)} q^{-|B(D)|-1} Z(G(D), w)$$

is an invariant of oriented links.

For a sketchy demonstration see [10, 13]. Note that only the normalization factor depends on the orientation of the link. A 5-tuple (X, W_+, W_-, a, q) where $a \neq 0$, $q^2 = |X|$ and W_+, W_- are matrices of $\mathcal{S}_X(\mathbb{C})$ satisfying (2)–(6) will be called a *symmetric spin model*, or SM for short. To deal with disconnected link diagrams, we should replace $|B(D)|$ by the Euler characteristic $\chi_b(D)$ of the set of black faces of D (of course, in the case of connected diagrams, these two definitions coincide).

2.6. Association schemes and Bose–Mesner algebras. The following facts concerning association schemes will be necessary in the sequel (see [1]). Let X be a set of cardinality n and let $R_i (i \in \{1, 2, \ldots, n\})$ be subsets of $X \times Y$ with the property that:

(i) $R_0 = \{(x, x) \mid x \in X\}$,
(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$
(iii) No R_i is empty and $R_i \cap R_j = \emptyset$ if $i \neq j$,
(iv) $\forall i \in \{0, 1, \ldots, d\}$, R_i is symmetric,
(v) For $i, j, k \in \{0, 1, \ldots, d\}$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is constant whenever $(x, y) \in R_k$. This constant is denoted by p_{ij}^k. Such a configuration is called a (*commutative*) *symmetric d-class association scheme on X*. Symmetry implies that $p_{ij}^k = p_{ji}^k$. The *adjacency matrices* A_i, $i \in \{0, 1, \ldots, d\}$ are defined by $A_i[x, y] = 1$ if $(x, y) \in R_i$ and $A_i[x, y] = 0$ if $(x, y) \notin R_i$. Then Eqns (i)–(v) can be reformulated as follows:

(i′) $A_0 = I$,
(ii′) $\sum_{0\le i\le d} A_i = J$
(iii′) $A_i \neq 0$ and $A_i \circ A_j = \delta_{i,j} A_i$,
(iv′) $\forall i \in \{0, 1, \ldots, d\}$, A_i is symmetric,
(v′) $A_i A_j = A_j A_i = \sum_{0\le i\le d} p_{ij}^k A_k$.

Let $\mathcal{A}$ denote the subspace of $\mathcal{M}_X(\mathbb{C})$ spanned by the matrices A_i, $i = 0, 1, \ldots, d$. By (iii′), these matrices are linearly independent and hence form a basis of $\mathcal{A}$. Then (iii′) and (ii′) imply that, under the Hadamard product, $\mathcal{A}$ is an associative commutative algebra with unit J, and the A_i's form a basis of orthogonal idempotents for this algebra. By (i′) and (v′), we deduce that under the ordinary matrix product, $\mathcal{A}$ is also an associative commutative algebra with unit I. The algebra $\mathcal{A}$ is called the *symmetric Bose–Mesner algebra* (or *SBM-algebra* for short) of the association scheme. Conversely, a classical result (see [3]) states that a $(d+1)$-dimensional subspace of $\mathcal{M}_X(\mathbb{C})$ which contains I and J, consisting of symmetric matrices, closed under the ordinary and Hadamard matrix product, is the SBM-algebra of some symmetric d-class association scheme. Such a subspace is called a *symmetric BM algebra* (or *SBM algebra* for short) *on* X.

2.7. Duality. A *duality* of a SBM algebra $\mathcal{A}$ is a linear map ψ from $\mathcal{A}$ (viewed as a vector space) to itself which satisfies the following properties:

$$\forall M \in \mathcal{A},\ \psi(\psi(M)) = nM \tag{7}$$

$$\forall M, N \in \mathcal{A}, \psi(MN) = \psi(M) \circ \psi(N). \tag{8}$$

It easily follows that

$$\psi(I) = J \qquad \text{and} \qquad \psi(J) = nI, \tag{9}$$

$$\forall M, N \in \mathcal{A}, \psi(M \circ N) = \frac{1}{n}\psi(M)\psi(N). \tag{10}$$

