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Amenability and co-amenability of algebraic quantum groups II

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Abstract

We continue our study of the concepts of amenability and co-amenability for algebraic quantum groups in the sense of A. Van Daele and our investigation of their relationship with nuclearity and injectivity. One major tool for our analysis is that every non-degenerate *-representation of the universal C*-algebra associated to an algebraic quantum group has a unitary generator which may be described in a concrete way.

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1. Introduction

The present paper is a continuation of our previous paper [3], where we initiated a study of the concepts of amenability and co-amenability for algebraic quantum groups (see also [2]). We gave there several equivalent formulations of co-amenability and showed that co-amenability of an algebraic quantum group (A, Δ) always implies amenability of its dual algebraic quantum group $(\hat{A}, \hat{\Delta})$. We also obtained some results concerning the relationship between co-amenability of

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(A, Δ), injectivity of the von Neumann algebra M associated to (A, Δ) and amenability of $(\hat{A}, \hat{\Delta})$. The algebra M is the von Neumann algebra generated by A_{r}, where $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ denotes the analytic extension of (A, Δ).

One may construct a unique universal C*-algebraic quantum group $(A_{\mathrm{u}}, \Delta_{\mathrm{u}})$ associated to an algebraic quantum group (A, Δ) (see [9]). We show in Section 4 of the present paper that co-amenability of (A, Δ) is equivalent to the fact that the canonical homomorphism from A_{u} onto A_{r} is injective (see [2] for the compact case). This generalizes a classical result in the case that (A, Δ) is the algebraic quantum group associated to the group algebra of a discrete group. Further, we establish the following result, which is also well-known in the group algebra case (see [14]):

Theorem 1.1. *Let (A, Δ) be an algebraic quantum group.*
Consider the following statements:

(1) (A, Δ) *is co-amenable*
(2) $(\hat{A}, \hat{\Delta})$ *is amenable*
(3) A_{u} *is nuclear*
(4) A_{r} *is nuclear*
(5) M *is injective*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$.
If (A, Δ) is compact and has a tracial Haar functional, then we also have $(5) \Rightarrow (1)$, *that is, all statements above are equivalent.*

The main new part of this result is the fact that (2) implies (3). It is possible to deduce that (1) implies (3) from Ng's paper [13] on Hopf C*-algebras. His proof is related to the one given by Blanchard ([4], see also [1]) in the setting of regular multiplicative unitaries. Our proof is quite different and relies on the characterization of the nuclearity of a C*-algebra B in terms of the injectivity of B^{**}. The equivalence between (1), (2) and (5) in the compact tracial case may be deduced from Ruan's main result (Theorem 4.5) in [15]. We propose a proof that (5) implies (1) in this case, which we believe is somewhat more accessible than his. The interesting question as to whether any (or all) of the statements (2),(3),(4) or (5) always implies (1) seems quite hard to answer. As a pendant to this question, we show that for a compact (A, Δ), injectivity of M always implies a kind of "perturbed" co-amenability, involving the notion of quantum dimension (of irreducible unitary corepresentations).

An important tool in our approach is the fact that any non-degenerate $*$-representation of A_{u} on some Hilbert space has a unitary "generator" which may be described in a concrete way. We present a self-contained proof of this Kirchberg-type result in Section 3. Similar results (using the universal corepresentation and a certain L^1-algebra) have been previously obtained by Kustermans in [9,10]. The classical result of Kirchberg for Kac algebras may be found in [7].

The reader should consult [3] for an introduction to this subject, including a more extensive list of references to related papers. Section 2 contains a review of most of the necessary background material required for understanding the present paper.

2. Preliminaries

We recall in this section some definitions and results from [2,11,18,19]. We also prove some technical lemmas that we need later on.

We begin with some terminology that will be used throughout the paper.

Every algebra will be a (not necessarily unital) associative algebra over the complex field $\mathbf{C}$. The identity map on a set V will be denoted by ι_V, or simply by ι, if no ambiguity is involved.

If V and W are linear spaces, V' denotes the linear space of linear functionals on V and $V \otimes W$ denotes the linear space tensor product of V and W. The *flip map* from $V \otimes W$ to $W \otimes V$ is the linear map sending $v \otimes w$ onto $w \otimes v$, for all $v \in V$ and $w \in W$. If V and W are Hilbert spaces, $V \otimes W$ denotes their Hilbert space tensor product; we denote by $B(V)$ and $B_0(V)$ the C*-algebras of bounded linear operators and compact operators on V, respectively. If $v \in V$ and $w \in W$, $\omega_{v,w}$ denotes the weakly continuous bounded linear functional on $B(V)$ that maps x onto $(x(v), w)$. We set $\omega_v = \omega_{v,v}$. We will often also use the notation ω_v to denote a restriction to a C*-subalgebra of $B(V)$ (the domain of ω_v will be determined by the context).

If V and W are algebras, $V \otimes W$ denotes their algebra tensor product. We sometimes denote this algebraic tensor product by $V \odot W$ if we feel there is some danger of confusion. If V and W are C*-algebras, then $V \otimes W$ will denote their C*-tensor product with respect to the minimal (spatial) C*-norm. If V and W are von Neumann algebras, then $V \bar{\otimes} W$ will denote their von Neumann algebra tensor product.

For a review of some results related to multiplier algebras, especially multiplier algebras of C*-algebras, and to slice maps, we refer to [2]. We will use repeatedly these results and also most of the terminology introduced in this paper. For the ease of the reader and to fix notation, we recall here some of the basic definitions and properties of algebraic quantum groups.

Let A be a non-degenerate $*$-algebra and let Δ be a non-degenerate $*$-homomorphism from A into the multiplier algebra $M(A \otimes A)$. Suppose that the following conditions hold:

(1) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
(2) The linear mappings defined by the assignments $a \otimes b \mapsto \Delta(a)(b \otimes 1)$ and $a \otimes b \mapsto \Delta(a)(1 \otimes b)$ are bijections from $A \otimes A$ onto itself.

Then the pair (A, Δ) is called a *multiplier Hopf $*$-algebra*.

In Condition (1), we are regarding both maps as maps into $M(A \otimes A \otimes A)$, so that their equality makes sense. It follows from Condition (2), by taking adjoints, that the maps defined by the assignments $a\otimes b\mapsto(b\otimes 1)\Delta(a)$ and $a\otimes b\mapsto(1\otimes b)\Delta(a)$ are also bijections from $A\otimes A$ onto itself.

Let (A,Δ) be a multiplier Hopf $*$-algebra and let ω be a linear functional on A and a an element in A. There is a unique element $(\omega\otimes\iota)\Delta(a)$ in $M(A)$ for which

$$(\omega\otimes\iota)(\Delta(a))b=(\omega\otimes\iota)(\Delta(a)(1\otimes b))$$

and

$$b(\omega\otimes\iota)(\Delta(a))=(\omega\otimes\iota)((1\otimes b)\Delta(a))$$

for all $b\in A$. The element $(\iota\otimes\omega)\Delta(a)$ in $M(A)$ is determined similarly. Thus, ω induces linear maps $(\omega\otimes\iota)\Delta$ and $(\iota\otimes\omega)\Delta$ from A to $M(A)$.

There exists a unique non-zero $*$-homomorphism ε from A to $\mathbf{C}$ such that, for all $a\in A$,

$$(\varepsilon\otimes\iota)\Delta(a)=(\iota\otimes\varepsilon)\Delta(a)=a.$$

The map ε is called the *co-unit* of (A,Δ). Also, there exists a unique anti-multiplicative linear isomorphism S on A that satisfies the conditions

$$m(S\otimes\iota)(\Delta(a)(1\otimes b))=\varepsilon(a)b$$

and

$$m(\iota\otimes S)((b\otimes 1)\Delta(a))=\varepsilon(a)b$$

for all $a,b\in A$. Here $m:A\otimes A\to A$ denotes the linearization of the multiplication map $A\times A\to A$. The map S is called the *antipode* of (A,Δ). The antipode is in general neither $*$-preserving, nor involutive; however, we have $S(S(a^*)^*)=a$ for all $a\in A$. We also have $\Delta S=\chi(S\otimes S)\Delta$, where χ denotes the (induced) flip map on $M(A\otimes A)$.

If $\omega\in A'$, we say ω is *left invariant* if $(\iota\otimes\omega)\Delta(a)=\omega(a)1$, for all $a\in A$. Right invariance is defined similarly. If a non-zero left-invariant linear functional on A exists, it is unique, up to multiplication by a non-zero scalar. A similar statement holds for a non-zero right-invariant linear functional. If φ is a left-invariant functional on A, the functional $\psi=\varphi S$ is right invariant.

If A admits a non-zero, left-invariant, positive linear functional φ, we call (A,Δ) an *algebraic quantum group* and we call φ a *left Haar integral* on (A,Δ). Faithfulness of φ is automatic.

Note that although $\psi=\varphi S$ is right invariant, it may not be positive. On the other hand, it is proved in [11] that a non-zero, right-invariant, positive linear functional on A—a *right Haar integral*—necessarily exists. As for a left Haar integral, a right Haar integral is necessarily faithful.

The left Haar functional φ is not necessarily tracial (or central). However, there is a unique bijective homomorphism $\rho:A\to A$ such that $\varphi(ab)=\varphi(b\rho(a))$, for all $a,b\in A$. It satisfies $\rho(\rho(a^*)^*)=a$ for all $a\in A$ and $\Delta\rho=(S^2\otimes\rho)\Delta$.

One useful property, which will be used several times in the sequel, is that every element of A has "compact support": given $a \in A$, there exists some $b \in A$ such that $ab = ba = a$. (In fact, a more general result is true [8].) Another property we will need is the so-called strong left invariance of φ, which is proved in [18, Proposition 3.11]. It says that

$$(\iota \otimes \varphi)((1 \otimes a)\Delta(b)) = S((\iota \otimes \varphi)(\Delta(a)(1 \otimes b)))$$

holds for all $a, b \in A$.

We now turn to a short discussion of duality. If (A, Δ) is an algebraic quantum group, denote by $\hat{A}$ the linear subspace of A' consisting of all functionals φa, where $a \in A$. Since $\varphi a = \rho(a)\varphi$, we have $\hat{A} = \{a\varphi \mid a \in A\}$. If $\omega_1, \omega_2 \in \hat{A}$, one can define a linear functional $(\omega_1 \otimes \omega_2)\Delta$ on A by setting $(\omega_1 \otimes \omega_2)\Delta(a) = (\varphi \otimes \varphi)((a_1 \otimes a_2)\Delta(a))$, where $\omega_1 = \varphi a_1$ and $\omega_2 = \varphi a_2$. Using this, the space $\hat{A}$ can be made into a non-degenerate $*$-algebra. The multiplication is given by $\omega_1\omega_2 = (\omega_1 \otimes \omega_2)\Delta$ and the involution is given by setting $\omega^*(a) = \omega(S(a)^*)^-$, for all $a \in A$ and $\omega_1, \omega_2, \omega \in \hat{A}$; it is clear that $\omega_1\omega_2, \omega^* \in A'$ but one can show that, in fact, $\omega_1\omega_2, \omega^* \in \hat{A}$.

One can realize $M(\hat{A})$ as a linear space by identifying it as the linear subspace of A' consisting of all $\omega \in A'$ for which $(\omega \otimes \iota)\Delta(a)$ and $(\iota \otimes \omega)\Delta(a)$ belong to A. (It is clear that $\hat{A}$ belongs to this subspace.) In this identification of $M(\hat{A})$, the multiplication and involution are determined by

$$(\omega_1\omega_2)(a) = \omega_1((\iota \otimes \omega_2)\Delta(a)) = \omega_2((\omega_1 \otimes \iota)\Delta(a))$$

and

$$\omega^*(a) = \omega(S(a)^*)^-$$

for all $a \in A$ and $\omega_1, \omega_2, \omega \in M(\hat{A})$.

Note that the co-unit ε of A is the unit of the $*$-algebra $M(\hat{A})$.

There is a unique $*$-homomorphism $\hat{\Delta}$ from $\hat{A}$ to $M(\hat{A} \otimes \hat{A})$ such that for all $\omega_1, \omega_2 \in \hat{A}$ and $a, b \in A$,

$$((\omega_1 \otimes 1)\hat{\Delta}(\omega_2))(a \otimes b) = (\omega_1 \otimes \omega_2)(\Delta(a)(1 \otimes b))$$

and

$$(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(a \otimes b) = (\omega_1 \otimes \omega_2)((a \otimes 1)\Delta(b)).$$

Of course, we are here identifying $A' \otimes A'$ as a linear subspace of $(A \otimes A)'$ in the usual way, so that elements of $\hat{A} \otimes \hat{A}$ can be regarded as linear functionals on $A \otimes A$.

The pair $(\hat{A}, \hat{\Delta})$ is an algebraic quantum group, called the *dual* of (A, Δ). Its co-unit $\hat{\varepsilon}$ and antipode $\hat{S}$ are given by $\hat{\varepsilon}(a\varphi) = \varphi(a)$ and $\hat{S}(a\varphi) = (a\varphi)\circ S$, for all $a \in A$.

There is an algebraic quantum group version of Pontryagin's duality theorem for locally compact abelian groups that asserts that (A, Δ) is canonically isomorphic to the dual of $(\hat{A}, \hat{\Delta})$; that is, (A, Δ) is isomorphic to its double dual $(A\hat{}\hat{}, \Delta\hat{}\hat{})$.

We now turn to the analytic theory of algebraic quantum groups. We first recall the concept of a GNS pair. Suppose given a positive linear functional ω on a $*$-algebra A. Let H be a Hilbert space, and let $\Lambda : A \to H$ be a linear map with dense range for which $(\Lambda(a), \Lambda(b)) = \omega(b^*a)$, for all $a, b \in A$. Then we call (H, Λ) a *GNS pair* associated to ω. As is well known, such a pair always exists and is essentially unique. For, if (H', Λ') is another GNS pair associated to ω, the map, $\Lambda(a) \mapsto \Lambda'(a)$, extends to a unitary $U : H \to H'$.

If φ is a left Haar integral on an algebraic quantum group (A, Δ), and (H, Λ) is an associated GNS pair, then it can be shown that there is a unique $*$-homomorphism $\pi : A \to B(H)$ such that $\pi(a)\Lambda(b) = \Lambda(ab)$, for all $a, b \in A$. Moreover, π is faithful and non-degenerate. We let A_{r} denote the norm closure of $\pi(A)$ in $B(H)$. Thus, A_{r} is a non-degenerate C*-subalgebra of $B(H)$. The $*$-representation $\pi : A \to B(H)$ is essentially unique, for if (H', Λ') is another GNS pair associated to φ, and $\pi' : A \to B(H')$ is the corresponding $*$-representation, then, as we observed above, there exists a unitary $U : H \to H'$ such that $U\Lambda(a) = \Lambda'(a)$, for all $a \in A$, and consequently, $\pi'(a) = U\pi(a)U^*$.

