In this section we fix terminology and recall some basic definitions. We also outline some of the results in the paper.

All rings are assumed to be commutative with identity and local rings are Noetherian rings with a unique maximal ideal. Let \((R, m)\) be a local ring with \(m\) as the maximal ideal such that \(\dim R = \text{ht}(m) = n\). There exists \(x = x_1, \ldots, x_n\) in \(R\) such that \((x)R\) has \(m\) as its radical; such a set \(x\) is called a system of parameters of \(R\), abbreviated “s.o.p.” Let \(M\) be an \(R\)-module, not necessarily finitely generated, and let \(y = (y_1, \ldots, y_n)\) be elements of \((R, m)\) such that

1. \((y)M \neq M\),
2. for each \(i, 0 \leq i < n, y_{i+1}\) is not a zero divisor in \(M/(y_1, y_2, \ldots, y_i)M\).

We say that \(M\) is \(y\)-regular, or that \(y\) forms an \(M\)-sequence, of length \(n\). Let \(I \subset m\) be an ideal, \(\text{depth}_I(M)\) is defined as the length of a maximal \(M\)-sequence contained in \(I\). The dimension of \(M\), \(\dim M\), is defined as the dimension of the ring \(R/\text{Ann}(M)\). We have in general \(\dim R \geq \dim M \geq \text{depth}_R M\) where \(\text{depth}_R M\) is just \(\text{depth}_m M\). In the case when all of them are equal, i.e., \(\text{depth}_R M = \dim R\), we say \(M\) is a “Cohen–Macaulay module” of \(R\). If \(M\) is finitely generated, it is called a small Cohen–Macaulay module for emphasis. If \((R, m)\) is a Cohen–Macaulay module over itself, we say that \(R\) is a Cohen–Macaulay ring. Here are some examples

(a) Zero-dimensional local rings are Cohen–Macaulay.
(b) Regular local rings are Cohen–Macaulay. (A regular local ring is a local ring where there is a s.o.p. which generates the maximal ideal.)
(c) Gorenstein rings are Cohen–Macaulay. (A Gorenstein ring is a local ring which has finite injective dimension over itself.)
(d) $k[[X, Y]]/(X^2, XY)$ is a one-dimensional ring that is not Cohen-Macaulay.

(e) A one-dimensional local domain is Cohen-Macaulay. In general if $\dim R \geq 2$ and $R$ is normal, then $\text{depth}_m(R) \geq 2$. The ring $k[X^4, X^3Y, XY^3, Y^4]_m$, where $m$ is the irrelevant ideal, is two-dimensional but it is neither normal nor Cohen-Macaulay (see [9]).

(f) $k[xs, ys, zs, xt, yt, zt]_m$ where $x^3 + y^3 + z^3 = 0$ is a three-dimensional normal domain but it is not Cohen-Macaulay. This ring will be the topic for Section 3.

The existence of big Cohen-Macaulay modules over local rings containing a field is shown by M. Hochster (see [5, 6, 7]). The proof involves "trivializing" relations among a given s.o.p. of $R$. This problem is closely related to the direct summand problem, namely, if $S$ is a module-finite ring extension of a domain $R$, when does $R$ sit inside $S$ as an $R$-direct summand? In the case when $R$ contains the rationals, then $R$ is normal if and only if for every finite ring extension $S$, $R \rightarrow S$ splits. The case when $R$ is regular and contains a field of characteristic $p$ is answered by M. Hochster in [8]. The case when the extension $S$ is gotten by adjoining $p$'th roots is studied in [1] by R. Fedder. We will show the failure of splitting in separable extensions (Artin-Schreier type) in the case when $R = k[X_1, \ldots, X_n]/(f)$ where $f$ is a homogeneous polynomial of degree $\geq n$ with the origin as an isolated singularity (see Proposition 1.11 and Corollary 1.12). Also we will show that under certain circumstances we can pass this question of splitting to the associated graded ring of $R$ (Proposition 1.9).

Very little is known of the existence of small Cohen-Macaulay modules over complete local rings (see [7]). We will show that small Cohen-Macaulay modules do exist for the coordinate rings of the products of smooth projective curves over an algebraically closed field (Proposition 2.3).

Finally in Section 3, we will trivialize relations on a s.o.p. for the ring $k[X, Y, Z]/(X^3 + Y^3 + Z^3) \otimes k[S, T]$ when characteristic of $k$ is $p = 2(\text{mod } 3)$ or 0, where $\otimes$ denotes the Segre Product. We shall construct big Cohen-Macaulay algebras with identity for these rings.

I. SPLITTING IN MODULE-FINITE EXTENSIONS

The splitting problem in this section originated from the following questions. Let $R$ be a local domain containing a field $k$ of characteristic $p > 0$. Let $K$ be the fraction field of $R$ and $\Omega$ be an algebraic closure of $K$. Let $\bar{R}$ be the integral closure of $R$ in $\Omega$. Is $\bar{R}$ Cohen-Macaulay over $R$?
More specifically, if \( R = k[[X_1, \ldots, X_n]] \), where the \( X \)'s are indeterminates, is \( \overline{R} \) Cohen–Macaulay?

In the case when \( R \) contains a field \( k \) of characteristic 0, the answer for the earlier question is negative by the following example. Let \( R \) be a non-Cohen–Macaulay normal domain containing the rationals and \( K \) its fraction field (see Example (f)). Let \( \overline{R} \) be the integral closure of \( R \) in the algebraic closure of \( K \). Suppose \( \overline{R} \) is Cohen–Macaulay over \( R \). Let \( x_1, \ldots, x_n \) be a system of parameters of \( R \) such that \( x_1, \ldots, x_n \) is an \( \overline{R} \)-sequence. Since \( R \) is not Cohen–Macaulay, there is an integer \( k, 1 \leq k < n \), and \( t \in R \) with \( t \notin (x_1, \ldots, x_k)R \), such that \( x_{k+1} \in (x_1, \ldots, x_k)R \). Since \( \overline{R} \) is Cohen–Macaulay over \( R \), \( t \) must be in \( (x_1, \ldots, x_k)\overline{R} \). Therefore there is a finite integral extension \( T \) of \( R \) such that \( t \in (x_1, \ldots, x_k)T \). But the characteristic of \( R \) is 0 and \( R \) is normal, and the field trace map divided by the degree of the field extension of the quotient field of \( T \) over the quotient field of \( R \) is a retraction from \( T \) to \( R \). This implies that \( R \) is a direct summand of \( T \) as an \( R \)-module. This says every ideal of \( R \) is a contracted ideal (see Lemma 1.3.). Thus \( t \in (x_1, \ldots, x_k)T \cap R = (x_1, \ldots, x_k)R \) and this is a contradiction.

If \( \text{char} \ R = p > 0 \), the trace map will not work in general since we might be dividing by \( p \). But the above example does suggest to us studying the problem of splitting in module-finite extensions. In a domain which is a direct summand in every module-finite ring extension, non-trivial relations among a system of parameters, if they exist, will persist in every module-finite extension. And for a domain which fails to be a direct summand, one may try to "trivialize" non-trivial relations among systems of parameters.

Henceforth, all rings will be commutative with identity, all subrings of a given ring contain the original identity, and all ring homomorphisms are assumed to preserve the identity.

A. Some General Reductions

**Definition.** Let \( R \) be a domain. \( R \) is a "splinter" if and only if \( R \) is a direct summand, as an \( R \)-module, of every module-finite ring extension.