A SBM-algebra will be called *self-dual* if it admits a duality. A classical result in linear algebra asserts that every SBM-algebra also has a (unique up to order) basis of idempotents for the ordinary matrix product, that will be denoted $(E_0, E_1, \ldots, E_d)$ with $E_0 = \frac{1}{n}J$. By (8), the $\psi(E_i)$ form a basis of orthogonal idempotents for the Hadamard product, and by unicity of such a basis, the $\psi(E_i)$ are the A_i in some order. By an appropriate indexing, we can take $\psi(E_i) = A_i$. Let P be the matrix of ψ with respect to the basis $(E_0, E_1, \ldots, E_d)$. P is called the *first eigenmatrix* of the association scheme. This matrix contains all the information about $\mathcal{A}$ viewed as an abstract algebra. Indeed, the following easy computation gives a formula expressing the multiplication table of the A_i as a function of the p_{ij}:

$$\begin{aligned}
(A_i A_j) \circ A_k &= \left(\left(\sum_u p_{ui} E_u\right) \sum_v p_{vj} E_v\right) \circ A_k \\
&= \left(\sum_u p_{ui} p_{uj} E_u\right) \circ A_k \\
&= 1/q^2 \left(\sum_u p_{ui} p_{uj} \sum_s p_{su} A_s\right) \circ A_k \\
&= 1/q^2 \left(\sum_u p_{ui} p_{uj} p_{ku}\right) A_k,
\end{aligned}$$

whence $p_{ij}^k = 1/q^2\left(\sum_u p_{ui} p_{uj} p_{ku}\right)$. The first row of P is related to the valency of the A_i viewed as graphs: we have $A_i E_0 = p_{i0} E_0$ so that p_{i0} is the eigenvalue of A_i corresponding to the all-one eigenvector.

Let us say that a subset of $\mathcal{M}_X(\mathbb{C})$ is *weakly Bose–Mesner* (WBM for short) if it is closed under complex linear combinations, the ordinary matrix product, the Hadamard product, and contains I, J. Thus SBM-algebras on X are exactly the WBM subsets of $\mathcal{M}_X(\mathbb{C})$ consisting only of symmetric matrices.

Clearly $\mathcal{M}_X(\mathbb{C})$ is WBM, and the intersection of WBM subsets is again WBM. Thus every subset F of $\mathcal{M}_X(\mathbb{C})$ has unique *WBM-closure* $Cl(F)$, which is the smallest WBM subset of $\mathcal{M}_X(\mathbb{C})$ containing it. The importance of self-dual SBM-algebras in the study of spin models is revealed by the following theorem, due to Jaeger (Theorem B of [11]):

THEOREM 3. *Let (X, W_+, W_-, a, q) be a symmetric spin model. $Cl(\{W_+\})$ is a SBM-algebra, containing W_+, W_-. Moreover, $Cl(\{W_+\})$ is self-dual and the map ψ: $\mathcal{M}_X(\mathbb{C}) \longrightarrow \mathcal{M}_X(\mathbb{C})$ defined by $\psi(M) = aW_- \circ (W_+(W_- \circ M))$ induces a duality on $Cl(\{W_+\})$.*

It follows from Eqns (3) and (5), that ψ as defined in this theorem satisfies

$$\psi(W_+) = qW_- \tag{11}$$

$$\psi(W_-) = qW_+. \tag{12}$$

3. THE RESULTS

3.1. 3-transformation and the Nakanishi–Montesinos conjecture. A *3-transformation* is performed by selecting a region in a link diagram inside which the diagram takes one of the forms described in Figure 1, and then replacing this local configuration by an equivalent one without changing the rest of the diagram.

Two links L, L' are *$3T$-equivalent* if the diagram of L can be obtained from the diagram of L' by a finite sequence of 3-transformations and non-oriented Reidemeister moves. The following conjecture asserts that the 3-transformation is an unknotting operation.

CONJECTURE (NAKANISHI–MONTESINOS). *Every link is $3T$-equivalent to a link consisting of unknotted trivial knots.*

3.2. NM-invariants and NM-spin models. We are now interested in spin models that lead to a partition function invariant under non-oriented Reidemeister moves and 3-transformations. Such an invariant will be called *NM-invariant*. The invariants considered so far were invariants of oriented diagrams. Due to 3-transformations being incompatible with orientation, we have to find invariants of non-oriented diagrams.