We shall use the symbol M to denote the von Neumann algebra generated by A_{r}. Of course, A_{r} and $\pi(A)$ are weakly dense in M.

Now observe that there exists a unique non-degenerate $*$-homomorphism $\Delta_{\mathrm{r}} : A_{\mathrm{r}} \to M(A_{\mathrm{r}} \otimes A_{\mathrm{r}})$ such that, for all $a \in A$ and all $x \in A \otimes A$, we have

$$\Delta_{\mathrm{r}}(\pi(a))(\pi \otimes \pi)(x) = (\pi \otimes \pi)(\Delta(a)x)$$

and

$$(\pi \otimes \pi)(x)\Delta_{\mathrm{r}}(\pi(a)) = (\pi \otimes \pi)(x\Delta(a)).$$

We also recall that

$$A_{\mathrm{r}} = [(\omega \otimes \iota)(\Delta_{\mathrm{r}}(x)) \mid x \in A_{\mathrm{r}},\ \omega \in A_{\mathrm{r}}^*] = [(\iota \otimes \omega)(\Delta_{\mathrm{r}}(x)) \mid x \in A_{\mathrm{r}},\ \omega \in A_{\mathrm{r}}^*].$$

The pair $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ is a reduced locally compact quantum group in the sense of Definition 4.1 of [12]; it is called the *analytic extension* of (A, Δ) associated to φ.

We also need to recall that there is a unique unitary operator W on $H \otimes H$ such that

$$W((\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1))) = \Lambda(a) \otimes \Lambda(b),$$

for all $a, b \in A$. This unitary satisfies the equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12};$$

thus, it is a multiplicative unitary, said to be *associated to* (H, Λ). Here we have used the leg numbering notation of [1]. One can show that $W \in M(A_r \otimes B_0(H))$, so especially $W \in (A_r \otimes B_0(H))'' = M \bar{\otimes} B(H)$, and that A_r is the norm closure of the linear space $\{(\iota \otimes \omega)(W) \mid \omega \in B_0(H)^*\}$. Also, $\Delta_r(a) = W^*(1 \otimes a)W$, for all $a \in A_r$.

Since the map Δ_r is unitarily implemented, it has a unique weakly continuous extension to a unital $*$-homomorphism $\Delta_r : M \to M \bar{\otimes} M$, given explicitly by $\Delta_r(a) = W^*(1 \otimes a)W$, for all $a \in M$. The Banach space M_* may be regarded as a Banach algebra when equipped with the canonical multiplication induced by Δ_r; thus, the product of two elements ω and σ is given by $\omega\sigma = (\omega \bar{\otimes} \sigma) \circ \Delta_r$.

We use the same symbol R to denote the anti-unitary antipode of A_r and of M, and we denote by τ the scaling group of (A_r, Δ_r) (see [11,12]).

Consider now the algebraic dual $(\hat{A}, \hat{\Delta})$ of (A, Δ). A right-invariant linear functional $\hat{\psi}$ is defined on $\hat{A}$ by setting $\hat{\psi}(\hat{a}) = \varepsilon(a)$, for all $a \in A$. Here $\hat{a} = a\varphi$ and ε is the co-unit of (A, Δ). Since the linear map, $A \to \hat{A}$, $a \mapsto \hat{a}$, is a bijection (by faithfulness of φ), the functional $\hat{\psi}$ is well defined. Now define a linear map $\hat{\Lambda} : \hat{A} \to H$ by setting $\hat{\Lambda}(\hat{a}) = \Lambda(a)$, for all $a \in \mathscr{A}$. Since $\hat{\psi}((\hat{b})^* \hat{a}) = \varphi(b^* a) = (\Lambda(a), \Lambda(b))$, for all $a, b \in A$, it follows that $(H, \hat{\Lambda})$ is a GNS-pair associated to $\hat{\psi}$. It can be shown that it is unitarily equivalent to the GNS-pair for a left Haar integral $\hat{\varphi}$ of $(\hat{A}, \hat{\Delta})$. Hence, we can use $(H, \hat{\Lambda})$ to define a representation of the analytic extension $(\hat{A}_r, \hat{\Delta}_r)$ of $(\hat{A}, \hat{\Delta})$ on the space H. There is a unique $*$-homomorphism $\hat{\pi} : \hat{A} \to B(H)$ such that $\hat{\pi}(a)\hat{\Lambda}(b) = \hat{\Lambda}(ab)$, for all $a, b \in \hat{A}$. Moreover, $\hat{\pi}$ is faithful and non-degenerate. Let $\hat{A}_r$ be the norm closure of $\hat{\pi}(A)$ in $B(H)$, so $\hat{A}_r$ is a non-degenerate C*-subalgebra of $B(H)$. One can show that $W \in M(B_0(H) \otimes \hat{A}_r)$ and that $\hat{A}_r$ is the norm closure of the linear space $\{(\omega \otimes \iota)(W) \mid \omega \in B_0(H)^*\}$. Define a linear map $\hat{\Delta}_r : \hat{A}_r \to M(\hat{A}_r \otimes \hat{A}_r)$ by setting $\hat{\Delta}_r(a) = W(a \otimes 1)W^*$, for all $a \in \hat{A}_r$. Then $\hat{\Delta}_r$ is the unique $*$-homomorphism $\hat{\Delta}_r : \hat{A}_r \to M(\hat{A}_r \otimes \hat{A}_r)$ such that, for all $a \in \hat{A}$ and $x \in \hat{A} \otimes \hat{A}$,

$$\hat{\Delta}_r(\hat{\pi}(a))(\hat{\pi} \otimes \hat{\pi})(x) = (\hat{\pi} \otimes \hat{\pi})(\hat{\Delta}(a)x)$$

and

$$(\hat{\pi} \otimes \hat{\pi})(x)\hat{\Delta}_r(\hat{\pi}(a)) = (\hat{\pi} \otimes \hat{\pi})(x\hat{\Delta}(a)).$$

Note that one can show that $W \in M(A_r \otimes \hat{A}_r)$ and $(\Delta_r \otimes \iota)(W) = W_{13}W_{23}$.

An algebraic quantum group (A, Δ) is of *compact type* if A is unital, and of *discrete type* if there exists a non-zero element $h \in A$ satisfying $ah = ha = \varepsilon(a)h$, for all $a \in A$. It is known that (A, Δ) is of compact type (respectively, of discrete type) if, and only if, its dual $(\hat{A}, \hat{\Delta})$ is of discrete type (respectively, of compact type).

We use the symbol $\hat{M}$ to denote the von Neumann algebra generated by $\hat{A}_r$, so that $\hat{A}_r$ and $\hat{\pi}(\hat{A})$ are weakly dense in $\hat{M}$. As with Δ_r, since $\hat{\Delta}_r$ is unitarily implemented, it has a unique extension to a weakly continuous unital $*$-homomorphism $\hat{\Delta}_r : \hat{M} \to \hat{M} \bar{\otimes} \hat{M}$, given explicitly by $\hat{\Delta}_r(a) = W(a \otimes 1)W^*$, for all $a \in \hat{M}$.

It should be noted that both M and $\hat{M}$ are in the standard representation. This follows easily from [11] and standard von Neumann algebra theory (see [16], for example). As a consequence, all normal states on these algebras are (restriction of) vector states.

We now recall the definition of co-amenability of an algebraic quantum group. Suppose that (A, Δ) is an algebraic quantum group and let (H, Λ) be a GNS pair associated to a left Haar integral. As the representation $\pi : A \to B(H)$ is injective, we can use it to endow A with a C*-norm by setting $||a|| = ||\pi(a)||$, for $a \in A$. Following [3] (see also [2]), we say that (A, Δ) is *co-amenable* if its co-unit ε is norm-bounded with respect to this norm. Several characterizations of co-amenability are given in [3]. We just remind the reader that the algebraic quantum group of compact type associated to the group algebra of a discrete group Γ is co-amenable according to this definition if, and only if, Γ is amenable. Also, co-amenability is automatic in the case of a discrete type algebraic quantum group.

We also recall from [3] the definition of amenability for an algebraic quantum group. Let (A, Δ) be an algebraic quantum group with von Neumann algebra M. A *right-invariant mean* for (A, Δ) is a state m on M such that

$$m((\iota \bar{\otimes} \omega)\Delta_{\mathrm{r}}(x)) = \omega(1)m(x)$$

for all $x \in M$ and $\omega \in M_*$. A *left-invariant mean* is defined analogously. We say that (A, Δ) is *amenable* if (A, Δ) admits a right-invariant mean. Using the existence of the anti-unitary antipode R on (M, Δ_{r}) ([11,12]), this is easily seen to be equivalent to requiring that (A, Δ) admits a left-invariant mean. The algebraic quantum group associated to the algebra of complex functions with finite support on a discrete group Γ is amenable if, and only if, the group Γ is amenable, by the very definition of the amenability of a group. Amenability is automatic for an algebraic quantum group (A, Δ) of compact type.

We end this section with some technical lemmas.

We denote by δ the modular "function" of (A, Δ). Especially, δ is an invertible, self-adjoint element of $M(A)$ satisfying

$$\Delta(\delta) = \delta \otimes \delta, \;\; \varepsilon(\delta) = 1, \;\; S(\delta) = \delta^{-1}.$$

Further, there exists $\mu \in \mathbf{T}$ such that

$$\varphi S(a) = \varphi(a\delta) = \mu\varphi(\delta a)$$

for all $a \in A$.

Lemma 2.1. *Let $a \in A$. Then we have*

(1) $(\hat{a})^* = (S(a)^*\delta)^\wedge$,
(2) $\hat{a}\delta^{-1} = \mu(a\delta^{-1})^\wedge$,
(3) $\widehat{S(a)}\rho^{-1}S = \widehat{a\delta}$,
(4) $\rho\widehat{(a^*)}S^{-1} = \mu^{-1}(S(a^*)\delta)^\wedge$.

Proof. (1) See [11, Lemma 7.14].

(2) Observe that

$$\varphi(\delta^{-1}b) = \varphi(\delta^{-1}b\delta^{-1}\delta) = \mu\varphi(\delta\delta^{-1}b\delta^{-1}) = \mu\varphi(b\delta^{-1})$$

for all $b \in A$. Hence we get

$$\begin{aligned}(\hat{a}\delta^{-1})(c) &= \hat{a}(\delta^{-1}c) = \varphi(\delta^{-1}ca)\\ &= \mu\varphi(ca\delta^{-1}) = \mu(a\delta^{-1})^\wedge(c)\end{aligned}$$

for all $c \in A$.

(3) We have

$$\begin{aligned}(\widehat{S(a)}\rho^{-1}S)(b) &= \varphi(\rho^{-1}(S(b))S(a)) = \varphi(S(a)S(b))\\ &= \varphi S(ba) = \varphi(ba\delta) = \widehat{a\delta}(b)\end{aligned}$$

for all $b \in A$.

(4) We have

$$\begin{aligned}(\rho\widehat{(a^*)}S^{-1})(b) &= \varphi(S^{-1}(b)\rho(a^*)) = \varphi(a^*S^{-1}(b))\\ &= \varphi S(a^*S^{-1}(b)\delta^{-1}) = \varphi(\delta bS(a^*)) = \mu^{-1}\varphi(bS(a^*)\delta)\\ &= \mu^{-1}(S(a^*)\delta)^\wedge(b)\end{aligned}$$

for all $b \in A$. $\square$

Let now $\hat{\rho}$ denote the automorphism of $\hat{A}$ satisfying

$$\hat{\psi}(\hat{a}\hat{b}) = \hat{\psi}(\hat{b}\hat{\rho}(\hat{a}))$$

for all $a, b \in A$. (The existence of $\hat{\rho}$ is proved in a similar way as the existence of ρ.)

Lemma 2.2. *Let $a \in A$. Then we have*

(1) $\hat{\rho}(\hat{a}) = (S^2(a)\delta^{-1})^\wedge$
(2) $\hat{\rho}(((S(a^*))^\wedge)^*) = \hat{a}$

Proof. (1) For all $b \in A$ we have

$$\begin{aligned}\hat{\psi}((\hat{b})^*\hat{\rho}(\hat{a})) &= \hat{\psi}(\hat{a}(\hat{b})^*) = \hat{\psi}((\hat{a}^*)^*(\hat{b})^*)\\ &= \hat{\psi}(((S(a)^*\delta)^\wedge)^*(S(b)^*\delta)^\wedge) \quad \text{(using Lemma 2.1, (1))}\end{aligned}$$

$$= \varphi((S(a)^*\delta)^* S(b)^*\delta) = \varphi(\delta^* S(a)S(b)^*\delta) = \varphi(\delta S(a)S^{-1}(b^*)\delta)$$
$$= \varphi S(\delta S(a)S^{-1}(b^*)) = \varphi(b^* S^2(a)\delta^{-1}) = \hat{\psi}((\hat{b})^*(S^2(a)\delta^{-1})^\wedge),$$

and the assertion follows from the faithfulness of $\hat{\psi}$.

(2) Observe that

$$S^2(S^2(a^*)^*) = S(S(S^2(a^*)^*)) = S(S^{-1}(S^2(a^*))^*)$$
$$= S(S(a^*)^*) = a.$$

Hence, using (1) from Lemma 2.1, and (1) above, we get

$$\hat{\rho}((((S(a^*))^\wedge)^*) = \hat{\rho}((S(S(a^*))^*\delta)^\wedge) = \hat{\rho}((S^2(a^*)^*\delta)^\wedge)$$
$$= (S^2(S^2(a^*)^*\delta)\delta^{-1})^\wedge = (S^2(S^2(a^*)^*)S^2(\delta)\delta^{-1})^\wedge = (S^2(S^2(a^*)^*))^\wedge = \hat{a}$$

as desired. $\square$

Lemma 2.3. *Define $F : \hat{A} \to \hat{A}$ by $F(\hat{a}) = \widehat{S(a^*)}$. Then F is antilinear, antimultiplicative and involutive.*

Proof. We only show antimultiplicativity as the other two properties are easily checked.