**Lemma 1.1.** Let \( R \subseteq S \) be rings such that \( S \) is finitely presented as an \( R \)-module. Then \( R \) is a direct summand of \( S \) if and only if \( R_p \) is a direct summand of \( S_p \) for every maximal ideal \( P \) of \( R \). If \( T \) is a faithfully flat \( R \)-algebra then \( R \) is a direct summand of \( S \) if and only if \( T = R \otimes_R T \) is a direct summand of \( S \otimes_R T \).

**Proof.** See [8].

**Lemma 1.2.** Let \( R \) be a domain and \( S \) an integral extension of \( R \). There is a prime ideal \( P \) of \( S \) such that \( P \cap R = (0) \), and if \( R \) is a direct summand of \( S/P \) (under the induced inclusion), then \( R \) is a direct summand of \( S \).
Proof. See [8].

**Lemma 1.3.** If \( R \subset S \) are rings such that \( R \) is a direct summand of \( S \) then for every ideal \( I \) of \( R \), \( IS \cap R = I \).

Proof. See [9].

Given a Noetherian domain \( R \), if we want to decide whether \( R \) is a splinter or not, Lemma 1.1 reduces to the case where \( R \) is local. Let \( R \) be a splinter, let \( a/b \) be an element in the quotient field of \( R \), \( a, b \in R \), such that \( a/b \) is integral over \( R \). Let \( \bar{R} \) be \( R[a/b] \). Then \( a/b \in \bar{R} \) and \( a \in b\bar{R} \cap R = bR \) by Lemma 1.3. Therefore \( a/b \in R \) and we have the next lemma.

**Lemma 1.4.** If \( R \) is a splinter, then \( R \) is normal.

In case \( R \) contains the rationals this is also a sufficient condition via the trace argument. Thus we will study the case where \( \text{char } R = p > 0 \), \( p \) is prime, and \( R \) is normal. Moreover, if the completion of \( R \) with respect to the maximal ideal is also a domain (e.g., \( R \) is normal and excellent), then by Lemma 1.1 we have that \( R \) is a splinter if \( \tilde{R} \) is a splinter. Therefore we focus on the case where \( R \) is a complete normal local domain and \( \text{char } R = p > 0 \). Given this, assume that \( R \subset S \) where \( S \) is a module-finite ring extension and \( R \) is not a direct summand of \( S \). Then by Lemma 1.2, we may assume \( S \) is a domain and \( S \) is contained in some finite normal field extension \( L \) over the quotient field \( K \) of \( R \). Enlarging \( S \) to be the integral closure of \( R \) in \( L \) gives us the following:

\[
\begin{array}{c}
R \\
\downarrow \quad \downarrow \\
K \\
\downarrow \quad \downarrow \\
L
\end{array}
\]

where \( L \) over \( K \) is finite, normal, and \( R \) is not a direct summand of its integral closure in \( L \). Let \( K^{1/p} \) be a purely inseparable extension of \( K \) so that \( L \cdot K^{1/p} \) is separable over \( K^{1/p} \). Then \( R^{1/p} \) is the integral closure of \( R \) in \( K^{1/p} \) and either \( R \) is a direct summand of \( R^{1/p} \) or it is not. The problem of whether a domain \( R \) with characteristic \( p > 0 \) is a direct summand of \( R^{1/p} \) or not is dealt with extensively in [1, 11]. Here we need the assumption that \( k \) is perfect, i.e., \( k^p = k \), to keep the finiteness condition by the following lemma.

**Lemma 1.5.** Let \( R \) be a finitely generated \( k \)-algebra over \( k \) where \( k \) is perfect. Then \( R^{1/p} \) is finitely generated as a module over \( R \).
Since the case of finite $k$-algebras is the one we are most interested in and $R$, Fedder gave a complete answer for the problem of splitting in purely inseparable extension for finite $k$-algebras where $k$ is perfect (see [1]), hence we will assume that $k$ is perfect. $R$ is a direct summand in $R^{1/p}$ (thus for $R^{1/p^e}$ for all $e > 0$), and proceed from here. Given this we see that $R^{1/p^e}$ has to fail to be a direct summand of $SR^{1/p^e}$ for a retraction from $SR^{1/p^e}$ would restrict to a retraction from $S$ to $R$. Thus $R^{1/p^e}$ fails to be a direct summand of its integral closure in a Galois (i.e., separable, finite, normal) extension over its quotient field. But $R$ is isomorphic to $R^{1/p^e}$ as a domain. Therefore we may assume that $R$ fails to be a direct summand in $\bar{R}$ where $\bar{R}$ is the integral closure of $R$ in a Galois extension $L$ over its quotient field $K$.

**Proposition 1.6.** $\text{Char } R = p > 0$, $R$ is a splinter if and only if

(a) $R$ splits in every purely inseparable extension and

(b) $R \to S$ splits for every $S$ where $S$ is the integral closure of $R$ in a Galois extension over the quotient field of $R$.

To sum up, let $R$ be a complete normal local domain with coefficient field $k$ where $k$ is perfect with characteristic of $k = p > 0$, let $L$ be a Galois extension degree $p^e$ over the quotient field of $R$, let $S$ be the integral closure of $R$ in $L$, and assume that $R$ is a direct summand of $R^{1/p}$. When does the map $R \to S$ split? The following lemma is crucial for splitting maps.

**Lemma 1.7.** If $R$ is an excellent domain, $S$ is a finite extension of $R$. Then $R \to S$ splits if and only if every ideal of $R$ is a contracted ideal.

*Proof.* See [8].

B. Relations between $R$ and $\text{Gr}_m R$

Let $(R, m)$ be a local Noetherian ring where $m$ is the maximal ideal. Let $\text{Gr}_m R = \bigoplus_{i=0}^{\infty} m^i/m^{i+1}$ denote the associated graded ring of $R$ with respect to $m$. If $x \in R - \{0\}$, let $O(x)$ be the order of $x$ with respect to $m$, i.e., the non-negative integer $d$ such that $x \in m^d$ but $x \notin m^{d+1}$. If $O(x) \geq d$, let $[x]_d = x + m^{d+1}$ and let $[x]$ denote the leading term of $x$ in $\text{Gr}_m R$. Let $l$ be a positive integer and $A$ be the set of $n$-tuples of non-negative integers whose sum is $l$. Let $x = \{x_1, \ldots, x_n\}$ be a subset of $R$, $a = (i_1, \ldots, i_n) \in A$, and define $x^a$ to be the monomial $x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n}$.

**Lemma 1.8.** Let $(R, m)$ be a Noetherian local ring and $x_1, \ldots, x_n$ in $R$ be
such that \( O(x_i) = k, \ k > 0, \) for \( i = 1, 2, \ldots, n \). Assume that \([x_1], \ldots, [x_n]\) form a regular sequence in \( \text{Gr}_m R \) and that \( \text{Gr}_m R \) is a domain. Suppose

\[
t = \sum_a r_a x^a \Mod(m^{kl + d}), \tag{*}
\]

\( t, \ r_a \in R, \) where \( d \geq O(t) - kl. \) Then there exist \( r'_a \in R \) such that

\[
t = \sum_a r'_a x^a \Mod(m^{kl + d})
\]

and \( O(r'_a) \geq O(t) - kl \) for all \( a \) in \( A. \)

**Proof:** Given (*), we may assume that for some \( r \)'s, \( O(r) < O(t) - kl. \) We will construct a new set of coefficients by modifying the ones with lowest order to higher orders and do this repeatedly if necessary.