One way to make Z' (defined in Theorem 2) an invariant of an unoriented diagram is to use in the normalization factor the sum (denoted T') of signs of *self-crossings* of the different components instead of the Tait number T: to compute T', one has to sum the signs of crossings where the two strings involved belong to the same component, instead of summing over all crossings as for T. So let Z'' be defined by

$$\forall D, Z''(D, W_+, W_-, a, q) = a^{-T'(D)} q^{-\chi_b(D)-1} Z(D, W_+, W_-, a, q).$$

We can now state the following

PROPOSITION 4. *Let (X, W_+, W_-, a, q) be a symmetric spin model. If*

$$a = \pm 1 \tag{13}$$
$$W_+^{\circ 3} = aJ \tag{14}$$
$$W_+^3 = aq^3 I \tag{15}$$
$$W_-^3 = aq^3 I \tag{16}$$
$$W_-^{\circ 3} = aJ \tag{17}$$

then $Z''(D, W_+, W_-, a, q)$ is a NM-invariant.

In fact, the following lemma shows that no generality is lost taking $a = 1$.

LEMMA 5. *$(X, W_+, W_-, 1, q)$ is a spin model iff $(X, -W_+, -W_-, -1, q)$ is a spin model. Then*

$$\forall D, Z''(D, -W_+, -W_-, -1, q) = Z''(D, W_+, W_-, 1, q).$$

A symmetric spin model $(X, W_+, W_-, 1, q)$ *satisfying conditions* (14)–(17) *will be called* a *NM-spin model.*

REMARK. Though there might exist spin models inducing a NM-invariant without (13)–(17), such objects would be exceptional and it is reasonable to define NM-spin models as above (a tedious calculation, which we do not reproduce here for the sake of conciseness, illustrates the fact that (13)–(17) are natural conditions in this sense).

COROLLARY 6. *Let $(X, W_+, W_-, 1, q)$ be a NM-spin model. Then the self-dual SBM-algebra $Cl(\{W_+\})$ has dimension at most* 4. *More precisely $Cl(\{W_+\})$ is the vector space $\langle I, J, W_+, W_- \rangle$.*

PROOF. By Theorem 3, $Cl(\{W_+\})$ is a self-dual SBM-algebra. Eqns (5) and (14) yield $W_+ \circ W_+ = W_-$. Similarly, (4) and (15) yield $W_+^2 = qW_-$. □

In the sequel $d + 1$ will denote the dimension of $Cl(\{W_+\})$. We are interested in NM-spin models with $d > 0$, because $d = 0$ implies $n = 1$ and the associated NM-invariant is trivial. More generally, we shall suppose $n \geq 2$ to avoid the degenerate case $n = 1$.

3.3. $\dim Cl(\{W_+\}) \leq 3$. First assume the algebra $Cl(\{W_+\})$ is two-dimensional. So W_+ and W_- are linear combinations of I and J. Let $W_+ = aI + b(J - I)$. Eqn (1) gives $a = 1$, (12) gives $b^3 = 1$, then (2) gives $q = 1 + b(n - 1)$. As $n \geq 2$, we have a contradiction. Thus there is no NM-spin model generating a two-dimensional BM algebra.

If $d = 2$, we denote by $(A_0 = I, A_1, A_2)$ the basis of idempotents for the Hadamard product of $Cl(\{W_+\})$. We have the following result:

PROPOSITION 7. *If* $\dim Cl(\{W_+\}) = 3$ *then* $q = -3$ *and the underlying graphs of* A_1 *and* A_2 *are isomorphic to the lattice graph* $L(2, 3)$.

We now have to study the invariant $Z'' : D \longrightarrow Z''(D, W_+, W_-, 1, -3)$ associated with this NM-spin model. It turns out to be a specialization of the well known *Kauffman invariant*, denoted F'_ϵ. The Kauffman invariant is a normalization of the Kauffman polynomial F_ϵ. For $\epsilon \in \{+1, -1\}$, F_ϵ is a mapping from the class of unoriented diagrams to the ring $\mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$, which is invariant under Reidemeister moves of type II and III, takes the value 1 on the diagram consisting of a single component with no crossing (denoted $\bigcirc$), and satisfies the rules:

$$F_\epsilon(\bigcirc) = 1 \tag{18}$$
$$F_\epsilon(D') = a^{-1} F_\epsilon(D), \qquad F_\epsilon(D'') = a^1 F_\epsilon(D) \tag{19}$$
$$F_\epsilon(D_+) + F_\epsilon(D_-) = z(F_\epsilon(D_0) + \epsilon F_\epsilon(D_\infty)) \tag{20}$$

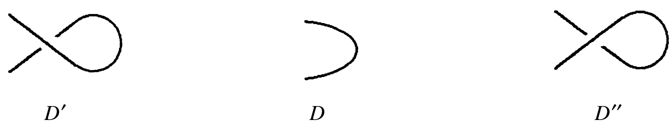

FIGURE 6. D', D and D''.