Let $a, b \in A$. Write $a \otimes b = \sum_{i=1}^n \Delta(p_i)(q_i \otimes 1)$ for some $p_i, q_i \in A$, $i = 1, \dots, n$. Then

$$(\hat{a}\hat{b}) = \sum_{i=1}^n \varphi(q_i)\hat{p}_i.$$

Indeed, we have

$$(\hat{a}\hat{b})(c) = (\hat{a} \otimes \hat{b})\Delta(c) = (\varphi \otimes \varphi)(\Delta(c)(a \otimes b))$$
$$= (\varphi \otimes \varphi)(\Delta(c)\sum_{i=1}^n \Delta(p_i)(q_i \otimes 1)) = \sum_{i=1}^n (\varphi \otimes \varphi)(\Delta(cp_i)(q_i \otimes 1))$$
$$= \sum_{i=1}^n \varphi(cp_i)\varphi(q_i) = \varphi\left(c\left(\sum_{i=1}^n \varphi(q_i)p_i\right)\right) = \left(\sum_{i=1}^n \varphi(q_i)\hat{p}_i\right)(c)$$

for all $c \in A$. Using this expression, we get

$$F(\hat{a}\hat{b})(c) = \left(\sum_{i=1}^n F(\varphi(q_i)\hat{p}_i)\right)(c) = \sum_{i=1}^n \varphi(q_i^*)\widehat{S(p_i^*)}(c) = \sum_{i=1}^n \varphi(q_i^*)\varphi(cS(p_i^*)).$$

On the other hand, recalling that χ denote the flip map on $M(A\otimes A)$, we have

$$
\begin{aligned}
(F(\hat{b})F(\hat{a}))(c) &= (\widehat{S(b^*)}\widehat{S(a^*)})(c) = (\widehat{S(b^*)}\otimes\widehat{S(a^*)})\Delta(c)\\
&= (\varphi\otimes\varphi)(\Delta(c)(S(b^*)\otimes S(a^*)))\\
&= \sum_{i=1}^{n} (\varphi\otimes\varphi)(\Delta(c)(S\otimes S)((1\otimes q_i^*)\chi\Delta(p_i^*)))\\
&= \sum_{i=1}^{n} (\varphi\otimes\varphi)(\Delta(c)((S\otimes S)\chi\Delta(p_i^*))(1\otimes S(q_i^*)))\\
&= \sum_{i=1}^{n} (\varphi\otimes\varphi)(\Delta(c)(\chi(S\otimes S)\Delta(p_i^*))(1\otimes S(q_i^*)))\\
&= \sum_{i=1}^{n} (\varphi\otimes\varphi)(\Delta(cS(p_i^*))(1\otimes S(q_i^*)))\\
&= \sum_{i=1}^{n} (\varphi S\otimes\varphi)(\Delta(cS(p_i^*))(\delta^{-1}\otimes S(q_i^*)))\\
&= \sum_{i=1}^{n} (\varphi S\otimes\varphi)(\Delta(cS(p_i^*)\delta^{-1})(1\otimes\delta S(q_i^*)))\\
&= \sum_{i=1}^{n} \varphi S(cS(p_i^*)\delta^{-1})\varphi(\delta S(q_i^*))\\
&= \sum_{i=1}^{n} \varphi(cS(p_i^*))\varphi S(q_i^*\delta^{-1}) = \sum_{i=1}^{n} \varphi(q_i^*)\varphi(cS(p_i^*))
\end{aligned}
$$

for all $c\in A$, and the antimultiplicativity of F follows. $\square$

Lemma 2.4. *Let $a\in A$. Pick $c\in A$ such that $\widehat{S(a^*)} = \hat{c}\widehat{S(a^*)}$. Then we have*

$$(\iota\otimes\varphi)((1\otimes c^*)\Delta(a)) = a.$$

Proof. Using Lemma 2.3, we get $\hat{a}\widehat{S(c^*)} = \hat{a}$. Now, to prove the assertion, it is clearly enough to show that

$$(\varphi\otimes\varphi)((b^*\otimes c^*)\Delta(a)) = \varphi(b^*a)$$

holds for all $b\in A$. This may be established as follows:

$$
\begin{aligned}
(\varphi\otimes\varphi)((b^*\otimes c^*)\Delta(a)) &= \varphi(b^*(\iota\otimes\varphi c^*)\Delta(a))\\
&= \varphi S(b^*((\iota\otimes\varphi c^*)\Delta(a))\delta^{-1}) = \varphi(\delta((S\otimes\varphi c^*)\Delta(a))S(b^*))
\end{aligned}
$$

$$= \varphi(\delta((S \otimes \varphi c^* S^{-1} S)\Delta(a))S(b^*))$$

$$= \varphi(\delta((\varphi c^* S^{-1} \otimes \iota)\Delta(S(a)))S(b^*))$$

$$= \mu^{-1}\varphi(((\varphi c^* S^{-1} \otimes \iota)\Delta(S(a)))S(\delta^{-1}b^*))$$

$$= \mu^{-1}\varphi(\rho^{-1}S(\delta^{-1}b^*)((\varphi c^* S^{-1} \otimes \iota)\Delta(S(a))))$$

$$= \mu^{-1}\varphi(((\varphi c^* S^{-1} S \otimes \iota)\Delta(\rho^{-1}S(\delta^{-1}b^*)))S(a))$$

(where we have used strong left invariance of φ)

$$= \mu^{-1}(\varphi c^* \otimes S(a)\varphi)\Delta\rho^{-1}S(\delta^{-1}b^*) = \mu^{-1}(\rho\widehat{(c^*)} \otimes \widehat{S(a)})\Delta\rho^{-1}S(\delta^{-1}b^*)$$

$$= \mu^{-1}(\rho\widehat{(c^*)} \otimes \widehat{S(a)})(S^{-2} \otimes \rho^{-1})\Delta S(\delta^{-1}b^*)$$

$$= \mu^{-1}(\widehat{S(a)} \otimes \rho\widehat{(c^*)})(\rho^{-1} \otimes S^{-2})(S \otimes S)\Delta(\delta^{-1}b^*)$$

$$= \mu^{-1}(\widehat{S(a)}\rho^{-1}S \otimes \rho\widehat{(c^*)}S^{-1})\Delta(\delta^{-1}b^*) = \mu^{-2}(\widehat{a\delta} \otimes S\widehat{(c^*)}\delta)\Delta(\delta^{-1}b^*)$$

(where we have used Lemma 2.1, (3) and (4))

$$= \mu^{-2}(\widehat{a\delta} \otimes S\widehat{(c^*)}\delta)(\delta^{-1} \otimes \delta^{-1})\Delta(b^*) = \mu^{-2}\mu^2(\hat{a} \otimes S\widehat{(c^*)})\Delta(b^*)$$

(where we have used Lemma 2.1, (2))

$$= (\hat{a}S\widehat{(c^*)})(b^*) = \hat{a}(b^*) = \varphi(b^*a),$$

where we have used that $\hat{a}S\widehat{(c^*)} = \hat{a}$.

This finishes the proof. $\square$

Lemma 2.5. *Let $a, b \in A$. Pick $a_1, b_1, \ldots, a_n, b_n \in A$ such that*

$$\Delta(b)(a \otimes 1) = \sum_{k=1}^{n} a_k \otimes b_k.$$

Then, for all $\omega \in \hat{A}$, we have

$$((a\omega)S)\hat{b} = \sum_{k=1}^{n} \omega(a_k)\hat{b_k}.$$

Proof. Let $c \in A, \omega \in f\hat{A}$. Then

$$(((a\omega)S)\hat{b})(c) = (((a\omega)S) \otimes (b\varphi))\Delta(c) = (a\omega)S(\iota \otimes (b\varphi))\Delta(c)$$
$$= (a\omega)S(\iota \otimes \varphi)(\Delta(c)(1 \otimes b)) = (a\omega)(\iota \otimes \varphi)((1 \otimes c)\Delta(b)),$$

using strong left invariance of φ. Hence,

$$\begin{aligned}(((a\omega)S)\hat{b})(c) &= (\omega \otimes \varphi)((1 \otimes c)\Delta(b)(a \otimes 1)) \\ &= (\omega \otimes \varphi)\left((1 \otimes c)\left(\sum_{k=1}^{n} a_k \otimes b_k\right)\right) \\ &= \sum_{k=1}^{n} \omega(a_k)\varphi(cb_k) = \left(\sum_{k=1}^{n} \omega(a_k)\hat{b_k}\right)(c).\end{aligned}$$

This shows the assertion. $\square$

Lemma 2.6. *Let $a, b, c \in A$. Write*

$$\Delta(b)(a \otimes 1) = \sum_{k=1}^{n} a_k \otimes b_k, \quad b \otimes c = \sum_{i=1}^{m} \Delta(p_i)(q_i \otimes 1)$$

for some $a_1, b_1, \dots, a_n, b_n, p_1, q_1, \dots, p_m, q_m \in A$. Further, for each $i = 1, \dots, m$, write

$$\Delta(p_i)(a \otimes 1) = \sum_{j=1}^{s(i)} x_{ij} \otimes y_{ij}$$

for some $x_{i1}, y_{i1}, \dots, x_{is(i)}, y_{is(i)} \in A$. Then

$$\sum_{i=1}^{m} \sum_{j=1}^{s(i)} \varphi(q_i)\hat{y}_{ij} \otimes x_{ij} = \sum_{k=1}^{n} \hat{b_k}\hat{c} \otimes a_k.$$

Proof. Set $X = \sum_{i=1}^{m} \sum_{j=1}^{s(i)} \varphi(q_i)\hat{y}_{ij} \otimes x_{ij}$ and $Y = \sum_{k=1}^{n} \hat{b_k}\hat{c} \otimes a_k$, which are both elements in $\hat{A} \otimes A$. To show that $X = Y$, it suffices, using separation, to prove that $(e \otimes (\varphi f))X = (e \otimes (\varphi f))Y$ for all $e, f \in A$. Note that we regard here e as an element of the double dual of A.

Let $e, f \in A$. Then

$$\begin{aligned}(e \otimes (\varphi f))X &= \sum_{i,j} \varphi(q_i)\varphi(ey_{ij})\varphi(fx_{ij}) \\ &= \sum_{i,j} (\varphi \otimes \varphi \otimes \varphi)((1 \otimes e \otimes f)(q_i \otimes y_{ij} \otimes x_{ij})) \\ &= \sum_{i} (\varphi \otimes \varphi \otimes \varphi)((1 \otimes e \otimes f)(q_i \otimes (\Delta_{\mathrm{op}}(p_i)(1 \otimes a)))) \\ &= (\varphi \otimes (\varphi e) \otimes (a\varphi f))\left(\sum_i q_i \otimes \Delta_{\mathrm{op}}(p_i)\right) \\ &= (\varphi \otimes (\varphi e) \otimes (a\varphi f))\left((\iota \otimes \Delta_{\mathrm{op}})\left(\sum_i q_i \otimes p_i\right)\right) \\ &= (\varphi \otimes (\varphi e) \otimes (a\varphi f))((\iota \otimes \Delta_{\mathrm{op}})(S^{-1} \otimes \iota)(\Delta(c))(b \otimes 1)),\end{aligned}$$

using here the formula established in [11, Proposition 2.2] at the last step.

Continuing this computation, we get

$$\begin{aligned}(e \otimes (\varphi f))X &= (((b\varphi)S^{-1} \otimes (\varphi e) \otimes (a\varphi f))((\iota \otimes \Delta_{\mathrm{op}})\Delta(c)) \\ &= (((b\varphi)S^{-1} \otimes (a\varphi f) \otimes (\varphi e))((\iota \otimes \Delta)\Delta(c)) \\ &= (((b\varphi)S^{-1} \otimes (a\varphi f) \otimes (\varphi e))((\Delta \otimes \iota)\Delta(c)) \\ &= (((b\varphi)S^{-1} \otimes (a\varphi f))\Delta((\iota \otimes (\varphi e))\Delta(c)) \\ &= (((b\varphi)S^{-1} \otimes (a\varphi f))\Delta((\iota \otimes \varphi)((1 \otimes e)\Delta(c))) \\ &= (((b\varphi)S^{-1} \otimes (a\varphi f))\Delta(S((\iota \otimes \varphi)(\Delta(e)(1 \otimes c)))),\end{aligned}$$

using strong left invariance of φ.

This gives

$$\begin{aligned}(e \otimes (\varphi f))X &= (((a\varphi f)S) \otimes (b\varphi))\Delta((\iota \otimes \varphi)(\Delta(e)(1 \otimes c))) \\ &= (((a\varphi f)S) \otimes (b\varphi))\Delta((\iota \otimes \hat{c})\Delta(e)) \\ &= (((a\varphi f)S) \otimes \hat{b} \otimes \hat{c}))(\Delta \otimes \iota)\Delta(e) \\ &= (((a\varphi f)S) \otimes \hat{b} \otimes \hat{c}))(\iota \otimes \Delta)\Delta(e) \\ &= (((a\varphi f)S) \otimes (\hat{b}\hat{c}))\Delta(e) = (((a\varphi f)S)\hat{b}\hat{c})(e).\end{aligned}$$

On the other hand, we have

$$(e\otimes(\varphi f))Y = \sum_{k=1}^{n}((\hat{b_k}\otimes\hat{c})\Delta(e))(\varphi f)(a_k)$$
$$= \sum_{k=1}^{n}(\hat{b_k}\hat{c})(e)(\varphi f)(a_k) = \left(\left(\sum_{k=1}^{n}(\varphi f)(a_k)\hat{b_k}\right)\hat{c}\right)(e).$$

Hence, the assertion will be proved if we can show that

$$((a\varphi f)S)\hat{b} = \sum_{k=1}^{n}(\varphi f)(a_k)\hat{b_k}$$

holds for all $f\in A$. But this clearly follows from Lemma 2.5, and the proof is finished. $\square$

3. Non-degenerate representations and unitary generators

We let (A,Δ) be an algebraic quantum group throughout this section and use notation and terminology introduced in the previous section.

Following [9], we first introduce the universal C*-algebraic quantum group $(A_{\mathrm{u}},\Delta_{\mathrm{u}})$ associated to (A,Δ). The C*-algebra A_{u} is the completion of A with respect to the C*-norm $||\cdot||_{\mathrm{u}}$ on A defined by

$$||a||_{\mathrm{u}} = \sup\{||\varphi(a)|| \mid \varphi \text{ is a } *\text{-homomorphism from } A \text{ into some C}^*\text{-algebra}\}.$$

(The non-trivial fact that this gives a well-defined norm on A_{u} is shown in [9].) Let π_{u} denote the identity mapping from A into A_{u}. The co-product map Δ_{u} is defined in such way that it is the unique non-degenerate $*$-homomorphism $\Delta_{\mathrm{u}}: A_{\mathrm{u}}\to M(A_{\mathrm{u}}\otimes A_{\mathrm{u}})$ satisfying

$$(\pi_{\mathrm{u}}\otimes\pi_{\mathrm{u}})(x)\Delta_{\mathrm{u}}(\pi_{\mathrm{u}}(a)) = (\pi_{\mathrm{u}}\otimes\pi_{\mathrm{u}})(x\Delta(a))$$

and

$$\Delta_{\mathrm{u}}(\pi_{\mathrm{u}}(a))(\pi_{\mathrm{u}}\otimes\pi_{\mathrm{u}})(x) = (\pi_{\mathrm{u}}\otimes\pi_{\mathrm{u}})(\Delta(a)x)$$

for all $a\in A$ and $x\in A\otimes A$.