Let \( \min_{a \in A} \{O(r_a)\} = c < O(t) - kl, \) and label the \( r \)'s with order \( c \) to be \( r_1, \ldots, r_s, \) and the corresponding \( x^a \) in (*) to be \( X_1, \ldots, X_s. \) Then we have

\[
\sum_{i=1}^{s} r_i X_i = 0 \Mod(m^{kl + c}).
\]

Since \( \text{Gr}_m R \) is a domain so is \( R. \) Thus the order function is a valuation and \( \sum_{i=1}^{s} [r_i][X_i] = 0 \) in \( \text{Gr}_m R. \) By assumption, \([x_1], \ [x_2], \ldots, [x_n]\) form a regular sequence and \([X_i]\) are monomials of degree \( l \) in \([x_1], \ [x_2], \ldots, [x_n].\) Therefore the \([r_i]\) came from the trivial relations (see Theorem 2.E, [10]). That is, let \( M_{ij} \) and \( M_{ji} \) be monomials in \([x_1], \ [x_2], \ldots, [x_n]\) of least degree such that \( M_{ij}[X_i] - M_{ji}[X_j] = 0, \ 1 \leq i < j \leq s. \)

Let \( \vec{V}_{ij} \in [\text{Gr}_m R]^* \) where

\[
\vec{V}_{ij} = (0 \ldots 0, M_{ij}, 0 \ldots 0, -M_{ji}, 0 \ldots 0), \quad 1 \leq i < j \leq S,
\]

(where \( M_{ij} \) is in the \( i \)th spot and \( M_{ji} \) is in the \( j \)th spot) so that \( \vec{V}_{ij} \cdot ([X_1], \ldots, [X_s]) = 0, \) using the notation of dot product. We have

\[
\sum_{i=1}^{s} [r_i][X_i] = [\vec{r}] \cdot [\vec{X}] = 0,
\]

where \([\vec{r}] = ([r_1], \ldots, [r_s])\) and \([\vec{X}] = ([X_1], \ldots, [X_s]).\) To say the \([r_i]\) came from the trivial relations means

\[
[\vec{r}] = \sum_{1 \leq i < j \leq S} h_{ij} \vec{V}_{ij},
\]
where \( h_{ij} \in \text{Gr}_m R, \) \( \text{deg } h_{ij} + \text{deg } M_{ij} = c. \) Thus

\[
[r_i] = \sum_{k=1}^{i-1} h_{ki} M_{ik} + \sum_{k=i+1}^{s} h_{ik} M_{ik}.
\]

Now choose \( g_{ij} \in R \) such that \([g_{ij}] = h_{ij},\) let \( m_{ij} \) and \( m_{ji} \) be the corresponding monomials in \( x_1, \ldots, x_n, \) such that

\[
m_{ij}x_i - m_{ji}x_j = 0, \quad 1 \leq i < j \leq s.
\]

Also set \( r_{ij} \in R^s \) where

\[
\bar{v}_{ij} = (0 \ldots 0, m_{ij}, 0 \ldots 0, -m_{ji}, 0 \ldots 0).
\]

Then \( (\sum g_{ij} \bar{v}_{ij}) \cdot (X_1, \ldots, X_s) = 0 = \sum b_i X_i \) in \( R \) where

\[
b_i = \sum_{k=1}^{i-1} g_{ki} m_{ik} + \sum_{k=i+1}^{s} g_{ik} m_{ik}.
\]

By the fact that the order function is a valuation we see \([b_i] = [r_i].\) Therefore, after collecting like terms on the right hand side of

\[
t = \sum r_i x^a - \sum b_i X_i \text{Mod}(m^{kt+d}),
\]

we see all the coefficients have order \( >c. \) Applying this procedure repeatedly if necessary we obtain the lemma.

Let \( R \) be an integral domain and \( S \) a finite extension of \( R. \) Let \( I \subset R \) be an ideal.

DEFINITION. The set \( \overline{IS} = \{ s \in S | s \text{ satisfies a polynomial equation } f(x) = x^n + r_1 x^{n-1} + \cdots + r_n \in R[X] \text{ where } r_k \in I^k \text{ for } k = 1, 2, \ldots, n \}. \) We say \( I \) is integrally closed if \( \overline{IR} = I, \) or equivalently \( \overline{IS} \cap R = I \) (see [4]).

PROPOSITION 1.9. Let \( (R, m) \) be a local domain with \( \text{Gr}_m R \) a domain also. Let \( x_1, \ldots, x_n \) be in \( R \) such that \( O(x_i) = k \) and \([x_1, \ldots, x_n]\) form an \( r \)-sequence in \( \text{Gr}_m R. \) Let \( S \) be the integral closure of \( R \) in a Galois extension \( L \) of the quotient field \( K \) of \( R. \) Let \( w \in R \) be such that \( w \notin (x_1, \ldots, x_n)R \) but \( w \in (x_1, \ldots, x_n)S, \) and \( O(w) = k + d \leq 2k. \) Furthermore, assume \( m^n \) is integrally closed for \( n = 1, 2, \ldots. \) Then \([w] \in ([x_1], \ldots, [x_n]) \cdot \text{Gr}_{\overline{[m^nS]}} S \) under the extension \( \text{Gr}_m R \rightarrow \text{Gr}_{\overline{[m^nS]}} S. \)
Proof. We are given that

$$w = u_1 x_1 + u_2 x_2 + \cdots + u_n x_n, \quad u_i \in S$$

and $u_i$ satisfies the equation

$$u_i^N + e_1 u_i^{N-1} + \cdots + e_N = 0,$$

where $e_i$ is $(-1)^i$-th symmetric function of $g_1(u_1), \ldots, g_N(u_1)$. Let $g$ act on the equation $w - u_1 x_1 = \sum_{i=2}^n u_i x_i$. We obtain $w - g(u_1)x_1 = \sum_{i=2}^n g(u_i)x_i$. This equation will be denoted as $E_g$. Summing up $E_g$'s, $g \in G$, we get

$$N \cdot w + e_1 x_1 = \sum_{i=2}^n r_i x_i,$$

where $r_i = \sum_G g(u_i)$. Each $r_i$ is invariant under $G$, integral over $R$, and since $R$ is normal, we see that $r_i \in R$. Since $O(w) = k + d$, we have

$$e_1 x_1 = \sum_{2}^n r_i x_i \text{ Mod}(m^k + d).$$

Assume $O(e_1) = c < d$. Then by Lemma 1.8 we can choose $r'_i$, $i = 2, 3, \ldots, n$, such that $O(r'_i) \geq c$. In $G_m R$, Eq. (2) becomes

$$[e_1][x_1] = \sum_{h \in J} [r'_h][x_h],$$

where $J$ is some subset of $\{2, 3, \ldots, n\}$. So $[e_1] (=[x_2, \ldots, x_n]; [x_1]) G_m R$. But since $[x_1], \ldots, [x_n]$ form an $R$-sequence in $G_m R$, we can treat the $[x_i]$'s as indeterminates in the colon operation, i.e., $[e_1] (=[x_2, \ldots, x_n]) G_m R$. Therefore $O(e_1) \geq k > d$ and this is a contradiction. So we must have $O(e_1) \geq d$.