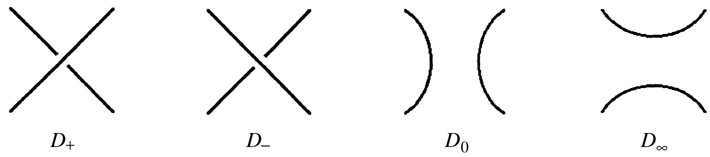

FIGURE 7. D_+, D_-, D_0 and D_∞.

where $\bigcirc$ is the diagram consisting of a single component with no crossing, D, D', D'' are identical diagrams except in a small disk depicted on Figure 6, and D_+, D_-, D_0, D_∞ are identical diagrams except in a small disk depicted on Figure 7.

The case $\epsilon = -1$ is called the *Dubrovnik form*, and the two forms are equivalent up to a change of variable [14]. We obtain from F_ϵ the invariant F'_ϵ of oriented links by normalizing by $a^{-T(L)}$.

PROPOSITION 8. *For every diagram D,* $Z''(D) = [F'_1(D)]_{(a=1.z=-1)}$.

For a complete characterization of the Kauffman polynomial specializations coming from spin models, see [8]. It is easy to see that the value of Z'' on p disjoint trivial knots is $(-3)^{p-1}$. The following proposition due to Brandt, Lickorish and Millet [15] asserts that Z'' is *trivial*, in the sense it does not distinguish the true links modulo 3-transformation from the unknots modulo 3-transformation. The contrary would have refuted the conjecture.

PROPOSITION 9. *For every diagram D,* $[F_1(D)]_{(a=1,z=-1)}$ *is a power of* -3.

The exponent of -3 is the dimension of the first homology with $\mathbb{Z}_3$ coefficients of the double cover of S^3 branched over L.

The following interpretation of this NM-spin model will be useful in the sequel: let Q_h be the quadratic form $\mathbb{Z}_3 \times \mathbb{Z}_3$ defined by $Q_h(x) = x_1^2 + x_2^2$ with $x = (x_1, x_2)$. We can consider that W_+ and W_- are indexed by $\mathbb{Z}_3 \times \mathbb{Z}_3$. It is easy to see that up to a permutation of the indices, we have $W_+[x, y] = \omega^{Q_h(x-y)}$ and $W_-[x, y] = \omega^{-Q_h(x-y)}$.

3.4. An infinite family of examples with $\dim Cl(\{W_+\}) = 4$. The only known (to us) example of a four-dimensional SBM-algebra generated by a NM-spin model is a generalization of the solution found in dimension 2 which can be constructed as follows (this example was communicated by François Jaeger).

Let $\omega = e^{2i\pi/3}$, let Q be a non-degenerate quadratic form in m variables on $GF(3)$, $X = GF(3)^m$, and let W_+, W_- be the matrices indexed by X such that $W_+[x, y] = \omega^{Q(x-y)}$ and $W_-[x, y] = W_+[x, y]^{-1}$.

PROPOSITION 10. *If m is even there exists q a square root of* $|X| = 3^m$ *such that* $(X, W_+, W_-, 1, q)$ *is a NM-spin model, and* $\dim Cl(\{W_+\}) = 4$ *unless* $m = 2$ *and* $Q : (x, y) \mapsto x^2 + y^2$.

The proof of this proposition will give explicit values for q. When q has such a value, the associated invariant does not distinguish true links from trivial ones.