The universality of A_{u} makes it possible to extend uniquely from A to A_{u} any $*$-homomorphism from A into some C*-algebra. Especially, the co-unit ε of (A,Δ) extends to a $*$-homomorphism $\varepsilon_{\mathrm{u}}: A_{\mathrm{u}}\to\mathbf{C}$ such that $\varepsilon_{\mathrm{u}}\circ\pi_{\mathrm{u}} = \varepsilon$. One easily checks that ε_{u} satisfies the co-unit property, that is,

$$(\varepsilon_{\mathrm{u}}\otimes\iota)\Delta_{\mathrm{u}}(a) = (\iota\otimes\varepsilon_{\mathrm{u}})\Delta_{\mathrm{u}}(a) = a, \quad a\in A_{\mathrm{u}}.$$

It follows immediately from this that Δ_{u} is injective. Also, there exists a unique $*$-homomorphism π_{r} from A_{u} onto A_{r} satisfying $\pi_{\mathrm{r}}\circ\pi_{\mathrm{u}} = \pi$. By construction, we have $(\pi_{\mathrm{r}}\otimes\pi_{\mathrm{r}})\circ\Delta_{\mathrm{u}} = \Delta_{\mathrm{r}}\circ\pi_{\mathrm{r}}$.

We remark that one easily verifies that our definition of co-amenability of (A, Δ) may now be rephrased as saying that ε_u is weakly contained in π_r, that is, $\ker(\pi_r) \subset \ker(\varepsilon_u)$.

One of the off-springs of [9] is that there is a bijective correspondence between non-degenerate $*$-homomorphisms of A_u and unitary corepresentations of $(\hat{A}_r, \hat{\Delta}_r)$ (this may be seen by combining results from Sections 7 and 13 in [9]). Kustermans has also established a similar result for more general locally compact quantum groups in [10]. For completeness, we recall the definition of a unitary corepresentation. Consider a C*-algebraic quantum semigroup (B, Γ), that is, a C*-algebra B equipped with a co-product map Γ. Then a *unitary corepresentation* of (B, Γ) on a (non-zero) Hilbert space K is a unitary element $U \in M(B \otimes B_0(K))$ such that $(\Gamma \otimes \iota)U = U_{13}U_{23}$.

In this section we show that any non-degenerate $*$-representation of A_u on a Hilbert space has a unitary "generator", that is, it arises from some unitary corepresentation of $(\hat{A}_r, \hat{\Delta}_{r,op})$. By $\hat{\Delta}_{r,op}$ we mean throughout this paper the co-product on $\hat{A}_r$ *opposite* to $\hat{\Delta}_r$. It may be seen as a matter of taste choosing to work with the opposite co-product on the dual side. However, one reason for this choice is that this is the one tacitly adopted by Kustermans and Vaes [12] in the setting of locally compact quantum groups: the "dual" of (A_r, Δ_r) in their sense is in fact precisely $(\hat{A}_r, \hat{\Delta}_{r,op})$.

For completeness, we recall how $\hat{\Delta}_{r,op}$ is defined. Let $\hat{\chi} : \hat{A}_r \otimes \hat{A}_r \to \hat{A}_r \otimes \hat{A}_r$ denote the flip map. We also denote by the same symbol its extension to a $*$-automorphism of $M(\hat{A}_r \otimes \hat{A}_r)$. Then $\hat{\Delta}_{r,op} = \hat{\chi} \circ \hat{\Delta}_r$.

One can also describe $\hat{\Delta}_{r,op}$ with the help of a multiplicative unitary $\hat{W}$ related to W, in the spirit of [1]. Indeed, let Σ denote the flip map on $H \otimes H$ and set $\hat{W} = \Sigma W^* \Sigma$. Then one checks readily that

$$\hat{\Delta}_{r,op}(y) = \hat{W}^*(1 \otimes y)\hat{W}$$

for all $y \in \hat{A}_r$. We may use this formula to extend $\hat{\Delta}_{r,op}$ to a map from $\hat{M}$ into $\hat{M} \bar{\otimes} \hat{M}$, which we also denote by the same symbol. The pair $(\hat{M}, \hat{\Delta}_{r,op})$ is then a Hopf von Neumann algebra.

We equip $\hat{A}_r{}^*$ (resp. $\hat{M}_*$) with the product induced by $\hat{\Delta}_{r,op}$, that is $\omega\eta = (\omega \otimes \eta)\hat{\Delta}_{r,op}$ (resp. $\omega\eta = (\omega \bar{\otimes} \eta)\hat{\Delta}_{r,op}$). It is then straightforward to check that the Banach space $\hat{A}_r{}^*$ (resp. $\hat{M}_*$) is a Banach algebra under this product.

Our approach relies on the following fundamental result, which takes advantage of the fact that we are dealing with algebraic quantum groups.

Proposition 3.1. *Let notation be as above. Then*:

(a) *There exists a* (*unique*) *injective homomorphism* $Q : A \to \hat{M}_*$ *satisfying*
$$Q(a)[\hat{\pi}(\hat{b})] = \varphi(S^{-1}(a)b)$$

for all $a, b \in A$. If $a \in A$, and $c \in A$ is chosen so that

$$\hat{c}\widehat{S(a^*)} = \widehat{S(a^*)},$$

then $Q(a) = \omega_{\Lambda(a),\Lambda(c)}$ (restricted to $\hat{M}$).

(b) *The algebra $Q(A)$ is norm-dense in $\hat{M}_*$.*

(c) *If Θ denotes the canonical $*$-homomorphism from (A, Δ) onto its double dual, then $Q(a) \circ \hat{\pi} = \Theta(S^{-1}(a))$ for all $a \in A$.*

(d) *Let Q_{r} denote the map from A into $\hat{A}_{\mathrm{r}}^*$ obtained by restricting each $Q(a)$ to $\hat{A}_{\mathrm{r}}$. Then Q_{r} is also an injective homomorphism.*

Proof. Let $a \in A$. We define a linear functional $P(a)$ on $\hat{\pi}(\hat{A})$ by

$$P(a)[\hat{\pi}(\hat{b})] = \varphi(S^{-1}(a)b)$$

for all $b \in A$. Now choose $c \in A$ such that $\hat{c}\ \widehat{S(a^*)} = \widehat{S(a^*)}$. Then we have

$$P(a)[\hat{\pi}(\hat{b})] = \omega_{\Lambda(a),\Lambda(c)}(\hat{\pi}(\hat{b}))$$

for all $b \in A$. Indeed, let $b \in A$. Then

$$\begin{aligned} P(a)[\hat{\pi}(\hat{b})] &= \varphi(S^{-1}(a)b) = \varphi((S^{-1}(a)^*)^*\ b) = \varphi(S(a^*)^*\ b) \\ &= \hat{\psi}((\widehat{S(a^*)})^*\ \hat{b}) = \hat{\psi}((\widehat{S(a^*)})^*\hat{c}^*/\hat{b}) = \hat{\psi}(\hat{c}^*\ \hat{b}\ \hat{\rho}((\widehat{S(a^*)})^*)) \\ &= \hat{\psi}(\hat{c}^*\ \hat{b}\ \hat{a}) \quad \text{(using Lemma 2.2)} \\ &= (\hat{\Lambda}(\hat{b}\hat{a}), \hat{\Lambda}(\hat{c})) = (\hat{\pi}(\hat{b})\hat{\Lambda}(\hat{a}), \hat{\Lambda}(\hat{c})) = (\hat{\pi}(\hat{b})\Lambda(a), \Lambda(c)) \\ &= \omega_{\Lambda(a),\Lambda(c)}(\hat{\pi}(\hat{b})) \end{aligned}$$

as asserted.

It follows clearly from the formula just established that $P(a)$ has a unique extension $Q_{\mathrm{r}}(a) \in \hat{A}_{\mathrm{r}}$ and also a unique extension to $Q(a) \in \hat{M}_*$, both determined by restricting suitably $\omega_{\Lambda(a),\Lambda(c)}$.

We show now that assertion (c) holds. Let $a \in A$. Then we have

$$\begin{aligned} (Q(a)\circ\hat{\pi})(\hat{b}) &= Q(a)(\hat{\pi}(\hat{b})) = \varphi(S^{-1}(a)\ b) \\ &= \hat{b}(S^{-1}(a)) = (\Theta(S^{-1}(a)))(\hat{b}) \end{aligned}$$

for all $b \in A$. Hence, $Q(a) \circ \hat{\pi} = \Theta(S^{-1}(a))$, as desired.

The map $Q : a \to Q(a)$ from A into $\hat{M}_*$ is clearly linear. We show that Q is multiplicative.

Let $a_1, a_2 \in A$. For all $b \in A$ we have

$$\begin{aligned}(Q(a_1)\ Q(a_2))(\hat{\pi}(\hat{b})) &= (Q(a_1) \bar{\otimes} Q(a_2))(\hat{\Delta}_{\mathrm{r,op}}(\hat{\pi}(\hat{b})))\\ &= (Q(a_2) \bar{\otimes} Q(a_1))(\hat{\Delta}_{\mathrm{r}}(\hat{\pi}(\hat{b}))) = (Q(a_2) \bar{\otimes} Q(a_1))((\hat{\pi} \otimes \hat{\pi})\hat{\Delta}(\hat{b}))\\ &= (\Theta(S^{-1}(a_2)) \otimes \Theta(S^{-1}(a_1)))\hat{\Delta}(\hat{b}) = \hat{\Delta}(\hat{b})(S^{-1}(a_2) \otimes S^{-1}(a_1))\\ &= \hat{b}(S^{-1}(a_2)\ S^{-1}(a_1)) = \hat{b}(S^{-1}(a_1 a_2))\\ &= \varphi(S^{-1}(a_1 a_2)\ b) = (Q(a_1 a_2))(\hat{\pi}(\hat{b})).\end{aligned}$$

As $\hat{\pi}(\hat{A})$ is weakly dense in $\hat{M}$, the multiplicativity of Q follows.

To finish the proof of (a), it remains only to show that Q is injective. Let $a \in A$ and assume that $Q(a) = 0$.

Then, for all $b \in A$, we have $0 = Q(a)(\hat{\pi}(\hat{b})) = \varphi(S^{-1}(a)b)$. Hence, inserting $b = (S^{-1}(a))^*$, we get $\varphi(S^{-1}(a)\ (S^{-1}(a))^*) = 0$, that is, $S^{-1}(a) = 0$ since φ is faithful on A. Thus, $a = 0$, as desired.

A little thought shows that assertion (d) is also established by the arguments given so far. We finally prove assertion (b).

We first show that $Q(A) = \{\omega_{\Lambda(e),\Lambda(f)|\hat{M}} \mid e, f \in A\}$.

The inclusion $\subset$ is obvious from what we already have seen. To prove the reverse inclusion, let $e, f \in A$.

Then, choosing $d \in A$ such that $(S^{-1}(d)^*)^\wedge = \hat{f}\hat{\rho}^{-1}(\hat{e})^*$, we have

$$\omega_{\Lambda(e),\Lambda(f)|\hat{M}} = Q(d).$$

Indeed, for all $b \in A$, we have

$$\begin{aligned}\omega_{\Lambda(e),\Lambda(f)}(\hat{\pi}(\hat{b})) &= (\hat{\pi}(\hat{b})\ \Lambda(e), \Lambda(f)) = (\hat{\pi}(\hat{b})\ \hat{\Lambda}(\hat{e}), \hat{\Lambda}(\hat{f}))\\ &= (\hat{\Lambda}(\hat{b}\ \hat{e}), \hat{\Lambda}(\hat{f})) = \hat{\psi}(\hat{f}^*\ \hat{b}\ \hat{e}) = \hat{\psi}(\hat{\rho}^{-1}(\hat{e})\ \hat{f}^*\ \hat{b})\\ &= \hat{\psi}((S^{-1}(d)^*)^\wedge\ \hat{b}) = \varphi(S^{-1}(d)b) = Q(d)(\hat{\pi}(\hat{b})).\end{aligned}$$

This proves the reverse inclusion. Now, since the action of $\hat{M}$ on H is standard, we have $\hat{M}_* = \{\omega_{u,v|\hat{M}} \mid u, v \in H\}$. Further, the following inequality, which is surely well-known, is easy to prove:

Let $u, v \in H, a, b \in A$. Then

$$||\omega_{u,v|\hat{M}} - \omega_{\Lambda(a),\Lambda(b)|\hat{M}}|| \leqslant ||u - \Lambda(a)||\ ||v|| + ||\Lambda(a)||\ ||v - \Lambda(b)||.$$

As $\Lambda(A)$ is dense in H, the norm-density of $Q(A)$ in $\hat{M}_*$ clearly follows. This finishes the proof of (b), and thereby of the proposition. $\square$

Theorem 3.2. *Let U be a unitary corepresentation of $(\hat{A}_{\mathrm{r}}, \hat{\Delta}_{\mathrm{r,op}})$ on a Hilbert space K. Define $\pi_U : A \to B(K)$ by*

$$\pi_U(a) = (Q_{\mathrm{r}}(a) \otimes \iota)\, U, \ a \in A.$$

Then π_U is a non-degenerate $*$-representation of A on K, that is, it is a $*$-homomorphism of A into $B(K)$ which is non-degenerate in the sense that $[\pi_U(A)K] = K$.

We shall also denote by π_U the associated non-degenerate $*$-representation of A_{u} on K, and call U the generator of π_U.

Proof. We write π instead of π_U in this proof.

Let $a \in A$. Since $Q_{\mathrm{r}}(a) \in \hat{A}_{\mathrm{r}}^*$, it is clear that $\pi(a) \in B(K)$. The linearity of π is evident. The multiplicativity of π follows from the corepresentation property of U and the multiplicativity of Q_{r}. Indeed, we have

$$\begin{aligned}\pi(ab) &= (Q_{\mathrm{r}}(ab) \otimes \iota)\, U = ((Q_{\mathrm{r}}(a) Q_{\mathrm{r}}(b)) \otimes \iota)\, U \\ &= ((Q_{\mathrm{r}}(a) \otimes Q_{\mathrm{r}}(b))\hat{\Delta}_{\mathrm{r,op}} \otimes \iota)\, U = (Q_{\mathrm{r}}(a) \otimes Q_{\mathrm{r}}(b) \otimes \iota)(\hat{\Delta}_{\mathrm{r,op}} \otimes \iota)\ U \\ &= (Q_{\mathrm{r}}(a) \otimes Q_{\mathrm{r}}(b) \otimes \iota)\, U_{13} U_{23} = ((Q_{\mathrm{r}}(a) \otimes \iota)\, U)((Q_{\mathrm{r}}(b) \otimes \iota)\, U) = \pi(a)\pi(b)\end{aligned}$$

for all $a, b \in A$.

