Asymptotic Bethe Ansatz on the GKP vacuum as a defect spin chain: Scattering, particles and minimal area Wilson loops

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Abstract

Moving from Beisert–Staudacher equations, the complete set of Asymptotic Bethe Ansatz equations and S-matrix for the excitations over the GKP vacuum is found. The resulting model on this new vacuum is an integrable spin chain of length $R = 2 \ln s$ ($s = $ spin) with particle rapidities as inhomogeneities, two (purely transmitting) defects and $SU(4)$ (residual R-)symmetry. The non-trivial dynamics of $\mathcal{N} = 4$ SYM appears in elaborated dressing factors of the 2D two-particle scattering factors, all depending on the 'fundamental' one between two scalar excitations. From scattering factors we determine bound states. In particular, we study the strong coupling limit, in the non-perturbative, perturbative and giant hole regimes. Eventually, from these scattering data we construct the $4D$ pentagon transition amplitudes (perturbative regime). In this manner, we detail the multi-particle contributions (flux tube) to the MHV gluon scattering amplitudes/Wilson loops (OPE or BSV series) and re-sum them to the Thermodynamic Bubble Ansatz.

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1. Introduction

The study of the energy of the excitations on a suitably chosen vacuum state is a problem which is common to very many physical theories. It often happens that most intriguing excitations arise over a vacuum state which is an intricate superposition of 'basic' states, *i.e.* a sort of Fermi sea of interacting 'pseudoparticles'. In general, this vacuum may be dubbed antiferromagnetic as the prototypical example in the realm of integrable models is the antiferromagnetic vacuum state of the Heisenberg spin chain. In their turn, important excitations on it are called spinons or solitons, whilst magnons are the (pseudo)particles forming the see on the ferromagnetic vacuum. In an easy Bethe Ansatz perspective [1], spinons may appear as holes in a distribution of a large number of real Bethe roots. As a consequence, these holes are constrained by quantisation conditions for their rapidities, which may anew be seen as Bethe(–Yang) equations for these new 'fundamental' particles. Of course, we expect this phenomenon to be of non-perturbative nature, so that integrability is the right realm to exploit it.

A similar, but obviously much richer situation, arises in the framework of Beisert–Staudacher Asymptotic Bethe Ansatz (ABA) equations determining, via a specific root configuration, the anomalous dimension (energy) of the single trace fields in planar $\mathcal{N} = 4$ SYM [2]. In this context one can choose as 'antiferromagnetic' vacuum the configuration which contains a large number, s, of type-4 roots and which describes, up to wrapping corrections [3], high spin ($= s$) twist two operators, namely, sketchily,

$$\mathcal{O} = \operatorname{Tr} Z D_+^s Z + \dots \quad , \tag{1.1}$$

where Z is one of the three (complex) scalars of the theory. In fact, this is likely the 'simplest' example of Wilson twist operator. It belongs to the paradigmatic $sl(2)$ sector of scalar operators, which are made up of only one (out of three) complex scalar Z and the (light-cone) covariant derivative D_+, so enjoying the sketchy form

$$\operatorname{Tr}(D_+^s Z^L) + \dots , \tag{1.2}$$

where dots stand for permutations. Built up in this selected way they result to be perturbatively closed under renormalisation, so forming a sector. These composite single trace operator have of course Lorentz spin s and twist (or length in the ferromagnetic/half-BPS vacuum perspective) L, with minimum value $L = 2$ for which (a descendant[1] of) the Gubser–Klebanov–Polyakov (GKP) 'vacuum' solution is realised [4]. Also, the AdS/CFT correspondence [5] relates an operator (1.2) to a spinning folded closed strings on AdS$_5 \times$S^5 spacetime, with angular momenta $s/\sqrt{\lambda}$ and $L/\sqrt{\lambda}$ on each space respectively, the 't Hooft coupling in the multi-colour $N_c \to \infty$ (planar) regime

$$\lambda \equiv N_c g_{YM}^2 \ ; \ g^2 \equiv \frac{\lambda}{8\pi^2}, \tag{1.3}$$

being connected to the string tension $T = \frac{\sqrt{\lambda}}{2\pi}$ [4,6]. On the other hand we may think of the operators (1.2) as obtained from the GKP vacuum (1.1) by adding scalar excitations on top of it. Of course, when $L > 2$ we can realise states with different energies, at fixed L, and typically the minimal energy has been more extensively studied for 'large size' $s \to +\infty$. In particular, the

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[1] In this case, the spin may be shifted by a finite amount which does not affect our analysis and results at high spin.

minimal anomalous dimension of (1.2) has been proven to enjoy at one loop the same leading behaviour $\sim \ln s$ at high spin (and fixed L) [7], as in the opposite string regime (strong coupling) [4]. Later on, the coefficient of this term[2] was obtained at all loops from the solution of a linear integral equation directly derived from the Beisert–Staudacher equations via the root density approach [10]. In very brief summary, as computed in [11], the high spin (asymptotic) expansion (at fixed g and L) enjoys the peculiar form

$$\gamma(g, s, L) = f(g) \ln s + f_{sl}(g, L) + \sum_{n=1}^{\infty} \gamma^{(n)}(g, L)(\ln s)^{-n} + O((\ln s)/s), \qquad (1.4)$$

in inverse integer powers of the size[3] $R \sim \ln s$, except the sub-leading $(\ln s)^0$ contribution $f_{sl}(g, L)$ (defect contribution). The latter, which reduces to the so-called virtual scaling function for $L = 2$, has been captured in [12] by a Non-Linear Integral Equation (NLIE) and in [13] by a linear integral equation (by means of which explicit strong coupling expansions can be performed [14], along the lines of those for the cusp [15]). Up to this order, we can be sure that this expansion enjoys the same form at all perturbative orders in QCD, or its Mellin transform, *i.e.* the evolution kernels [16]. Moreover, in the supersymmetric case similar linear integral equations hold for all the coefficients in (1.4) [11] and also for the next order $O((\ln s)/s)$ [17], and all these, – importantly the first two $f(g)$ and $f_{sl}(g, L)$, – are now believed to be exactly given by the ABA without wrapping,[4] also thanks to these recent studies.