PROPOSITION 11. *Let Q be a non-degenerate quadratic form on $GF(3)^m$ (m even), W_+, W_- defined as above and q defined as in the proof of Proposition* 10. *Then for every diagram D, there exists a positive integer p such that $Z''(D, W_+, W_-, q) = Z''(p \cdot \bigcirc, W_+, W_-, 1, q)$ where $p \cdot \bigcirc$ denotes the disjoint union of p unknots.*

The previous proposition is used to show that Z'' does not depend on the quadratic form Q. Then taking $Q = Q_h$ and applying Proposition 8, we get the following

PROPOSITION 12. *For every Q, $Z''(D, W_+, W_-, 1, q) = [F_1'(D)]_{(a=1,z=1)}$.*

3.5. Parameters of $Cl(\{W_+\})$ when $\dim Cl(\{W_+\}) = 4$. Let $(X, W_+, W_-, 1, q)$ be a NM-spin model such that $\dim Cl(\{W_+\}) = 4$, and let $(A_0 = I, A_1, A_2, A_3)$ be a basis of idempotents (for the Hadamard product) of $Cl(\{W_+\})$.

LEMMA 13. *Let $(\alpha, \beta, \delta, \gamma) \in \mathbb{C}^4$ be the coordinates of W_+ in the basis (A_0, A_1, A_2, A_3). Then $\alpha = 1$ and $\{\beta, \delta, \gamma\} = \{1, \omega, \omega^2\}$.*

By reindexing the A_i's if necessary, we can suppose that $\beta = 1$, $\delta = \omega$ and $\gamma = \omega^2$. Then we can compute the parameters of $Cl(\{W\})$:

PROPOSITION 14. *The first eigenmatrix of the SBM-algebra $Cl(\{W_+\})$ is*

$$P = \begin{bmatrix} 1 & (q-1)(q+3)/3 & q(q-1)/3 & q(q-1)/3 \\ 1 & 2q/3-1 & -q/3 & -q/3 \\ 1 & -q/3-1 & -q/3 & 2q/3 \\ 1 & -q/3-1 & 2q/3 & -q/3 \end{bmatrix} \tag{21}$$

and the p_{ij}^k's are (with obvious notations):

$$p_{ij}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (q-1)(q+3)/3 & 0 & 0 \\ 0 & 0 & q(q-1)/3 & 0 \\ 0 & 0 & 0 & q(q-1)/3 \end{pmatrix},$$

$$p_{ij}^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & (q^2+6q-18)/9 & q^2/9 & q^2/9 \\ 0 & q^2/9 & q(q-3)/9 & q^2/9 \\ 0 & q^2/9 & q^2/9 & q(q-3)/9 \end{pmatrix},$$

$$p_{ij}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & q(q+3)/9 & q^2/9-1 & q(q+3)/9 \\ 1 & q^2/9-1 & q^2/9 & q(q-3)/9 \\ 0 & q(q+3)/9 & q(q-3)/9 & q(q-3)/9 \end{pmatrix},$$

$$p_{ij}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & q(q+3)/9 & q(q+3)/9 & q^2/9-1 \\ 0 & q(q+3)/9 & q(q-3)/9 & q(q-3)/9 \\ 1 & q^2/9-1 & q(q-3)/9 & q^2/9 \end{pmatrix}.$$

Each p_{ij}^k being a non-negative integer, q must be divisible by 3, and $q \geq 3$ or $q \leq -9$. The study of strong and triple regularity of the underlying graphs of the A_i's, in the next section, will lead us to the following strengthening of these conditions:

PROPOSITION 15. *q is divisible by* 9, *and odd. Furthermore, A_1 (resp. A_2, A_3) is of type C.*19 *of* [7], *for $m = q/3$ (resp. $m = q + 3$).*

3.6. Regularity properties. We now state two combinatorial results which are a digression from the study of NM-spin models. These results might be useful, but we have not been able to make use of them to classify NM-spin models.

The first one concerns the strong regularity of the underlying graphs of the scheme. It is an easy fact that in a 2-class association scheme, the graphs of the relations are strongly regular. In fact this situation extends to association schemes for NM-spin models:

PROPOSITION 16. *Each of the underlying graphs of A_1, A_2, A_3 is strongly regular, with respective parameters* $\left(\frac{q^2+2q-3}{3}, \frac{q^2+6q-18}{9}, \frac{q^2+3q}{9}\right)$, $\left(\frac{q^2-q}{3}, \frac{q^2}{9}, \frac{q^2-3q}{9}\right)$ *and* $\left(\frac{q^2-q}{3}, \frac{q^2}{9}, \frac{q^2-3q}{9}\right)$.