To prove that π is $*$-preserving, we have to adapt some arguments from [12].

We set $\hat{W} = \Sigma W^* \Sigma$. As pointed out before, we have

$$\hat{\Delta}_{\mathrm{r,op}}(y) = \hat{W}^*(1 \otimes y)\hat{W}, \ y \in \hat{A}_{\mathrm{r}}.$$

It follows that

$$U_{13}\ U_{23} = (\hat{\Delta}_{\mathrm{r,op}} \otimes \iota) U = \hat{W}_{12}^*\ U_{23}\ \hat{W}_{12},$$

hence that

$$(*) \quad \hat{W}_{12}\ U_{13} = U_{23}\ \hat{W}_{12}\ U_{23}^*.$$

Let $\rho \in B_0(H)^*, \eta \in B_0(K)^*$. Then define $\gamma \in B_0(H)^*$ by

$$\gamma(x) = (\rho \otimes \eta)(U\,(x \otimes 1)\ U^*), \quad x \in B_0(H).$$

Applying $\iota \otimes \rho \otimes \eta$ to $(*)$ above, we get

$$((\iota \otimes \rho)\ \hat{W})((\iota \otimes \eta) U) = (\iota \otimes \gamma)\ \hat{W}.$$

As in [12], we can then conclude that $((\iota\otimes\rho)\ \hat{W})((\iota\otimes\eta)\ U)\in D(\hat{S}_{\mathrm{r,op}})$, and

$$\begin{aligned}\hat{S}_{\mathrm{r,op}}(((\iota\otimes\rho)\ \hat{W})((\iota\otimes\eta)\ U)) &= (\iota\otimes\gamma)(\hat{W}^*)\\ &= (\iota\otimes\rho\otimes\eta)(U_{23}\ \hat{W}^*_{12}\ U^*_{23})\\ &= (\iota\otimes\rho\otimes\eta)(U^*_{13}\ \hat{W}^*_{12})\quad (\text{using } (*))\\ &= (\iota\otimes\eta)(U^*)(\iota\otimes\rho)(\hat{W}^*)\\ &= (\iota\otimes\eta)(U^*)\hat{S}_{\mathrm{r,op}}((\iota\otimes\rho)\ \hat{W}).\end{aligned}$$

Now, as the set $\{(\iota\otimes\rho)\ \hat{W}\mid\rho\in B_0(K)^*\}$ is a core for $\hat{S}_{\mathrm{r,op}}$, and $\hat{S}_{\mathrm{r,op}}$ is closed, this implies that $x((\iota\otimes\eta)\ U)\in D(\hat{S}_{\mathrm{r,op}})$ and

$$\hat{S}_{\mathrm{r,op}}(x((\iota\otimes\eta)\ U)) = (\iota\otimes\eta)(U^*)\hat{S}_{\mathrm{r,op}}(x),\quad \forall x\in D(\hat{S}_{\mathrm{r,op}}).$$

From this, we can conclude that $(\iota\otimes\eta)U\in D(\overline{\hat{S}_{\mathrm{r,op}}})$ and

$$\overline{\hat{S}_{\mathrm{r,op}}}((\iota\otimes\eta)U) = (\iota\otimes\eta)(U^*)$$

(see [12, Remark 5.44]).

Let $a\in A$. We define $Q_r(a)^*$ to be the linear functional on $\hat{\pi}_r(\hat{A})$ given by

$$Q_r(a)^*(\hat{\pi}_r(\hat{b})) = \overline{Q_r(a)(\hat{\pi}_r(\hat{S}_{\mathrm{op}}(\hat{b})^*))},\quad b\in A.$$

Then we have

$$\begin{aligned}Q_r(a)^*(\hat{\pi}_r(\hat{b})) &= \overline{(\hat{S}_{\mathrm{op}}(\hat{b})^*)(S^{-1}(a))}\\ &= (\hat{S}_{\mathrm{op}}(\hat{b}))(S(S^{-1}(a))^*) = (\hat{S}_{\mathrm{op}}(\hat{b}))(a^*) = (\hat{S}^{-1}(\hat{b}))(a^*)\\ &= \hat{b}(S^{-1}(a^*)) = \varphi(S^{-1}(a^*)b)\\ &= Q_r(a^*)(\hat{\pi}_r(\hat{b}))\end{aligned}$$

for all $b\in A$. This shows that $Q_r(a)^*$ extends to an element of $\hat{A}^*_{\mathrm{r}}$, which is in fact equal to $Q_r(a^*)$.

Now, let $\overline{Q_r(a)}$ have its usual meaning, that is $\overline{Q_r(a)}\in\hat{A}^*_{\mathrm{r}}$ is defined by

$$\overline{Q_r(a)}(y) = \overline{Q_r(a)(y^*)},\quad y\in\hat{A}_{\mathrm{r}}.$$

We have then

$$Q_r(a)^*(x) = \overline{Q_r(a)}(\hat{S}_{\mathrm{r,op}}(x)),\quad x\in D(\hat{S}_{\mathrm{r,op}}).$$

Since $D(\hat{S}_{\mathrm{r,op}})$ is a strict bounded core for $\hat{S}_{\mathrm{r,op}}$ (see [12, Remark 5.44] again), we get

$$Q_r(a)^*(x) = \overline{Q_r(a)}(\overline{\hat{S}_{\mathrm{r,op}}}(x)), \quad x \in D(\overline{\hat{S}_{\mathrm{r,op}}}).$$

Combining this with what we have seen previously, we obtain

$$\begin{aligned}\eta((Q_r(a)^* \otimes \iota)U) &= Q_r(a)^*((\iota \otimes \eta)U) \\ &= \overline{Q_r(a)}(\overline{\hat{S}_{\mathrm{r,op}}}((\iota \otimes \eta)U)) = \overline{Q_r(a)}((\iota \otimes \eta)(U^*)) \\ &= \eta((\overline{Q_r(a)} \otimes \iota)(U^*)) = \eta(((Q_r(a) \otimes \iota)U)^*),\end{aligned}$$

hence

$$\begin{aligned}\eta(\pi(a^*)) &= \eta((Q_r(a^*) \otimes \iota)U) \\ &= \eta((Q_r(a)^* \otimes \iota)U) = \eta(((Q_r(a) \otimes \iota)U)^*) \\ &= \eta(\pi(a)^*).\end{aligned}$$

As this holds for all $\eta \in B_0(K)^*$, we can conclude that $\pi(a^*) = \pi(a)^*$, that is, π is $*$-preserving.

Finally, we prove that π is non-degenerate. Let $v \in K$ be such that $\pi(a)v = 0$ for all $a \in A$. Using that π is $*$-preserving, it is then enough to prove that $v = 0$.

Let $a, b \in A, w \in K$. Set

$$L = (U(\Lambda(a) \otimes v), \Lambda(b) \otimes w) = (((\omega_{\Lambda(a),\Lambda(b)} \otimes \iota)U)v, w).$$

Now, it follows from the proof of Proposition 3.1 that we may pick $d \in A$ such that $Q_{\mathrm{r}}(d)$ is equal to the restriction of $\omega_{\Lambda(a),\Lambda(b)}$ to $\hat{A}_{\mathrm{r}}$. Hence, we get

$$L = (((Q_{\mathrm{r}}(d) \otimes \iota)U)v, w) = (\pi(d)v, w) = 0.$$

As this holds for all $b \in A, w \in K$, this implies that $U(\Lambda(a) \otimes v) = 0$ for all $a \in A$. Thus, $\Lambda(a) \otimes v = U^*U(\Lambda(a) \otimes v) = 0$ for all $a \in A$. Since $\Lambda(A)$ is dense in H, this implies $v = 0$, as desired. □

Remark. Let $U \in \hat{M} \bar{\otimes} B(K)$ be a unitary such that $(\hat{\Delta}_{\mathrm{r,op}} \bar{\otimes} \iota)U = U_{13}U_{23}$. Then the map $\tilde{\pi}_U : A \to B(K)$ defined by $\tilde{\pi}_U(a) = (Q(a) \bar{\otimes} \iota)U, \ a \in A$, may also be seen to be a non-degenerate $*$-homomorphism, by a similar proof. This $*$-homomorphism extends by universality to a $*$-representation of A_{u} on K. It will follow from our next result that we in fact have $U \in M(\hat{A}_{\mathrm{r}} \otimes B_0(K))$. This means that U is indeed a unitary corepresentation of $(\hat{A}_{\mathrm{r}}, \hat{\Delta}_{\mathrm{r,op}})$ and $\tilde{\pi}_U = \pi_U$.

We now show that every non-degenerate $*$-representation of A_{u} has a unitary generator. Alternatively, one may formulate this result for non-degenerate $*$-representations of A.

Theorem 3.3. *Let ϕ be a non-degenerate $*$-representation of A_{u} on some Hilbert space K. Set $A_\phi = \phi(A_{\mathrm{u}}) = \overline{\phi(A)} \subset B(K)$. Then there exists a unique unitary corepresentation $U = U(\phi)$ of $(\hat{A}_{\mathrm{r}}, \hat{\Delta}_{\mathrm{r,op}})$ on K such that*

$$\phi(a) = (Q_{\mathrm{r}}(a) \otimes \iota)U, \quad a \in A.$$

Hence, in the notation of Theorem 3.2, we have $\pi_{U(\phi)} = \phi$.

The norm-closure of $\{(\omega \otimes \iota)U \mid \omega \in B_0(H)^*\}$ is equal to A_ϕ and U belongs to $M(\hat{A}_{\mathrm{r}} \otimes A_\phi)$.

Finally, we have $U(\pi_{\mathrm{r}}) = \hat{W}$ and $U(\varepsilon_{\mathrm{u}}) = 1_H \otimes 1$, which may be equivalently written as $\pi_{\hat{W}} = \pi_{\mathrm{r}}$ and $\pi_{1_H \otimes 1} = \varepsilon_{\mathrm{u}}$.

Proof. Let $v \in K$ and define $\Lambda_v : A \to K$ by

$$\Lambda_v(c) = \phi(c)v, \quad c \in A.$$

Now let $a, b \in A$ and choose $a_1, b_1, \ldots, a_n, b_n \in A$ such that

$$(*) \quad \Delta(a)(b \otimes 1) = \sum_{i=1}^{n} b_i \otimes a_i.$$

Then observe that

$$\sum_{i=1}^{n} \Lambda(a_i) \otimes \phi(b_i)v = (\Lambda \odot \Lambda_v)\left(\sum_{i=1}^{n} a_i \otimes b_i\right) = (\Lambda \odot \Lambda_v)(\chi(\Delta(a)(b \otimes 1))).$$

So the left-hand side above is independent of the choice of the a_i's and b_i's as long as they satisfy $(*)$. We set therefore

$$U(\Lambda(a) \otimes \phi(b)v) = \sum_{i=1}^{n} \Lambda(a_i) \otimes \phi(b_i)v.$$

Observe now that

$$\begin{aligned}\sum_{i,j=1}^{n} b_j^* b_i \varphi(a_j^* a_i) &= (\iota \otimes \varphi)\left(\left(\sum_{j=1}^{n} b_j^* \otimes a_j^*\right)\left(\sum_{i=1}^{n} b_i \otimes a_i\right)\right) \\ &= (\iota \otimes \varphi)((\Delta(a)(b \otimes 1))^*(\Delta(a)(b \otimes 1))) \\ &= (\iota \otimes \varphi)((b^* \otimes 1)\Delta(a^* a)(b \otimes 1)) \\ &= b^* b \varphi(a^* a) \quad \text{(by left invariance of } \varphi).\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
||U(\Lambda(a)\otimes\phi(b)v)||^2 &= \sum_{i,j=1}^{n} (\Lambda(a_i)\otimes\phi(b_i)v, \Lambda(a_j)\otimes\phi(b_j)v) \\
&= \sum_{i,j=1}^{n} (\Lambda(a_i),\Lambda(a_j))(\phi(b_i)v,\phi(b_j)v) \\
&= \sum_{i,j=1}^{n} \varphi(a_j^*a_i)(\phi(b_j^*b_i)v,v) \\
&= \left(\phi\left(\sum_{i,j=1}^{n} b_j^*b_i\varphi(a_j^*a_i)\right)v,v\right) = (\phi(b^*b\varphi(a^*a))v,v) \\
&= \varphi(a^*a)(\phi(b^*b)v,v) = ||\Lambda(a)\otimes\phi(b)v||^2.
\end{aligned}
$$

If we now extend U by linearity to a map from $\Lambda(A)\odot\phi(A)K$ into itself, the same kind of argument as above shows that U is a well-defined isometry on $\Lambda(A)\odot\phi(A)K$, and thereby that the map U is well defined. Since ϕ is assumed to be non-degenerate, we can extend U to a linear isometry on $H\otimes K$. Moreover, using the cancellation properties of (A,Δ), one easily checks that U is surjective, hence that it is a unitary on $H\otimes K$.

We now show that $U\in M(\hat{A}_{\mathrm{r}}\otimes A_\phi)$.

Let $a,b\in A$. Write $\Delta(b)(a\otimes 1)=\sum_{k=1}^{n} a_k\otimes b_k$ for some $a_1,b_1,\dots,a_n,b_n\in A$. Then

$$
(**)\quad U(\hat{\pi}(\hat{b})\otimes\phi(a)) = \sum_{k=1}^{n} \hat{\pi}(\hat{b_k})\otimes\phi(a_k).
$$

To prove this, consider $c\in A, v\in K$. It suffices to show that

$$
(U(\hat{\pi}(\hat{b})\otimes\phi(a)))(\hat{\Lambda}(\hat{c})\otimes v) = \left(\sum_{k=1}^{n} \hat{\pi}(\hat{b_k})\otimes\phi(a_k)\right)(\hat{\Lambda}(\hat{c})\otimes v),
$$

that is,

$$
(***)\quad U(\hat{\Lambda}(\hat{b}\hat{c})\otimes\phi(a)v) = \sum_{k=1}^{n} \hat{\Lambda}(\hat{b_k}\hat{c})\otimes\phi(a_k)v.
$$

Let L denote the left side of equation $(***)$. Write $b\otimes c=\sum_{i=1}^{r} \Delta(p_i)(q_i\otimes 1)$ for some $p_1,q_1,\dots,p_r,q_r\in A$. Then we have $\hat{b}\hat{c}=\sum_{i=1}^{r} \varphi(q_i)\hat{p_i}$ (as in the proof of Lemma 2.3). Further, for each $i=1,\dots,r$, write $\Delta(p_i)(a\otimes 1)=\sum_{j=1}^{s(i)} x_{ij}\otimes y_{ij}$ for some $x_{i1},y_{i1},\dots,x_{is(i)},y_{is(i)}\in A$.