Now do induction on the orders of the $e$'s. Assume $O(e_1) \geq d$, $O(e_{l-1}) \geq (l-1)d$. By taking the sum of all possible $l$-fold products of the $E_g$'s where each product consists of $l$ distinct factors, we obtain

$$c_0 w^l + c_1 e_1 w^{l-1} x_1 + \cdots + c_j e_j w^{l-j} x_1^j + \cdots + e_j x_1^j \in (x_2, \ldots, x_n)^l R,$$
where $c_j = \begin{pmatrix} N-j \\ j \end{pmatrix}$, $j = 0, 1, \ldots, l - 1$. Except for $e_i x^l$, a general term by the induction hypothesis has order

$$O(c_j e_j w^{l-j} x^j) \geq jd + (k + d)(l - j) + kj = (k + d)l.$$  

Using the notation of Lemma 1.8 we have

$$e_i x^l = \sum r_a x^a \bmod (m^{(k+d)}l).$$

Assume $O(e_i) = c < dl$. Then by Lemma 1.8 we may choose $r_a$ such that

$$e_i x^l = \sum r_a x^a \bmod (m^{(k+d)}l).$$

Passing this equation to $\text{Gr}_m R$ and using a similar argument for the case of $e_1$, we have $O(e_1) \geq kl > dl$ which is a contradiction. So $O(e_i) \geq c$ and

$$e_i x^l = \sum r_a x^a \bmod (m^{(k+d)}l).$$

The above proposition tells us that under some very specific hypotheses the problem of splitting can be passed on to the associated graded ring. Of course what one would like to have is the general case. More specifically, is it true that if $R$ is a local normal domain with char $R = p$ and $\text{Gr}_m R$ is a Cohen–Macaulay domain, and if $R$ is not a splinter, then $\text{Gr}_m R$ has a finite graded extension such that $\text{Gr}_m R$ is not a direct summand?

**Proposition 1.10.** Let $R$ be a non-negatively grade Cohen–Macaulay normal domain of dimension 2 and $R_0 = K$ where $K$ is field of characteristic $p > 0$. Let $x, y$ be an $R$-sequence where each is a $k$-form and $w$ is a $k+c$ form where $c \leq k$. If $w \in (x, y)S$ where $S$ is a graded extension of $R$ in a degree $p^s = N \bmod \Gal(L/K)$ over the quotient field $K$ of $R$, then

$$w^N \in (x^N, y^N, x^{N-1}y^{N-1}, \ldots, w)R.$$ 

**Proof.** We may assume $w \notin (x, y)R$. Let $w = ux - vy$. Since $w$ is a $k+c$ form, $u, v$ are $c$-forms in $S$. Let $G$ be the Galois group of $L$ over $K$. Let $N = p^s = O(G)$. Then $u$ and $v$ satisfy

$$u^N + e_1 u^{N-1} + \cdots + e_N = 0$$

$$v^N + d_1 v^{N-1} + \cdots + d_N = 0,$$
where $e_i, d_i$ are the $(-1)^i$-th elementary symmetric functions of \( \{ g(u) \} \) and \( \{ g(v) \} \), $g \in G$, respectively, and $e_i, d_i$ are forms of degree $\geq i \cdot c$ for $i = 1, 2, ..., N$. Applying $g \in G$ to the equation $w - ux = -vy$, we get the equation

$$w - g(u)x = -g(v)y \quad g \in G.$$  

Summing up $E_g$'s, we obtain $e_1x = d_1y$. Since $x, y$ is an $R$-sequence consisting of $k$-forms and $e_1$ and $d_1$ are $c$-forms, $k \geq c$, we see that $e_1 = k_0y$, $d_1 = k_0x$ for some $k_0 \in K$. Then by taking the sum of all possible 2-fold products of $E_g$'s where each product consists of distinct factors, we obtain

$$(p - 1)e_1wx + e_2x^2 = d_2y^2$$

$$\Rightarrow e_1wx \in (x^2, y^2)R$$

$$\Rightarrow (k_0y)wx \in (x^2, y^2)R$$

$$\Rightarrow w \in [(x^2, y^2): k_0xy].$$

Since $x, y$ is an $R$-sequence and $w \notin (x, y)R$, we have $[(x^2, y^2): k_0xy] \neq R$ and $k_0 = 0$, i.e., $e_1 = 0 = d_1$. Thus $e_2x^2 = d_2y^2$ and $e_2 = k_0y^2$, $d_2 = k_0x^2$ for $k_0 \in K$ by a similar argument. Now do induction in similar fashion by taking the sum of all $i$-fold products of $E_g$'s for $i = 1, 2, ..., N - 1$. We obtain that $e_1 = d_1 = 0$, ..., $e_{N-2} = d_{N-2} = 0$, $e_{N-1} = k_0y^{N-1}$, $d_{N-1} = k_0x^{N-1}$ for some $k_0 \in K$. Finally the $N$-fold product of the $E_g$ yields

$$w^N + e_{N-1}wx^{N-1} + e_Nx^N = d_Ny^N$$

or

$$w^N = d_Ny^N - e_Nx^N - k_0x^{N-1}y^{N-1}, w,$$

where $d_N, e_N \in R$, and $k_0 \in K$.

C. Artin–Schreier Type Extensions

**Definition.** Let $R$ be a domain of characteristic $p > 0$. $S$ is said to be an Artin–Schreier type extension of $R$ if $S = R[u_1, ..., u_n]$ where $u_i$ satisfies a polynomial $P_i(T)$ over $R[T]$ of the form

$$P_i(T) = T^N + r_iT + s_i \quad \text{where} \quad N = p^e \quad \text{and} \quad r_i, s_i \in R.$$  

Although the conditions for $R$ to be a splinter are not easy to find, there is a simple sufficient condition for $R$ not to be a splinter because splitting fails in an Artin–Schreier type extension.
**Proposition 1.11.** Let $R$ be a domain of characteristic $p > 0$. Let $N = p^e$ for some integer $e > 0$ and

$$w^N \in \left( x_1^N, x_2^N, ..., x_n^N, w \cdot \prod_{i=1}^{n} x_i^{N-1} \right) R, \quad w \in R.$$  

There is an extension $S$ of $R$ such that $w \in (x_1, ..., x_n)_S$.

**Proof.** Let $w^N = \sum_{i=1}^{n} A_i x_i^N + B \cdot w \cdot \prod_{i=1}^{n} x_i^{N-1}, A_i, B \in R$ and $f_i(T) = T^N - (B \cdot \prod_{i \neq i} x_i^{N-1}) \cdot T - A_i$ be polynomials in $R[T]$ for $i = 1, 2, ..., n$. Let $s_i$ be a root of $f_i(T)$ so that $s_i^N - (B \cdot \prod_{i \neq i} x_i^{N-1}) s_i - A_i = 0$

$$\Rightarrow x_i^N s_i^N - x_i^N \left( B \cdot \prod_{j \neq i} x_j^{N-1} \right) s_i - x_i^N A_i = 0$$

$$\Rightarrow (x_i s_i)^N - \left( B \cdot \prod_{j=1}^{n} x_j^{N-1} \right) (x_i s_i) - x_i^N A_i = 0.$$