Then we easily obtain the following

COROLLARY 17. *The algebras $\langle A_0, A_1, A_2 + A_3\rangle$, $\langle A_0, A_1 + A_2, A_3\rangle$, $\langle A_0, A_1 + A_3, A_2\rangle$ are SBM-algebras.*

The second result deals with the so-called triple regularity. Let α, β, γ be elements of X. Say they form a uvw-triangle if (α, β) is in the relation R_u, (α, γ) in R_v, and (β, γ) in R_w. Denote by $K_{ijk}^{\alpha\beta\gamma}(uvw)$ the number of $x \in X$ such that $(x, \alpha) \in R_i$, $(x, \beta) \in R_j$, and $(x, \gamma) \in R_k$ (in this case we say that α, β, γ, x form an *i, j, k-star*). In other words,

$$\sum_{x \in X} A_i[\alpha, x] \cdot A_j[\beta, x] \cdot A_k[\gamma, x] = K_{ijk}^{\alpha\beta\gamma}(uvw).$$

An association scheme is said to be *triply regular* if there exists an integer $K_{ijk}(uvw)$ such that for any $\alpha, \beta, \gamma \in X$ forming a uvw-triangle, $K_{ijk}^{\alpha\beta\gamma}(uvw)$ is independent of α, β, γ and is equal to $K_{ijk}(uvw)$. The BM-algebra of the association scheme will also be said to be *triply regular*. In some way, triple regularity is a higher order version of the property (v) of the association schemes.

In the presence of triple regularity, we may verify the Star-Triangle equation using the constant of triple regularity, for example, Nomura was able to prove that every Hadamard graph gives rise to a spin model in a uniform manner by using triple regularity. Without triple regularity, there is no general technique to verify the Star-Triangle equation without checking the sum.

The triple regularity property is nice for other reasons. For example, it was shown in [9] that the problem of finding a spin model in a triply regular and self-dual scheme reduces the problem to verifying certain properties (planar duality and planar reversibility) on a small example (the complete graph on four vertices). Furthermore, the scheme corresponding to a spin model whose link invariant is one of the classical polynomial link invariants (Homfly of Kauffmann) is triply regular. Indeed, the skein relations satisfied by these polynomials give rise to a linear equation among W_+, W_-, I and J, which in turn implies that the corresponding Bose–Mesner algebra has dimension at most three. The following result of Jaeger can be found in [11].

PROPOSITION 18. *Any three-dimensional SBM-algebra generated by a spin model is triply regular.*

Unfortunately, we shall see that NM-spin models give rise to association schemes which are not triply regular (except when $q = 3$). This shows that examples of NM-spin models should be difficult to produce.

We first state a lemma, which proves that in the presence of triple regularity there are some non-trivial relations among the constants of triple regularity.

LEMMA 19. *Let X, $R_0, \ldots, R_d$ define a d-class association scheme. Assume it is triply regular and let p_{ij}^k, $K_{ijk}(uvw)$ be defined as above. Then*

$$\forall i, k, u, v, w \in \{0, 1, 2, 3\}, \qquad p_{vw}^u K_{ijk}(uvw) = p_{ij}^u K_{vwk}(uij). \tag{22}$$

As a consequence, we have, as announced:

PROPOSITION 20. *Let $(X, W_+, W_-, 1, q)$ be a NM-spin model with $q \neq 3$. Then $Cl(\{W_+\})$ is not triply regular.*

In fact we can do a little better; the following proposition implies Proposition 20:

PROPOSITION 21. *Let $(X, W_+, W_-, 1, q)$ be a NM-spin model with $q \neq 3$, and let A_0, A_1, A_2, A_3 be the Hadamard idempotents of the SBM-algebra $Cl(\{W_+\})$. Then the SBM-algebra $\langle A_0, A_1, A_2 + A_3\rangle$ is not triply regular.*

We have not been able to decide whether the two other SBM-algebras $\langle A_0, A_1 + A_2, A_3\rangle$ and $\langle A_0, A_1 + A_3, A_2\rangle$ are triply regular. But from the previous result it is possible to answer a natural question in our framework.

COROLLARY 22. *The SBM-algebra $\langle A_0, A_1, A_2 + A_3\rangle$ is not generated by a spin model when $q \neq 3$.*

PROOF. As $\langle A_0, A_1, A_2 + A_3\rangle$ is not triply regular, Proposition 18 implies that it is not generated by a spin model. □

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