Then, using the definition of U at the second step, we get

$$\begin{aligned} L &= \sum_{i=1}^{m} \varphi(q_i) U(\Lambda(p_i) \otimes \phi(a)v) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{s(i)} \varphi(q_i)(\Lambda(y_{ij}) \otimes \phi(x_{ij})v) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{s(i)} \hat{\Lambda}(\varphi(q_i)\hat{y}_{i,j}) \otimes \phi(x_{ij})v. \end{aligned}$$

Set $X = \sum_{i=1}^{m} \sum_{j=1}^{s(i)} \varphi(q_i)\hat{y}_{ij} \otimes x_{ij} \in \hat{A} \odot A$. Then, to show that $(***)$ holds, it clearly suffices to prove that $X = \sum_{k=1}^{n} \hat{b_k}\hat{c} \otimes a_k$. But this is precisely what is established in Lemma 2.6. Hence, we have shown that $(**)$ holds.

Let $\mathscr{F} : A \to \hat{A}$ denote the "Fourier transform", that is $\mathscr{F}(a) = \hat{a},\ a \in A$. Then $(**)$ may be rewritten as

$$U(\hat{\pi}(\hat{b}) \otimes \phi(a)) = (\hat{\pi} \odot \phi)(\mathscr{F} \odot \iota)\chi(\Delta(b)(a \otimes 1)), \quad a, b \in A.$$

This means that $U(\hat{\pi}(\hat{A}) \odot \phi(A)) \subset \hat{\pi}(\hat{A}) \odot \phi(A)$. Since $\mathscr{F}$ is bijective and $\Delta(A)(1 \otimes A) = A \otimes A$, we get in fact equality. A continuity argument gives then $U(\hat{A}_{\mathrm{r}} \otimes A_\phi) = (\hat{A}_{\mathrm{r}} \otimes A_\phi)$. As U is unitary, we also get $U^*(\hat{A}_{\mathrm{r}} \otimes A_\phi) = (\hat{A}_{\mathrm{r}} \otimes A_\phi)$. Applying the $*$-operation in $B(H \otimes K)$, we then get $(\hat{A}_{\mathrm{r}} \otimes A_\phi)U = (\hat{A}_{\mathrm{r}} \otimes A_\phi)$. Hence, we have shown that $U \in M(\hat{A}_{\mathrm{r}} \otimes A_\phi)$.

Next we show that $U \in M(\hat{A}_{\mathrm{r}} \otimes B_0(K))$. To this end, we first prove that $U(\hat{A}_{\mathrm{r}} \otimes B_0(K)) \subset \hat{A}_{\mathrm{r}} \otimes B_0(K)$. Now, for all $a, b, c \in A$ and $u, v \in K$, we have

$$U(\hat{\pi}(\hat{bc}) \otimes (\cdot, u)\phi(a)v) = U(\hat{\pi}(\hat{b}) \otimes \phi(a))(\hat{\pi}(\hat{c}) \otimes (\cdot, u)v) \in \hat{\pi}(\hat{A}) \odot B_0(K)$$

as we have seen that $U(\hat{\pi}(\hat{b}) \otimes \phi(a)) \in \hat{\pi}(\hat{A}) \odot \phi(A)$. Since $\hat{A}^2 = \hat{A}$ and ϕ is non-degenerate, it follows from a continuity argument that $U(\hat{A}_{\mathrm{r}} \otimes B_0(K)) \subset \hat{A}_{\mathrm{r}} \otimes B_0(K)$, as desired. Now, using that $U^*(\hat{\pi}(\hat{A}) \odot \phi(A)) \subset \hat{\pi}(\hat{A}) \odot \phi(A)$, we get similarly that $U^*(\hat{A}_{\mathrm{r}} \otimes B_0(K)) \subset \hat{A}_{\mathrm{r}} \otimes B_0(K)$. Taking adjoints, we get $(\hat{A}_{\mathrm{r}} \otimes B_0(K))U \subset \hat{A}_{\mathrm{r}} \otimes B_0(K)$. Hence, we have shown that $U \in M(\hat{A}_{\mathrm{r}} \otimes B_0(K))$.

We now establish the following formula:

$$(****) \quad (\omega_{\Lambda(a),\Lambda(b)} \otimes \iota)U = \phi((\iota \otimes \varphi)((1 \otimes b^*)\Delta(a))), \quad a, b \in A.$$

Let $d \in A$ and $v, w \in K$.

Write $\Delta(a)(d\otimes 1)=\sum_{i=1}^{n} d_i\otimes a_i$ for $a_1,d_1,\ldots,a_n,d_n\in A$. Then

$$\begin{aligned}(\omega_{\Lambda(a),\Lambda(b)}\otimes\iota)\,U(\,\phi(d)v,w) &= (U(\Lambda(a)\otimes\phi(d)v),\Lambda(b)\otimes w)\\ &= \sum_{i=1}^{n}\,(\Lambda(a_i)\otimes\phi(d_i)v,\Lambda(b)\otimes w)\\ &= \sum_{i=1}^{n}\,\varphi(b^*a_i)(\phi(d_i)v,w)\\ &= \left(\phi\left(\sum_{i=1}^{n}\,\varphi(b^*a_i)d_i\right)v,w\right)\\ &= \left(\phi\left((\iota\otimes\varphi)\left(\sum_{i=1}^{n}\,d_i\otimes b^*a_i\right)\right)v,w\right)\\ &= (\phi((\iota\otimes\varphi)((1\otimes b^*)\Delta(a)(d\otimes 1)))v,w)\\ &= (\phi((\iota\otimes\varphi)((1\otimes b^*)\Delta(a)))\,\phi(d)v,w),\end{aligned}$$

which shows $(****)$.

Using this formula, the norm-closure of $\{(\omega\otimes\iota)U\mid\omega\in B_0(H)^*\}$ is easily seen to be equal to A_ϕ.

We are now in position to prove the formula relating ϕ and U, that is,

$$\phi(a)=(Q_{\mathrm{r}}(a)\otimes\iota)U,\quad a\in A.$$

Let $a\in A$. Pick $c\in A$ such that

$$\hat{c}\widehat{S(a^*)}=\widehat{S(a^*)}\hat{c}=\widehat{S(a^*)}.$$

Then we have

$$\begin{aligned}(Q_{\mathrm{r}}(a)\otimes\iota)U &= (\omega_{\Lambda(a),\Lambda(c)}\otimes\iota)U\quad\text{(using Proposition 3.1)}\\ &= \phi((\iota\otimes\varphi)((1\otimes c^*)\Delta(a)))\quad\text{(using $(****)$ above)}\\ &= \phi(a)\quad\text{(using Lemma 2.4)},\end{aligned}$$

as desired.

Once this fundamental formula is established, the corepresentation property and the uniqueness of U follow readily from the norm-density of $Q(A)$ in $\hat{M}_*$. For example, regarding $(\hat{\Delta}_{\mathrm{r,op}}\otimes\iota)U$ and $U_{13}U_{23}$ as lying in $\hat{M}\bar{\otimes}\hat{M}\bar{\otimes}B(K)$, as we may, we have

$$
\begin{aligned}
&(Q(a)\bar{\otimes} Q(b)\bar{\otimes}\iota)((\hat{\Delta}_{\mathrm{r,op}}\bar{\otimes}\iota)U)\\
&\quad = (((Q(a)\bar{\otimes} Q(b))\hat{\Delta}_{\mathrm{r,op}})\bar{\otimes}\iota)U\\
&\quad = (Q(ab)\bar{\otimes}\iota)U = (Q_{\mathrm{r}}(ab)\otimes\iota)U = \phi(ab) = \phi(a)\phi(b)\\
&\quad = ((Q_{\mathrm{r}}(a)\otimes\iota)U)((Q_{\mathrm{r}}(b)\otimes\iota)U)\\
&\quad = ((Q(a)\bar{\otimes}\iota)U)((Q(b)\bar{\otimes}\iota)U)\\
&\quad = (Q(a)\bar{\otimes} Q(b)\bar{\otimes}\iota)(U_{13}U_{23})
\end{aligned}
$$

for all $a, b\in A$.

Finally, we check the last assertion of the theorem. Let $a, b, c\in A$ and choose a_i's and b_i's such as $(*)$ holds. Then we have

$$
\begin{aligned}
\hat{W}(\Lambda(a)\otimes\pi_{\mathrm{r}}(b)\Lambda(c)) &= (\Sigma W^*\Sigma)(\Lambda(a)\otimes\Lambda(bc)) = (\Sigma W^*)(\Lambda(bc)\otimes\Lambda(a))\\
&= \Sigma(\Lambda\odot\Lambda)(\Delta(a)(bc\otimes 1)) = \Sigma(\Lambda\odot\Lambda)\left(\sum_{i=1}^{n} b_i c\otimes a_i\right)\\
&= \sum_{i=1}^{n} (\Lambda\odot\Lambda)(a_i\otimes b_i c)\\
&= \sum_{i=1}^{n} \Lambda(a_i)\otimes\pi_{\mathrm{r}}(b_i)\Lambda(c) = U(\pi_{\mathrm{r}})(\Lambda(a)\otimes\pi_{\mathrm{r}}(b)\Lambda(c))
\end{aligned}
$$

and

$$
\begin{aligned}
(1_H\otimes 1)(\Lambda(a)\otimes\varepsilon_{\mathrm{u}}(b)1) &= \Lambda(a\varepsilon(b)) = \Lambda((\varepsilon\otimes\iota)(\Delta(a)(b\otimes 1)))\\
&= \Lambda\left(\sum_{i=1}^{n} a_i\varepsilon(b_i)\right) = \sum_{i=1}^{n}\Lambda(a_i)\otimes\varepsilon_{\mathrm{u}}(b_i)1\\
&= U(\varepsilon_{\mathrm{u}})(\Lambda(a)\otimes\varepsilon_{\mathrm{u}}(b)1).
\end{aligned}
$$

This clearly implies that $\hat{W} = U(\pi_{\mathrm{r}})$ and $1_H\otimes 1 = U(\varepsilon_{\mathrm{u}})$, as desired. □

Remark. It is clear that Theorems 3.2 and 3.3 together provide a bijective correspondence between unitary corepresentations of $(\hat{A}_{\mathrm{r}}, \hat{\Delta}_{\mathrm{r,op}})$ and non-degenerate $*$-representations of A_{u}. In a similar way, one may prove that there is a bijective correspondence between unitary corepresentations of $(\hat{A}_{\mathrm{r}}, \hat{\Delta}_{\mathrm{r}})$ and non-degenerate $*$-representations of A_{u}, as proved in [9] in a quite different way. Alternatively, one may use here that $U \to U^*$ gives a bijective correspondence between unitary corepresentations of $(\hat{A}_r, \hat{\Delta}_{r,op})$ and unitary corepresentations of $(\hat{A}_r, \hat{\Delta}_r)$.

We also mention that Theorems 3.2 and 3.3 may easily be dualized to produce a bijective correspondence between unitary corepresentations of (A_r, Δ_r) and non-degenerate $*$-representations of $\hat{A}_u$.

Remark. Let $V \in \hat{M} \bar{\otimes} B(K)$ be a unitary such that $(\hat{\Delta}_{r,op} \bar{\otimes} \iota) V = V_{13} V_{23}$ and let $\tilde{\pi}_V$ be the associated $*$-representation of A_u defined in our previous remark. As a consequence of Theorem 3.3 we then get

$$\begin{aligned}(Q(a) \bar{\otimes} \iota) V &= \tilde{\pi}_V(a) \\ &= (Q_r(a) \otimes \iota) U(\tilde{\pi}_V) = (Q(a) \bar{\otimes} \iota) U(\tilde{\pi}_V)\end{aligned}$$

for all $a \in A$. This implies that $V = U(\tilde{\pi}_V)$.

Especially, we have $V \in M(\hat{A}_r \otimes B_0(K))$, as mentioned in a previous remark.

Remark. Let U be a unitary corepresentation of $(\hat{A}_r, \hat{\Delta}_{r,op})$ on K. We can define a representation Φ_U of the Banach algebra $\hat{M}_*$ on K by $\Phi_U(\omega) = (\omega \bar{\otimes} \iota) U$, $\omega \in \hat{M}_*$. Then we have $(\Phi_U \circ Q)(a) = \pi_U(a)$, $a \in A$, so $\Phi_U \circ Q$ is $*$-preserving and non-degenerate. One easily sees that the map $U \to \Phi_U$ gives a bijective correspondence between all unitary corepresentations of $(\hat{A}_r, \hat{\Delta}_{r,op})$ and all representations Φ of $\hat{M}_*$ such that $\Phi \circ Q$ is $*$-preserving and non-degenerate.

4. Amenability, co-amenability and nuclearity

We prove in this section the results stated in the Introduction.

Theorem 4.1. *Let (A, Δ) be an algebraic quantum group. Then (A, Δ) is co-amenable if and only if $A_u = A_r$, that is, the canonical map π_r from A_u onto A_r is injective.*

Proof. Assume that $A_u = A_r$. Then π_r is an isometry.

Let $a \in A$. Then we have $\|\pi_u(a)\|_u = \|\pi_r(\pi_u(a))\| = \|\pi(a)\|$. Hence,

$$|\varepsilon(a)| = |\varepsilon_u(\pi_u(a))| \leqslant \|\pi_u(a)\|_u = \|\pi(a)\|.$$

This shows that ε is bounded on A with respect to the reduced norm, that is, (A, Δ) is co-amenable.

Assume now that (A, Δ) is co-amenable, that is, $|\varepsilon(a)| \leqslant \|\pi(a)\|$ for all $a \in A$. Using Theorem 3.3, we can express this as

$$|(Q(a) \bar{\otimes} \iota)(1_H \otimes 1)| \leqslant |(Q(a) \bar{\otimes} \iota) \hat{W}|, \quad a \in A.$$

Using the norm-density of $Q(A)$ in $\hat{M}_*$ (cf. Proposition 3.1) and a continuity argument, we can conclude that

$$|\omega(1_H)| = |(\omega \bar{\otimes} \iota)(1_H \otimes 1)| \leqslant ||(\omega \bar{\otimes} \iota)\hat{W}||, \quad \omega \in \hat{M}_*.$$

To show that $A_{\mathrm{u}} = A_{\mathrm{r}}$, it is enough to show that π_{r} is isometric on $A = \pi_{\mathrm{u}}(A)$, or, equivalently, that $||a||_{\mathrm{u}} \leqslant ||\pi(a)||$, $a \in A$ (since the reverse inequality always holds by definition of the universal norm). To show this inequality, it suffices to show that

$$||\phi(a)|| \leqslant ||\pi(a)||$$

for any given non-degenerate $*$-representation ϕ of A on some Hilbert space K and any given $a \in A$.