Call this equation $E_i$ for $i = 1, 2, ..., n$. Summing up the $E_i$'s we get

$$\left( \sum_{i=1}^{n} s_i x_i \right)^N = \left( B \cdot \prod_{i=1}^{n} x_i^{N-1} \right) \left( \sum_{i=1}^{n} s_i x_i \right) + \sum_{i=1}^{n} A_i x_i^N.$$

Then subtracting

$$w^N = \left( B \cdot \prod_{i=1}^{n} x_i^{N-1} \right) \cdot w + \sum_{i=1}^{n} A_i x_i^N$$

from the corresponding sides, we get

$$\left( w - \sum_{i=1}^{n} s_i x_i \right)^N = \left( B \cdot \prod_{i=1}^{n} x_i^{N-1} \right) \left( w - \sum_{i=1}^{n} s_i x_i \right)$$

$$\Rightarrow w = \sum_{i=1}^{n} s_i x_i \text{ or } w - \sum_{i=1}^{n} s_i x_i = \Theta B^{1/N} \prod_{i=1}^{N} x_i,$$

where $\Theta$ is an $(N-1)$th root of unity

$$\Rightarrow w \in (x_1, ..., x_n)_S \quad \text{where} \quad S = R[s_1, s_2, ..., s_n, \Theta, B^{1/N}].$$

**Corollary 1.12.** Let $R = k[X_1, X_2, ..., X_n]/(f)$ where $f$ is a homogeneous polynomial of degree $\geq n$. $R$ is not a splinter.

**Proof.** Let degree $f = k \geq n$. We may assume $f = X_1^k + f_1 X_1^{k-1} + \cdots + f_k$, where $f_i$ is a polynomial in $X_2, ..., X_n$ of degree $i$. Let $x_i$ be the image of $X_i$ in $R$, and set $x_i^{k-1}$ as $w$ in the above proposition. Then $(x_i^{k-1})^p =$
h_1 x_1^k-1 + \cdots + h_{k-i} x_i^k + \cdots + h_{k-1} x_1 + h_k where h_{k-i} is a homogeneous polynomial of degree \((k-1)p-i\) in \(x_2, \ldots, x_n, \quad i = k-1, \ldots, 1, 0\). Since \(k-1 \geq n-1\), we see \(h_1, \ldots, h_k\) are in \((x_2^p, \ldots, x_n^p) R\), and \(h_1 x_1^{k-1} \in (x_2^p, \ldots, x_n^p, x_1^{k-1}, x_2^{k-1} \cdots x_n^{k-1})\). So there is a finite extension \(S\) such that \(x_1^{k-1} \in (x_2, \ldots, x_n) S\).

**Question.** If \(n \geq 3\), \(d = 2\) or \(n > d \geq 3\), is \(R = k[X_0, \ldots, X_n]/(f)\) a splINTER where \(d = \text{degree } f\)?

### II. SMALL COHEN–MACAULAY MODULES

The existence of small Cohen–Macaulay modules (i.e., the finitely generated ones) over a complete local ring is known for \(\dim R = 2\), or \(\dim R = 3\) where \(R\) is the completion of a finitely generated non-negatively graded algebra over a field \(k\) with \(\text{char } k = p, p > 0\) (see [7, 13]). We will show the existence of small Cohen–Macaulay modules for coordinate rings of products of smooth projective curves over an algebraically closed field \(k\). Most of the language and basic facts in this section can be found in [2, 3]. We will recall some of it for the sake of completeness. Throughout this section, \(k\) will be an algebraically closed field.

Let \(R\) be a non-negatively graded finitely generated \(k\)-algebra and \(X = \text{Proj } R\). Let \(M\) be a graded \(R\)-module and \(\widetilde{M}\) the sheaf associated to \(M\) on \(X\). Let \(M_d\) be the \(k\)-subspace of \(M\) consisting of degree \(d\) homogeneous elements. Then for any \(t \in \mathbb{Z}\), we define the twisted module \(M(t)\) by shifting \(t\) places to the left, i.e., \(M(t)_d = M_{t+d}\). Correspondingly we obtain the twisted sheaf \([M(t)]\) and it is isomorphic to \(M \otimes_{\mathcal{O}_X} \mathcal{O}_X(t)\) which will be denoted as \(\widetilde{M}(t)\). Given any sheaf \(F\) on \(X, U\) an open set in \(X\), we will denote by \(\Gamma(U, F)\) the sections of \(F\) on \(U\). Then we can form the following graded module: \(\bigoplus_{t \in \mathbb{Z}} \Gamma(X, F \otimes_{\mathcal{O}_X} \mathcal{O}_X(t)) = \Gamma_*(F)\), the module associated with the sheaf \(F\). There is a natural homomorphism, the Serre map, \(s: M \rightarrow \Gamma_*(\widetilde{M})\). Let \(R\) be a graded ring such that \(R\) is finitely generated by \(R_1\) as a \(R_0\)-algebra and \(R_0\) is a finite \(k\)-algebra. Then \(\widetilde{M}\) is isomorphic to the sheaf associated with \(\Gamma_*(\widetilde{M})\) on \(X\). Furthermore, given a coherent sheaf \(F\) on \(X\), one can find a finitely generated graded module, namely \(\Gamma_*(F)\), over \(R\) such that \(\Gamma_*(F)\) is isomorphic to \(F\). In short, there is an equivalence between the category of coherent sheaves on \(X\) and the category of finitely generated graded modules over \(R\) modulo with a certain equivalence relation on maps.

Let \(R = k[X_0, X_1, \ldots, X_n]/I, \quad X = \text{Proj } R, \quad m = (x_1, \ldots, x_n) R\) and \(M\) be a graded module of \(R\). Then \(\text{depth}_m(M) = i\) if and only if the local cohomology \(H^j_m(M) = 0\) for \(0 \leq j \leq i-1\) and \(H^i_m(M) \neq 0\). The local
cohomology can be computed via the following cohomological Čech complex (which is the direct limit of the Koszul complex),

\[ K^*: 0 \rightarrow M \xrightarrow{\psi_0} \bigoplus_{i=0}^{n} M_{x_i} \xrightarrow{\psi_1} \bigoplus_{i<j} M_{x_ix_j} \rightarrow \cdots \rightarrow M_{x_0,x_1,\ldots,x_n} \rightarrow 0, \]

where the module at the \( t \)'th spot is \( \bigoplus_{k_1 < k_2 < \ldots < k_t} M_{x_{k_1}\ldots x_{k_t}} \) and the coboundary map is given by

\[ \psi_t(m/(x_{k_1}\ldots x_{k_t})) = \bigoplus_{s \in S} (-1)^t (mx_{k_1}\ldots x_s\ldots x_{k_t})^t, \]

where \( S = \{0, 1, 2, \ldots, n\} - \{k_1, k_2, \ldots, k_t\} \). Then \( H^i_m(M) = H^i(K^*) = \ker \psi_{t-1}/I_m \psi_t \). Considering the sheaf cohomology \( \tilde{H}^i(X, \tilde{M}(t)) \) of \( M(t) \) on \( X \) with respect to the open covering \( U = \{D_+(x_i)\}, i=0, 1, 2, \ldots, n \), for \( t \in \mathbb{Z} \).

This can be calculated by the cohomology of the Čech complex on \( \tilde{M}(t) \),

\[ C^*_t: 0 \rightarrow +[M(t)]_{(x_i)} \rightarrow +[M(t)]_{(x_ix_j)} \cdots \rightarrow [M(t)]_{(x_0x_1\ldots x_n)} \rightarrow 0, \]

where \( N^0_{(x)} \) denotes the degree zero piece in the localization at \( x \in R \). Comparing \( K^* \) and \( C^*_t \) for \( t \in \mathbb{Z} \), we see that \( \tilde{H}^i(X, \tilde{M}(t)) = H^i(C^*_t) \) is the \( t \)'th graded piece of \( H^{i+1}_m(M) \) (= \( H^{i+1}_m(K^*) \)), for \( i = 1, 2, \ldots, n \), and \( \bigoplus_{t \in \mathbb{Z}} H^0(C^*_t) = \bigoplus_{t \in \mathbb{Z}} \Gamma(X, \tilde{M}(t)) = \Gamma(\tilde{M}) = \ker \psi_1 \). One checks easily that the map \( \psi_0: M \rightarrow \ker \psi_1 = \Gamma(\tilde{M}) \) is precisely the Serre map \( s \). Since \( \tilde{M} \) is quasi-coherent and \( X \) is a separated Noetherian scheme, we may replace the Čech cohomology \( \tilde{H} \) by the sheaf cohomology. Therefore we obtain that

\[ H^{i+1}_m(M) = \bigoplus_{t \in \mathbb{Z}} H^i(X, \tilde{M}(t)) \quad \text{for} \quad i = 1, 2, \ldots, n-1. \]

Also we have that \( s: M \rightarrow \bigoplus_{t \in \mathbb{Z}} H^0(X, \tilde{M}(t)) = \Gamma(\tilde{M}) = \ker \psi_1 \) where \( \ker(s) = H^{0}_m(\tilde{M}), \Gamma(\tilde{M})/\text{Im}(s) = H^1_m(M) \). Putting everything together we obtain

**Lemma 2.1.** Let \( R = k[X_0, X_1, \ldots, X_n]/I \), where \( I \) is a homogeneous ideal and let \( M \) be a graded module over \( R \). Let \( m = (x_1, x_2, \ldots, x_n)R \) where \( x_i = x_i \mod I \). Let \( s: M \rightarrow \Gamma(\tilde{M}) \) be the Serre map. Then

(a) \( \text{depth}_m(M) \geq 1 \) if and only if \( s \) is injective.

(b) \( \text{depth}_m(M) \geq 2 \) if and only if \( s \) is an isomorphism.
Let $C$ be a smooth projective curve over $k$. There is a one-to-one correspondence between invertible sheaves on $C$ and divisors class on $C$. Let $D$ be a divisor. We denote the invertible sheaf associated with $D$ by $L(D)$ and $\dim_k \Gamma(C, L(D))$ by $l(D)$. We come to the set need:

Proof. Choose any divisor $D'$ of $C$ such that degree $D' > 2g - 2$. Then by the Riemann–Roch Theorem we have

$$l(D') - l(K - D') = \deg D' + 1 - g = n.$$  

Since $\deg(K - D') = 2g - 2 - \deg D' \leq 0$, we see that $l(K - D') = 0$. Therefore $l(D') = n$ and we can choose $P_1, \ldots, P_n$ such that $l(D' - \sum_{i=1}^n P_i) = l(D') - n = 0$. Set $D = D' - \sum_{i=1}^n P_i$, then $\deg(D) = \deg D' - n = g - 1$ and $l(D) = 0$.

Let $C_1, C_2, \ldots, C_n$ be smooth projective curves over $k$ with $R_1, R_2, \ldots, R_n$ as coordinate rings, respectively. The coordinate ring for $C_1 \times C_2 \times \cdots \times C_n$ is $R_1 \otimes R_2 \cdots \otimes R_n = \bigoplus_{d=0}^\infty R_1(d) \otimes \cdots \otimes R_n(d) \subset R_1 \otimes \kappa R_2 \cdots \otimes R_n$. The next proposition is the main result of this section.

**Proposition 2.3.** Let $R$ be the coordinate ring of $C_1 \times C_2 \times \cdots \times C_n$ where each $C_i$ is a smooth projective curve over $k$. $R$ has a finitely generated Cohen–Macaulay module.

Proof. Let $g_i = \text{genus of } C_i$, $i = 1, 2, \ldots, n$. Let $D_i$ be a divisor of $C_i$ such that $\deg D_i = g_i - 1$ and $\dim_k \Gamma(C_i, L(D_i)) = 0$. Their existence is given by Lemma 2.2. We denote $\prod_i^*(L(D_i))$ by $L_i$ on $C_1 \times C_2 \times \cdots \times C_n$ where $\prod_i^*$ is the projection map from $C_1 \times C_2 \cdots \times C_n$ to $C_i$ and $\prod_i^*$ is the pull-back map. Let $M = \Gamma^*_\kappa(C_1 \times C_2 \cdots \times C_n, L_1 \otimes L_2 \cdots \otimes L_n)$. $M$ is a finitely generated graded module over $R$. We claim that $\text{depth}_n(M) = n + 1$ where $m$ is the irrelevant ideal in $R$. Since $\dim R = \dim(C_1 \times C_2 \cdots \times C_n) + 1 = n + 1$, this gives what we want.

By Lemma 2.1(c), we need to show that for all $i \in \mathbb{Z}$, $H^i(C_1 \times C_2 \cdots \times C_n, \tilde{M}(t)) = 0$ for $i = 1, 2, \ldots, n - 1$, and $s$ is an isomorphism. Using the Kunneth formula, we have for all $i \in \mathbb{Z}$,

$$H^i(C_1 \times C_2 \times \cdots \times C_n, \tilde{M}(t)) = H^i(C_1 \times C_2 \cdots \times C_n, (L_1 \otimes \cdots \otimes L_n)(t)) = \bigoplus_{j_1 + j_2 + \cdots + j_n = i} \bigotimes_{j_i > 0} H^j(C_i, L_i(t)) = \bigoplus_{j_1 \geq 1 \text{ or } 0} \bigotimes_{j_i = i} H^j(C_i, L_i(t)).$$
The last equality is true because $C_i$'s are curves. For $i \leq n - 1$, each term of
the direct sum contains a factor $H^1(C_p, L_p(t)) \otimes H^0(C_q, L_q(t))$ for some $P$
and $q$.

If $t = 0$, then $H^0(C_q, L_q) = 0$ by our choice of $L_q$ and the term in
question is 0.

If $t < 0$, the divisor associated with $L_q(t)$ is strictly less than $L_q$.
Therefore $\dim_k \Gamma(C_q, L_q(t)) \leq \dim_k \Gamma(C_q, L_q) = 0$ and the term in question
is 0.

If $t > 0$, $H^1(C_p, L_p(t)) = H^0(C_p, (L_p \otimes O_{C_p}(t))')$ by the Serre
Duality Theorem. Using the Riemann–Roch Theorem and the choice of
$L_p$, we have

$$l(D_p) - l(K_p - D_p) = \deg D_p + 1 - g_p$$

$$\Rightarrow 0 - l(K_p - D_p) = 0 \Rightarrow l(K_p - D_p) = 0,$$

where $K_p$ is the canonical divisor associated to $w_p$. But if $t > 0$, we have the
divisor associated with $(L_p \otimes O_{C_p}(t))' \otimes w_p = L_p' \otimes w_p \otimes O_{C_p}(-t)$ is
strictly less than the divisor $K_p - D_p$. Therefore $H^1(C_p, L_p(t)) = H^0(C_p, (L_p \otimes O_{C_p}(t))' \otimes w_p)' = 0$ and the term in question is 0.