Now, let $U = U(\phi)$ be a generator for ϕ (extended to A_{u} to be pedantic), according to Theorem 3.3. Then this amounts to show

$$(*) \quad ||(Q(a) \bar{\otimes} \iota)U|| \leqslant ||(Q(a) \bar{\otimes} \iota)\hat{W}||.$$

To show $(*)$, we adapt an argument from [4, Proposition 5.5] (where Blanchard characterizes the amenability of regular multiplicative unitaries).

Let $v, w \in K$, $||v|| = ||w|| = 1$. Define $\omega \in \hat{M}_*$ by

$$\omega(x) = (Q(a) \bar{\otimes} \omega_{v,w})((x \otimes 1)U), \quad x \in \hat{M}.$$

Then

$$\omega(1_H) = (Q(a) \bar{\otimes} \omega_{v,w})U = \omega_{v,w}((Q(a) \bar{\otimes} \iota)U).$$

Hence we have

$$(**) \quad |\omega_{v,w}((Q(a) \bar{\otimes} \iota)U)| = |\omega(1_H)| \leqslant ||(\omega \bar{\otimes} \iota)\hat{W}||.$$

Now, recall (from the proof of Theorem 3.2) that we have

$$\hat{W}^*_{12} U_{23} \hat{W}_{12} = U_{13} U_{23}.$$

Therefore, applying $\iota \bar{\otimes} \sigma$ to this equation, where σ denotes the flip map from $B(H \otimes K)$ to $B(K \otimes H)$, we get

$$U_{32} \hat{W}_{13} U^*_{32} = \hat{W}_{13} U_{12}.$$

Using this, we obtain

$$\begin{aligned}(\omega \bar{\otimes} \iota)\hat{W} &= (Q(a) \bar{\otimes} \omega_{v,w} \bar{\otimes} \iota)(\hat{W}_{13} U_{12}) \\ &= (Q(a) \bar{\otimes} \omega_{v,w} \bar{\otimes} \iota)(U_{32} \hat{W}_{13} U^*_{32}) \\ &= (\omega_{v,w} \bar{\otimes} \iota)(\sigma(U)(1_K \bar{\otimes} (Q(a) \bar{\otimes} \iota)\hat{W})\sigma(U)^*),\end{aligned}$$

which implies that

$$(***) \quad \|(\omega \bar{\otimes} \iota)\hat{W}\| \leqslant \|(Q(a) \bar{\otimes} \iota)\hat{W}\|.$$

Combining $(**)$ and $(***)$, we get

$$|\omega_{v,w}((Q(a) \bar{\otimes} \iota)U)| \leqslant \|(Q(a) \bar{\otimes} \iota)\hat{W}\|.$$

As this holds for all $v, w \in K$, $\|v\| = \|w\| = 1$, this implies that $(*)$ holds, which finishes the proof. $\square$

Remark. In [1, Appendice], Baaj and Skandalis introduce the notions of amenability and co-amenability for regular multiplicative unitaries (see also [4]). These notions may be adapted to multiplicative unitaries associated to algebraic quantum groups as follows. We first remark that, from the point of view adopted in [1], it is quite natural to consider $V = \hat{W}$ as *the* multiplicative unitary associated with an algebraic quantum group (A, Δ); this point of view is supported by the fact that $\pi_{\hat{W}} = \pi_{\mathrm{r}}$, which we pointed out in Theorem 3.3. However, this is essentially a matter of convention. The adapted Baaj–Skandalis definition of co-amenability of $V = \hat{W}$ amounts then to require that $\pi_{\hat{W}} : A_{\mathrm{u}} \to A_{\mathrm{r}}$ is injective, in which case one also says that W is amenable. Co-amenability of W and amenability of V may be defined similarly by considering W to be *the* multiplicative unitary associated with $(\hat{A}, \hat{\Delta})$. Using this terminology, Theorem 4.1 just says that (A, Δ) is co-amenable if, and only if, $V = \hat{W}$ is co-amenable (resp. W is amenable).

Before stating our next result, we recall that a von Neumann algebra N acting on a Hilbert space K is called injective [14] if there exists a linear, norm one projection map from $B(K)$ onto N.

Theorem 4.2. *Assume that (A, Δ) is an algebraic quantum group such that $(\hat{A}, \hat{\Delta})$ is amenable. Let ϕ be any non-degenerate $*$-representation of A_{u} on some Hilbert space K. Then the von Neumann algebra $N = \phi(A_{\mathrm{u}})'' \subset B(K)$ is injective.*

Proof. By a classical result of Tomiyama [14,17], we can equivalently show that $N' = \phi(A_{\mathrm{u}})' = \phi(A)'$ is injective, that is, we have to construct a linear, norm one projection of $B(K)$ onto $\phi(A)'$.

Let U be a unitary generator for ϕ, so $U \in M(\hat{A}_{\mathrm{r}} \otimes B_0(K)) \subset \hat{M} \bar{\otimes} B(K)$, according to Theorem 3.3. We introduce the unital (injective) normal $*$-homomorphism $\alpha : B(K) \to \hat{M} \bar{\otimes} B(K)$ given by

$$\alpha(x) = U^*(1 \otimes x)U, \quad x \in B(K).$$

Then α is an action of $\hat{M}$ on $B(K)$, in the sense that $(\iota\bar{\otimes}\alpha)\alpha = (\hat{\Delta}_{\mathrm{r,op}}\bar{\otimes}\iota)\alpha$. Indeed, we have

$$\begin{aligned}
(\iota\bar{\otimes}\alpha)\alpha(x) &= (\iota\bar{\otimes}\alpha)(U^*(1\otimes x)U) = ((\iota\bar{\otimes}\alpha)U)^*(1\otimes\alpha(x))((\iota\bar{\otimes}\alpha)U)\\
&= (U_{23}^*U_{13}U_{23})^*(1\otimes U^*(1\otimes x)U)(U_{23}^*U_{13}U_{23})\\
&= (U_{23}^*U_{13}^*U_{23})U_{23}^*(1\otimes 1\otimes x)U_{23}(U_{23}^*U_{13}U_{23})\\
&= U_{23}^*U_{13}^*(1\otimes 1\otimes x)U_{13}U_{23} = (U_{13}U_{23})^*(1\otimes 1\otimes x)U_{13}U_{23}\\
&= ((\hat{\Delta}_{\mathrm{r,op}}\bar{\otimes}\iota)U)^*(\hat{\Delta}_{\mathrm{r,op}}(1)\otimes x)(\hat{\Delta}_{\mathrm{r,op}}\bar{\otimes}\iota)U\\
&= (\hat{\Delta}_{\mathrm{r,op}}\bar{\otimes}\iota)(U^*(1\otimes x)U) = (\hat{\Delta}_{\mathrm{r,op}}\bar{\otimes}\iota)\,\alpha(x)
\end{aligned}$$

for all $x\in B(K)$.

Set

$$B(K)^\alpha = \{x\in B(K)\mid \alpha(x) = 1\otimes x\}.$$

Then, using the density of $Q(A)$ in $\hat{M}_*$ and Theorem 3.3, we have

$$\begin{aligned}
B(K)^\alpha &= \{x\in B(K)\mid (1\otimes x)U = U(1\otimes x)\}\\
&= \{x\in B(K)\mid (Q(a)\bar{\otimes}\iota)((1\otimes x)U) = (Q(a)\bar{\otimes}\iota)(U(1\otimes x)),\ \forall a\in A\}\\
&= \{x\in B(K)\mid x((Q(a)\bar{\otimes}\iota)U) = ((Q(a)\bar{\otimes}\iota)U)x,\ \forall a\in A\}\\
&= \{x\in B(K)\mid x\phi(a) = \phi(a)x,\ \forall a\in A\}\\
&= \phi(A)'.
\end{aligned}$$

We shall now adapt an argument of Enock and Schwartz given in the proof of [7, Theorem 3.1] to construct a linear, norm one projection from $B(K)$ onto $B(K)^\alpha = \phi(A)'$.

Using our amenability assumption, we can pick a right-invariant mean m for $(\hat{A},\hat{\Delta}_{\mathrm{op}})$ (picking first a left-invariant mean for $(\hat{A},\hat{\Delta})$ and combining it with the anti-unitary antipode of $\hat{M}$).

Using that $|m((\iota\bar{\otimes}\eta)\alpha(x))|\leqslant\|\eta\|\,\|x\|$ for all $\eta\in B(K)_*,\ x\in B(K)$, one easily sees that there exists a linear contraction map $E: B(K)\to B(K)$ such that

$$\eta(E(x)) = m((\iota\bar{\otimes}\eta)\alpha(x)),\quad \eta\in B(K)_*,\ x\in B(K).$$

For $\omega \in \hat{M}_*$, $\eta \in B(K)_*$, we have

$$
\begin{aligned}
(\omega \bar{\otimes} \eta)(\alpha(E(x)) &= ((\omega \bar{\otimes} \eta) \circ \alpha)(E(x)) \\
&= m((\iota \bar{\otimes} (\omega \bar{\otimes} \eta) \circ \alpha)\alpha(x)) = m((\iota \bar{\otimes} \omega \bar{\otimes} \eta)(\iota \bar{\otimes} \alpha)\alpha(x)) \\
&= m((\iota \bar{\otimes} \omega \bar{\otimes} \eta)(\hat{\Delta}_{\mathrm{r,op}} \bar{\otimes} \iota)\alpha(x)) = m((\iota \bar{\otimes} \omega)\hat{\Delta}_{\mathrm{r,op}}((\iota \bar{\otimes} \eta)\alpha(x))) \\
&= m((\iota \bar{\otimes} \eta)\alpha(x))\omega(1) \quad \text{(using right-invariance of } m) \\
&= \eta(E(x))\omega(1) = (\omega \bar{\otimes} \eta)(1 \bar{\otimes} E(x)).
\end{aligned}
$$

It follows that $\alpha(E(x)) = 1 \otimes E(x)$ for all $x \in B(K)$, hence that E maps $B(K)$ into $B(K)^{\alpha}$. Further, if $x \in B(K)^{\alpha}$, that is, $\alpha(x) = 1 \otimes x$, then

$$\eta(E(x)) = m((\iota \bar{\otimes} \eta)(1 \otimes x)) = m(1)\eta(x) = \eta(x)$$

for all $\eta \in B(K)_*$. Thus, $E(x) = x$ for all $x \in B(K)^{\alpha}$. It clearly follows that E is a norm one projection from $B(K)$ onto $B(K)^{\alpha}$, which finishes the proof. $\square$

Corollary 4.3. *Assume that (A, Δ) is an algebraic quantum group such that $(\hat{A}, \hat{\Delta})$ is amenable. Then A_{u} is nuclear.*

Proof. By applying Theorem 4.2 to the universal $*$-representation Φ of A_{u}, we obtain that the second dual $A_{\mathrm{u}}^{**} = \Phi(A_{\mathrm{u}})''$ is injective. By a famous result of Connes, Choi and Effros (see [14, 2.35] for references), this is equivalent to the nuclearity of A_{u}. $\square$

We shall now give a simplified proof of a result which is essentially due to Ruan (see [15, Theorem 4.5]).

Theorem 4.4. *Assume that (A, Δ) is an algebraic quantum group such that its associated von Neumann algebra $M \subset B(H)$ is injective. Assume further that (A, Δ) is compact with unit* 1 *and has a tracial Haar functional* (*that is, equivalently*, (M, Δ_{r}) *is a compact Kac algebra* [7]).

Then (A, Δ) is co-amenable.

Proof. As usual in the compact case, we work with the normalized Haar functional φ of (A, Δ). It is known [1,22] that the traciality of φ is equivalent to $S^2 = \iota$, or, equivalently, to S being $*$-preserving.

Using the traciality assumption and the fact that φ is S-invariant, it is straightforward to check that the linear map $V_0 : \Lambda(A) \to \Lambda(A)$ defined by $V_0(\Lambda(a)) = \Lambda(S(a))$, $a \in A$, is an isometry, which extends to a self-adjoint unitary V on H. (See [1, Proposition 5.2] for a similar statement in the non-tracial case, which we will use in the proof of our next result). A simple calculation gives

$V\pi(a)V\pi(b) = \pi(b)V\pi(a)V$ for all $a, b \in A$. Hence, $\mathrm{Ad}(V)$ maps A_{r} (and M) into $\pi(A)' = M'$.

Now, we recall that injectivity of M implies that the $*$-homomorphism $P : M \odot M' \to B(H)$ determined by

$$P(x \otimes y) = xy, \quad x \in M, \quad y \in M'$$

has a bounded extension $\tilde{P} : M \otimes M' \to B(H)$, where we stress that $\otimes$ denotes the minimal tensor product (as opposed to the von Neumann algebra tensor product). Note that this deep result is not mentioned explicitly in [14]. It may be deduced from the literature as follows. Injectivity is equivalent to semidiscreteness, as first shown by Connes [5] in the factor case. A direct proof of the forward implication due to Wassermann may be found in [20]. The backward implication is shown by Effros and Lance [6], who also show that semidiscreteness is equivalent to the above property.

We use $\tilde{P}$ to define a map $\varepsilon_0 : A_{\mathrm{r}} \to \mathbf{C}$ by

$$\varepsilon_0(x) = (((\tilde{P} \circ (\iota \otimes \mathrm{Ad}(V)) \circ \Delta_{\mathrm{r}})(x))\, \Lambda(1), \Lambda(1)), \quad x \in A_{\mathrm{r}}.$$

Clearly, ε_0 is a state on A_{r}. Further, we have $\varepsilon_0(\pi(a)) = \varepsilon(a), a \in A$. Before establishing this fact, we point out that it clearly implies that ε is bounded with respect to the reduced norm of A_{r}, that is, (A, Δ) is co-amenable.