Thus for all $t$ and $i = 1, 2, \ldots, n - 1$, $H^0(C_1 \times C_2 \cdots \times C_n, \tilde{M}(t)) = 0$. The
Kunneth formula also implies that the Serre map is an isomorphism. Therefore $M$ is Cohen–Macaulay over $R$.

### III. Examples of Big Cohen–Macaulay Algebras with Identity

Let $S$ be an $R$-algebra where $(R, m)$ is a Noetherian local ring. If
$\text{depth}_m(S) = n = \dim R$, then we say $S$ is a Cohen–Macaulay algebra of $R$.
We say $S$ is a big Cohen–Macaulay algebra of $R$ if $S$ is not finitely
generated as an $R$-module. We do not require that $S$ contains an identity
element. The following theorem concerning their existence is due to
Hochster [5].

**Theorem (3.1).** Let $A$ be any commutative ring such that $A_{\text{red}}$ contains a
field. Let $x = x_1, x_2, \ldots, x_n$ be a system of parameters in $A$. Then there exists
a non-negatively graded $A$-algebra $B = \bigoplus B_i$ and an element $b \in B_1$ such
that for each $i \geq 1$, $B_i$ is $x$-regular and $b^i \notin (x)B_i$.

The algebra $B$ in Theorem (3.1) is without identity. But in this section,
we will construct big Cohen–Macaulay algebras with identities for certain
rings.

Let $A = k[X, Y, Z]/(X^3 + Y^3 + Z^3)$ and $B = k[S, T]$ where $k$ is an
algebraically closed field of characteristic $p$ with $p = 2(\mod 3)$. Then
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A (§) B = R can be realized as \( k[x_s, y_s, z_s, x_t, y_t, z_t] \) with \( x^3 + y^3 + z^3 = 0 \). Let \( I \) be the irrelevant ideal of \( R \). Then \( R_I \) is not Cohen–Macaulay for there is the following non-trivial relation on the system of parameters \( x_s, y_t, z(s + t) \) for \( R_I \):

\[
x_s(x^3t(s + t)) + yt(y^2s(s + t)) + z(s + t)(z^2st) = (x^3 + y^3 + z^3) \cdot st(s + t) = 0.
\]

We claim that \( R_I' = \{ R_I \text{ with all the } p^e \text{th roots adjoined for all } e > 0 \} \) is a big Cohen–Macaulay algebra identity for \( R \). That is, \( \text{depth}_{IR_I}(R_I') = 3 = \text{dim } R_I \), in particular \( a = x_s, b = y_t, c = z(s + t) \), is an \( R_I \)-sequence. We make the observation that if there is any non-trivial relation in \( R_I' \) of \( a, b, c \), then there is a non-trivial relation in \( R' = \{ R \text{ adjoined with all the } p^e \text{th roots for all } e > 0 \} \) on \( a, b, c \). Therefore we can work with \( R' \) instead. For fixed \( e > 0 \), let \( ^e \mathcal{R} \) be the ring \( R \) adjoined with all its \( p^e \text{th roots} \). \( ^e \mathcal{R} \) as an \( R \)-algebra can be viewed as \( R \)–an \( R \)-algebra through the \( e \)th power of the Frobinus map. That is, \( R \) is an \( R \)-algebra via the map \( R \to ^e \mathcal{R} \) where \( F^e(r) = r^p^e \). Or we may view it as \( R \)–an \( R^{p^e} \)-algebra. \( R' \) can be viewed as the direct limit of the following sequence:

\[
R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} \ldots
\]

The elements \( a \) and \( b \) are still a regular sequence for \( R' \). For if not, then there exists \( ^e \mathcal{R} \) for some \( e > 0 \) such that \( a, b \) fails to be a regular sequence in \( ^e \mathcal{R} \). But \( a, b \) is a regular sequence in \( ^0 \mathcal{R} = R \), therefore in \( ^e \mathcal{R} \), so the above is impossible. Thus all we need to show is that \( c \) is a non-zero divisor on \( R'/\langle a, b \rangle R' \). In particular if \( H^0_0(R'/\langle a, b \rangle R') = \{ r \in R'/\langle a, b \rangle R' \text{, and } F^e r = 0 \text{ for some } t > 0 \} - 0 \), then \( z \) is a non-zero divisor on \( R'/\langle a, b \rangle R' \). Using the following short exact sequences

\[
0 \to R' \xrightarrow{a} R' \to R'//aR' \to 0
\]

\[
0 \to R'/aR' \xrightarrow{b} R'/aR' \to R'/\langle a, b \rangle R' \to 0
\]

and the functorial properties of the local cohomology, we get \( H^2_t(R') = H^0_0(R'/\langle a, b \rangle R') \). Therefore we need to show \( H^2_t(R') = 0 \). Let \( C = \text{Proj } k[X, Y, Z]/(X^3 + Y^3 + Z^3), Y = \text{Proj } k[S, T] \), then

\[
H^2_t(R) = \bigoplus_{t \in Z} H^1(C \times Y, O_{C}(t) \otimes O_Y(t))
\]

\[
= \bigoplus_{t \in Z} [H^1(C, O_C(t)) \otimes H^0(Y, O_Y(t)) \oplus H^0(C, O_X(t)) \otimes H^1(Y, O_Y(t))].
\]
$C$ is a curve of genus 1. Its canonical divisor has degree equal to zero and the degree of the divisor associated with $O_C(t)$ is 3. Therefore, by the Serre Duality Theorem, we have $H^1(C \times Y, O_C(t) \otimes O_Y(t)) = 0$ for $t > 0$ because each term involves a factor that is associated with a divisor of negative degree. Similarly for $t < 0$ we have that $H^1(C \times Y, O_C(t) \otimes O_Y(t)) = 0$. For $t = 0$, we have

$$
H^1(C, O_C) = k, \quad H^0(Y, O_Y) = k
$$
$$
H^1(Y, O_Y) \cong H^0(Y, w_Y)^\vee = 0.
$$

Therefore $H^2_*(R) = k \otimes k = k$. In fact one can easily exhibit a generator for $H^2_*(R)$ in the cohomological Koszul complex,

$$
0 \to R \to R_a \oplus R_b \oplus R_c \xrightarrow{\psi} R_{ab} \oplus R_{bc} \oplus R_{ac} \to R_{abc} \to 0,
$$

namely,

$$(u, v, w) = (-z^2st/ab, y^2s(s + t)/ac, -x^2t(s + t)/bc)
$$

in $R_{ab} \oplus R_{ac} \oplus R_{bc}$. Its image in the cohomology generates $H^2_*(R) = k$. Since $R' = \lim R \to F R \to F R \to \cdots$, we have $H^2_*(R') = \lim H^2_*(R) \to F^* H^2_*(R) \to F^* \cdots$ where $F^*$ is induced by

$$
R_a \oplus R_b \oplus R_c \xrightarrow{\phi} R_{ab} \oplus R_{bc} \oplus R_{ac} \xrightarrow{\psi} R_{abc}
$$

and $F(u, v, w) = (u^p, v^p, w^p)$. Since $p = 3k + 2$, $z^2p^p s^t t^p = (-y^3 - x^3)^{2k+1}$. $z^2s^t t^p = \alpha x^p t^p + \beta y^p t^p$, $\alpha$, $\beta$ in $R$. Furthermore, we have that

$$
x^p s^p (x^{2p} t^p (s^p + t^p)) + y^p t^p (y^{2p} s^p (s^p + t^p)) + z^p (s^p + t^p) (z^{2p} s^p t^p) = 0,
$$

i.e.,

$$
x^p s^p [x^{2p} t^p (s^p + t^p) + \alpha z^p (s^p + t^p)] + y^p t^p [y^{2p} s^p (s^p + t^p)] + \beta z^p (s^p + t^p) = 0.
$$