Let $a \in A$ and write $\Delta(a) = \sum_{i=1}^{n} a_i \otimes b_i$ for some $a_1, b_1, \ldots, a_n, b_n \in A$. Then

$$\begin{aligned}
&((\tilde{P} \circ (\iota \otimes \mathrm{Ad}(V)) \circ \Delta_{\mathrm{r}})(\pi(a)))\, \Lambda(1) \\
&\quad = ((\tilde{P} \circ (\iota \otimes \mathrm{Ad}(V)) \circ (\pi \otimes \pi)(\Delta(a)))\Lambda(1) \\
&\quad = \left(\sum_{i=1}^{n} \pi(a_i)\right) \mathrm{Ad}(V)(\pi(b_i))\Lambda(1) \\
&\quad = \Lambda\left(\sum_{i=1}^{n} a_i S(b_i)\right) = \Lambda(m(\iota \otimes S)\Delta(a)) \\
&\quad = \Lambda(\varepsilon(a)1) = \varepsilon(a)\Lambda(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
\varepsilon_0(\pi(a)) &= (((\tilde{P} \circ (\iota \otimes \mathrm{Ad}(V)) \circ \Delta_{\mathrm{r}})(\pi(a))\, \Lambda(1), \Lambda(1)) \\
&= (\varepsilon(a)\Lambda(1), \Lambda(1)) = \varepsilon(a),
\end{aligned}$$

as asserted. $\square$

It would be interesting to know whether the traciality assumption in Theorem 4.4 is redundant. We shall now prove a related result, saying that injectivity of M implies a kind of perturbed co-amenability of (A, Δ). We recall first some more facts about the compact case.

Let (A, Δ) be a compact algebraic quantum group with unit 1. It is immediate that $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ is a compact quantum group in the sense of Woronowicz [21,22], with Haar state given by the restriction of $\omega_{\Lambda(1)}$ to A_{r}. The unique dense Hopf $*$-subalgebra [2] of $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ may be identified with $(A, \Delta, \varepsilon, S)$ (via the Hopf $*$-algebra isomorphism π). Using this identification, we may introduce the remarkable family $(f_z)_{z \in \mathbf{C}}$ of multiplicative linear functionals on A constructed by Woronowicz (see [21,22]).

Some of the properties of this family are $f_0 = \varepsilon$; $f_z * f_{z'} = f_{z+z'}$, where $\omega * \eta = (\omega \otimes \eta)\Delta$, $\omega, \eta \in A'$; the maps $a \to f_z * a = (\iota \otimes f_z)\Delta(a)$ and $a \to (f_z \otimes \iota)\Delta(a)$ are automorphisms of A; we have $f_z^* = f_{-\bar{z}}$ and $f_z \circ S = f_{-z}$; for all $a, b \in A$, we have $\varphi(ab) = \varphi(b(f_1 * a * f_1))$ and $S^2(a) = f_{-1} * a * f_1$.

We also mention that the following three conditions are equivalent:

φ is tracial; $f_z = \varepsilon$ for all $z \in \mathbf{C}$; $f_1 = \varepsilon$.

Theorem 4.5. *Assume that (A, Δ) is a compact algebraic quantum group such that its associated von Neumann algebra M is injective. Let (u^α) denote a complete set of pairwise inequivalent irreducible unitary corepresentations of the compact quantum group $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ and let n_α (resp. d_α) denote the ordinary (resp. quantum) dimension of u^α.*

Then there exists a state ε_1 on A_{r} such that

$$\varepsilon_1(u^\alpha_{ij}) = \frac{n_\alpha}{d_\alpha}\delta_{ij}, \quad 1 \leqslant i,\ j \leqslant n_\alpha.$$

Proof. We recall first that $d_\alpha = \sum_{i=1}^{n_\alpha} f_1(u^\alpha_{ii})$. In other words, d_α is the trace of the matrix $F_\alpha = (f_1 \odot \iota)u^\alpha$.

Next, we define $\tilde{S} : A \to A$ by $\tilde{S}(a) = f_1 * S(a)$, $a \in A$. (This map is sometimes called the twisted antipode of (A, Δ)). Using the properties of the f_z's mentioned above, one checks easily that $\tilde{S}$ is an involutive anti-automorphism of A.

Further, as shown in the proof of [1, Proposition 5.2], the linear map $U_0 : \Lambda(A) \to \Lambda(A)$ defined by $U_0(\Lambda(a)) = \Lambda(\tilde{S}(a)), a \in A$, is an isometry, which extends to a self-adjoint unitary U on H satisfying

$$(\mathrm{Ad}(U)(\pi(a)))\Lambda(b) = \Lambda(b\tilde{S}(a)), \quad a, b \in A.$$

It follows readily that $\mathrm{Ad}(U)(x) \in \pi(A)' = M'$, for all $x \in M$. (In fact, one may check that $\mathrm{Ad}(U)(x) = JR(x^*)J$ for all $x \in M$, where $J : H \to H$ denotes the Tomita–Takesaki map such that (H, J) is standard for M.)

Now, let $\tilde{P}$ be as in the proof of Theorem 4.4. We then define a state ε_1 on A_{r} by

$$\varepsilon_1(x) = (((\tilde{P} \circ (\iota \otimes \mathrm{Ad}(U)) \circ \Delta_{\mathrm{r}})(x))\, \Lambda(1), \Lambda(1)), \quad x \in A_{\mathrm{r}}.$$

Using the orthogonality relations [21,22] for the u^α_{ij}'s, one checks that ε_1 satisfies the stated property. More precisely, the computation goes as follows. Fix α, set

$n = n_\alpha, d = d_\alpha$ and write $u^\alpha_{ij} = \pi(v_{ij}),\ v_{ij} \in A$. The matrix (v_{ij}) is then an $n \times n$ unitary matrix over A, and one of the orthogonality relations for the u^α_{ij}'s gives

$$\varphi(v_{ik}v^*_{js}) = (1/d)\delta_{ij}f_1(v_{sk}), \quad i,j,k,s \in \{1, \ldots, n\}.$$

Using this, we get

$$\begin{aligned}
\varepsilon_1(u^\alpha_{ij}) &= (((\tilde{P}\circ(\imath \otimes \mathrm{Ad}(U))\circ(\pi \otimes \pi)(\Delta(v_{ij}))\,\Lambda(1), \Lambda(1)) \\
&= \sum_{k=1}^{n} (\pi(v_{ik})(\mathrm{Ad}(U)(\pi(v_{kj})))\Lambda(1), \Lambda(1)) \\
&= \sum_{k=1}^{n} (\Lambda(v_{ik}\tilde{S}(v_{kj})), \Lambda(1)) = \sum_{k=1}^{n} \varphi(v_{ik}\tilde{S}(v_{kj})) \\
&= \sum_{k=1}^{n} \varphi(v_{ik}(f_1 * v^*_{jk})) = \sum_{k=1}^{n} \varphi(v_{ik}((\imath \odot f_1)\Delta(v^*_{jk}))) \\
&= \sum_{k,s=1}^{n} \varphi(v_{ik}v^*_{js})f_1(v^*_{sk}) = (1/d) \sum_{k,s=1}^{n} \delta_{ij}f_1(v_{sk})f_1(v^*_{sk}) \\
&= (1/d)\delta_{ij}f_1\left(\sum_{k,s=1}^{n} v_{sk}v^*_{sk}\right) = (1/d)\delta_{ij}f_1(n1) = (n/d)\delta_{ij}
\end{aligned}$$

as desired. $\square$

Remark. Assume the existence of a state ε_1 satisfying the statement of Theorem 4.5. If we also assume that $d_\alpha = n_\alpha$ for all α (especially, if we also assume that (A, Δ) has a tracial Haar functional), then $\varepsilon_1\circ\pi$ coincides with the co-unit ε of (A, Δ), and we can then conclude that (A, Δ) is co-amenable. Hence, Theorem 4.4 is just a special case of Theorem 4.5.

In the general case, it is known that the ordinary dimension is always smaller than the quantum dimension (which is always positive and less than infinity). Thus $q_\alpha = n_\alpha/d_\alpha \in \langle 0, 1]$. However, for the relevant examples (like quantum $SU(2)$ etc) it tends exponentially to zero with 'increasing' α's. Of course, one may wonder whether it is possible to use the existence of the state ε_1 to deduce that ε is bounded.

One natural way to proceed is to consider ε_1 as an element of the Banach algebra A^*_r and try to use spectral calculus to "press" up the values q_α to 1. For any function f which is analytic on a region in the complex plane containing the closed unit disk with center at the origin and satisfies $f(0) = 0$, one may check that

$$f(\varepsilon_1)u^\alpha_{ij} = f(q_\alpha)\delta_{ij}$$

for all α and i,j. However, it seems difficult to proceed further without introducing some other assumptions. We also mention that A^*_r is in fact a Banach $*$-algebra with

$*$-operation given by

$$\psi^*(a) = \overline{\psi(R(a^*))}$$

for all $\psi \in A_r^*$ and $a \in A$. One may show that $\varepsilon_1 R = \varepsilon_1$, hence that $\varepsilon_1^* = \varepsilon_1$, but it is not clear that this may be of any help.

Another possible approach is to consider the bounded linear map from A_r into itself given by $\psi_1 = (\iota \otimes \varepsilon_1)\Delta_r$. It is not difficult to show that it is injective. If one could show surjectivity of ψ_1, then, appealing to the Open Mapping Theorem, ψ_1^{-1} would be bounded. Further, we would have $\varepsilon \circ \pi^{-1} = \varepsilon_1 \circ \psi_1^{-1}$ on $\pi(A)$. Hence, we would then be able to conclude that (A, Δ) is co-amenable. We are so far only able to see that ψ_1 has dense range, as it contains $\pi(A)$.

Remark. Let J and ∇ be the Tomita–Takesaki maps associated to $\Lambda(A)$, considered as a left Hilbert algebra in H, so that

$$J\nabla^{1/2}\Lambda(a) = \Lambda(a^*)$$

for all $a \in A$. Further, let τ denote the scaling group of (A, Δ) (see [11]).

For $z \in \mathbf{C}$, define a map $Q_z : \pi(A) \to M'$ by

$$Q_z\pi(a) = J\nabla^{-iz}(R\tau_{-z}\pi(a))^*\nabla^{iz}J$$

for all $a \in A$.

Clearly, Q_z is unital, multiplicative and linear. Setting $z = i/2$ gives,

$$Q_{i/2}\pi(a) = J\Delta^{1/2}\pi(S(a)^*)\Delta^{-1/2}J$$

for all $a \in A$.

If $z = t$ is real, then Q_t is $*$-preserving and bounded, and it may be extended to M. Note also that

$$Q_0(x) = JR(x^*)J, \quad x \in M.$$

Now, define a unital linear functional ϕ_z on $\pi(A)$ by

$$\phi_z\pi(a) = (P(\pi \odot Q_z\pi)\Delta(a)\Lambda(1), \Lambda(1))$$

for all $a \in A$ and $z \in \mathbf{C}, P : M \odot M' \to B(H)$ being defined as in the proof of Theorem 4.4.

For general $z \in \mathbf{C}$, one may easily show that

$$\phi_z\pi(u_{lj}^\alpha) = \sum_k \varphi(u_{lk}^\alpha f_{1+i(z-\bar{z})} * ((u_{jk}^\alpha)^*) * f_{-i(z+\bar{z})})$$

for all α and all l, j.

From this we see that

$$\phi_{i/2}\pi(u_{ij}^{\alpha}) = \delta_{ij} = \varepsilon(u_{ij}^{\alpha})$$

for all α and i,j, so $\phi_{i/2}\pi = \varepsilon$. Hence, co-amenability of (A,Δ) is equivalent to the boundedness of $\phi_{i/2}$.

Now, observe that when $z = t$ is real and M is assumed to be injective, then ϕ_t may be extended to a state on A_{r} such that

$$\phi_t(x) = (\tilde{P}(\imath \otimes Q_t)\Delta_r(x)\Lambda(1), \Lambda(1))$$

for all $x \in A_r$, $\tilde{P}$ being defined as in the proof of Theorem 4.5. Note that ϕ_0 is then just equal to the state ε_1 obtained in this theorem. One may wonder whether some analytic continuation argument could be used in this situation to deduce that $\phi_{i/2}$ is bounded.

Co-amenability of (A,Δ) may be characterized by the existence of a non-zero multiplicative linear functional on A_{r} [3]. However, when $A_{\mathrm{r}} = SU_q(2), q \in (0,1)$, we have checked that none of the ϕ_t are multiplicative, even though $SU_q(2)$ is known to be co-amenable.

Remark. Some of the essence of Theorems 4.4 and 4.5 may be presented in a more conceptual manner. Assume that (A,Δ) is a compact algebraic quantum group. We define the *adjoint* representation C of A on $B(H)$ as follows. Let P be the map introduced in the proof of Theorem 4.4 and U be the unitary on H introduced in the proof of Theorem 4.5. Then set

$$C(a) = ((P \circ (\imath \odot \mathrm{Ad}(U)) \circ \Delta_{\mathrm{r}})(\pi(a)), \quad a \in A.$$

(A more explicit way of defining C is

$$C(a)\Lambda(b) = \sum_{i=1}^{n} \Lambda(a_i b \tilde{S}(a_i'))$$

for $a, b \in A$, and $\Delta(a) = \sum_{i=1}^{n} a_i \otimes a_i'$.)

Using the map $\tilde{P}$ introduced in the proof of Theorem 4.4, one easily deduces that the injectivity of M implies that C is weakly contained in π, that is, more precisely, that the associated $*$-representation C_u of A_{u} is weakly contained in π_{r}. On the other hand, if the Haar state of (A,Δ) is tracial, then $\tilde{S} = S$ and the last part of the proof of Theorem 4.4 shows that ε_u is weakly contained in C_u. Combining these two assertions reproves Theorem 4.4. An open question is then whether ε_u is always weakly contained in C_u. A negative answer to this question is not unlikely, and it would then be of interest to find a more general condition than traciality of the Haar state ensuring the weak containment of ε_u in C_u.

We conclude with a proof of Theorem 1.1.

Proof of Theorem 1.1. (1) implies (2): This result is shown in [3]. For the ease of the reader, we sketch the argument. When (A, Δ) is co-amenable, then $(A_\mathrm{r}, \Delta_\mathrm{r})$ has a bounded co-unit ε_r which is a state on A_r and satisfies $(\varepsilon_\mathrm{r} \otimes \iota)(W) = 1$. A right-invariant mean for $(\hat{A}, \hat{\Delta})$ is then obtained by considering the restriction to $\hat{M}$ of any state extension of ε_r to $B(H)$.

(2) implies (3): This is Corollary 4.3.

(3) implies (4): As A_r is a quotient of A_u, this follows from the fact that quotients of nuclear C*-algebras are nuclear [6].

(4) implies (5): As $M = A_\mathrm{r}''$, this follows from the fact that any von Neumann algebra generated by a nuclear C*-algebra is injective (this is easily seen by using that the double dual of a nuclear C*-algebra is injective, as pointed out in the proof of Theorem 4.2).

Assume that (A, Δ) is compact and has a tracial Haar functional. Then (5) implies (1) is shown in Theorem 4.4. □

Finally, we remark that different proofs of (1) implies (5), and of (5) implies (2) in the compact tracial case, were given in [3].

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