Since $x^p s^p$ and $y^p t^p$ is an $R$-sequence, we see

$$
x^{2p} t^p (s^p + t^p) + \alpha z^p (s^p + t^p) = ry^p t^p
$$
$$
y^{2p} s^p (s^p + t^p) + \beta x^p (s^p + t^p) = -rx^p s^p
$$

for some $r \in R$. Therefore the element $e = (-\beta | a^p, \alpha | b^p, -r | c^p)$ in
$R_r \oplus R_s \oplus R_c$ will map to $(u^p, v^p, w^p)$ under $g$. That is, $g(e) = F((u, v, w)) = (u^p, v^p, w^p)$. Therefore $F_\ast(u, v, w) = 0$ which means $H_f^1(R') = 0$ and depth$_f(R') = 3$. Therefore $R_j$ gives us a big Cohen–Macaulay algebra with identity for $R_f$.

In this example, we will show the ring $(Q[X, Y, Z]/(X^3 + Y^3 + Z^3) \otimes Q[S, T])$, has a big Cohen–Macaulay algebra with identity ($Q =$ rationals). The modification procedure that is used in this section is similar to that used by Hochster [5]. Again we will construct an algebra over $Q[X, Y, Z]/(X^3 + Y^3 + Z^3) \otimes Q[S, T]$ which is $x = (x_1, x_2, x_3)$-regular where $x_1 = x_s, x_2 = y_t, x_3 = z(s + t)$ and this will give what we want.

**Definition 3.2.** Let $S$ be an $R$-algebra, $x = x_1, x_2, ... , x_n$ be a sequence of elements of $R$. Then a $k + 1$-tuple $s = (s_1, s_2, ... , s_{k+1})$ is called a type $k$ relation on $(x)$ if $s_{k+1}x_{k+1} = \sum s_i x_i$.

Let $R$ be $Z[X, Y, Z]/(X^3 + Y^3 + Z^3) \otimes Z[S, T] \cong Z[x_s, y_t, z_s, x_t, y_t, z_t]$ such that $x^3 + y^3 + z^3 = 0$. Then $x = x_1, x_2, x_3$ is a system of parameters for the irrelevant ideal. Let $r = (r_1, ... , r_{i+1})$ be a type $i$ relation on $(x)$ for $i = 1$ or 2. Define $R_1$, the first modification of $R$ with respect to $r$, as

$$R_1 = R[X_1, ... , X_i]/\left(r_{i+1} - \sum r_j X_j\right).$$

That is, if $r = (r_1, r_2)$ then $R_1 = R[X_1]/(r_2 - r_1X_1)$, if $r = (r_1, r_2, r_3)$ then $R_1 = R[X_1, X_2]/(r_3 - r_2X_2 - r_1X_1)$. $R_1$ has a natural $R$-algebra structure. Let $r' = (r_1, ... , r_{i+1}), r' \in R^n$, a type $i$ relation of $R^i$ on $(x)$. We can define the modification $R_2$ of $R_1$ with respect to $r$ by

$$R_2 = R[X_1, ... , X_i]/\left(r_{i+1} - \sum r_j X_j\right).$$

Continuing in this fashion, we have a sequence of modifications

$$R = R_0 \to R_1 \to R_2 \cdots . \quad (\ast)$$

We may form this sequence in such a way that all type 1 or type 2 relations of $R_k$ on $x$ are trivialized in some later modification for all $k \geq 0$. This is true because the set of all $r$'s form a finitely generated submodule in $R_k^2$ or $R_k^3$. Therefore we can enumerate the generators and modify them one by one somewhere down the line. Assume this is done. Set $A = \lim(R_0 \to R_1 \to \cdots )$, then $x_1 A: x_2 = x_1 A$ and $(x_1, x_2) A: x_3 = (x_1, x_2) A$. This is true because if there is a non-trivial relation of type 1 or 2 of $A$, it must come from some $R_k$. But we would have consequently trivialized it later on which is impossible. Tensoring the sequence $(\ast)$ with $Q$, we have
Let $R_i \otimes Q = R_i'$ and $Q \otimes_k A = A'$. Then $x_1 A' : x_2 A' = x_1 A'$ and $(x_1, x_2) A' : x_3 A' = (x_1, x_2) A'$ are still true. We claim that $x$ is a regular sequence of $A'$. What is needed for this claim is that $(x) A' \neq A'$, i.e., $1 \notin (x) A'$. We will show this by contradiction. Assume that $1 \in (x) A'$, then for some $R_i = R_i \otimes Q$, $1 \in (x) R_i$ or $n \in (x) R_i$ for some $n$. Choose $p = 2 (\mod 3)$ such that $p$ does not divide $n$ and $k$ an algebraically closed field of characteristic $p$. Then the sequence $R_0 \otimes k \to R_1 \otimes k \to \cdots R_i \otimes k$ is a sequence of modifications for the ring $R_0 \otimes k = k[X, Y, Z]/(X^3 + Y^3 + Z^3)$ and $1 \in (x) \in R_i \otimes k$. By the result of the last section, we know $R_0 \otimes k$ has $(x)$-regular algebra $B$ with identity. We claim we can lift the map $\psi_0$ to $\psi_1, \psi_2, \ldots, \psi_i$, in the following diagram:

We do this inductively. Suppose $\psi_d$ is defined for $0 \leq d < i$. To define $\psi_{d+1}$, we need to fill in the following diagram:

where $f$ is the modification map and

$$R_{d+1} \otimes k = (R_d \otimes k)[X_1, \ldots, X_j]/\left(r_{j+1} - \sum_{i=1}^j r_i X_i\right),$$

where $r = (r_1, \ldots, r_{j+1})$ is a type $j$ relation for $j = 1$ or $2$. Since $\psi_d(x_1 r_1 + \cdots + r_{j+1} X_j) = \sum_{i=1}^j x_i \psi_d(r_i) = 0$ in $B$ and $(x)$ is a regular sequence on $B$. We see that $\psi_d(r_{j+1}) = x_1 m_1 + \cdots + x_j m_j$ for some $m_1, \ldots, m_j$ in $B$. Define $\psi_{d+1} : R_{d+1} \otimes k \to B$ by $\psi_{d+1}(X_i) = m_i, l = 1, \ldots, j$. Then $\psi_{d+1}$ is a lift of $\psi_d$ and the diagram commutes. Therefore $\psi_i$ exists and we have the following diagram:
But this is impossible since $\psi_0(1) \neq 1 \psi(x) B$, but $\psi_0(1) = (\psi \circ f)(1) = \psi(f(1)) \in (x) B$. Therefore $x$ is a regular sequence for $A'$ and the ring $(\mathbb{Q}[X, Y, Z]/(X^3 + Y^3 + Z^3) \otimes \mathbb{Q}[S, T])$ has a big Cohen–Macaulay algebra with identity.

REFERENCES

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