The latter were focused on the same scalar, Z, added to (1.1), but we can generalise to the other fields: indeed, elementary one-particle excitations may correspond to inserting one of the other fields *i.e.*, besides the other two scalars, a gauge field (gluon) or a Fermi field (gaugino).[5] In other words, they are the lowest twist (=three) operators/states with the form

$$\mathcal{O}_{1\text{-}particle} = \text{Tr}\, Z D_+^{s-s'} \varphi D_+^{s'} Z + \dots \quad , \qquad (1.5)$$

where $\varphi = Z, W, X$, the scalars, or $\varphi = F_{+\perp}, \bar{F}_{+\perp}$, the two components of the gauge field, or $\varphi = \Psi_+, \bar{\Psi}_+$, the $4 + 4$ (anti-)fermions, respectively. Besides the energy, one can determine also the momentum of an operator through the Beisert–Staudacher ABA equations. Along this line, the one-particle dispersion relations of the excitations (1.5) have been receiving much attention in the different coupling regimes (*cf.* for instance [19] and the references therein); but recently they have been summarised, corrected and put forward in an illuminating work by Basso [20] (also reference therein).

On the same footing, we start wondering in [21] about the scattering S-matrix which may be attached to the two-particle states (of, at least, twist-4)

$$\mathcal{O}_{2\text{-}particles} = \text{Tr}\, Z D_+^{s-s_1-s_2} \varphi_1 D_+^{s_1} \varphi_2 D_+^{s_1} Z + \dots \quad , \qquad (1.6)$$

where φ_1 and φ_2 may be any general elementary local field as φ in (1.5), whereas in [21] we confined our attention to the peculiar (*cf.* below) case $\varphi_1 = \varphi_2 = Z$. In fact, as argued above,

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[2] This is the so-called *universal scaling function*, $f(g)$, which does not depend on L and equals twice the *cusp anomalous dimension* (renormalisation divergence [8]) of a light-like Wilson cusp, as in QCD[9].

[3] In fact, it is consistent with the length of the long classical string $R \sim \ln(s/\sqrt{\lambda})$ [4,6].

[4] For instance in [18] wrapping corrections to ABA start to contribute at order $e^{-R} = e^{-2\ln s} = 1/s^2$, inducing to think of a factor 2 in the size of the folded string $R = 2\ln s + \dots$.

[5] Notice that in the half-BPS vacuum description this state would belong to a longer spin chain of length $L = 3$.

we expect the Beisert–Staudacher quantisation conditions to give correct results at leading $\ln s$ and next to leading order $(\ln s)^0$. And then, regarding $R \sim \ln s$ as the size of the system, these orders are exactly the ones we need to write down $2D$ (many-particle) scattering amplitudes, *i.e.* (on-shell) quantisation conditions, for rapidities of excitations on the GKP vacuum. Generalising to all the other scalars, [22] have deduced the entire $SO(6)$ scattering, while we have computed in [23] all the g-depending scalar factors of the different scattering channels, neglecting the $SU(4)$ representation structure.

Moving from this lack, we shall make here our analysis deeper, by computing explicitly the matrix structures of the different $SU(4)$ representations carried by the 'elementary' particles and by their bound states. We will not only consider the two-body scattering, but also in general the multi-particle 2D scattering amplitudes. As a byproduct we will see a well know characterisation of integrable theories, namely the elasticity and factorisation, *i.e.* the determination of many-particle scattering by the two-particle one. Besides the traditional name of Bethe–Yang equations, we can call these quantisation conditions Asymptotic Bethe Ansatz equations as well, but now the term 'Asymptotic' refers to the new length $\sim \ln s$, which measures the validity of the equations (and to the 'new' vacuum). More precisely, from the BMN (ferromagnetic) vacuum [24] (no roots) we will switch on, in the Beisert–Staudacher equations, the configurations corresponding to the GKP (antiferromagnetic) vacuum and to all possible 'elementary' excitations over the GKP vacuum; to accomplish this, we will be using the idea of converting many (Bethe) algebraic equations describing an excited state into few non-linear integral equations (NLIEs) [25–27,12,13]. In this way, we will obtain the quantisation conditions of all the 'elementary' excitations over the GKP vacuum and show that the structure of these equations coincides with Bethe equations of an inhomogeneous spin chain of length $R = 2 \ln s$ with two identical (purely transmitting) defects and an $SU(4)$ symmetry in different representations (where the particle rapidities represent the inhomogeneities). Of course, the scalar pre-factors in front of the above $SU(4)$ matrix structure are dependent on g and characteristic of the theory (and GKP vacuum). Nevertheless, we can express all in terms of the scalar–scalar one [23]. Moreover, we will discuss in many details the consequences of switching to a different vacuum which basically means that any elementary particle interacts with the sea of covariant derivatives namely the type-4 roots. For instance, the poles of the new 2D scattering factors of these particle imply the entrance of bound states thereof into the spectrum and then the existence of new scattering amplitudes for the latter particles. As anticipated, not only the 2D scattering amplitudes, but also many physical quantities assume novel expressions, as for instance the energy, momentum [20] and all the other conserved charges carried by a single elementary or composite excitation (*cf.* below). Furthermore, the scattering of any particle onto two defects arises, as anticipated in [22,23], though they were absent in the ferromagnetic setup, and they are likely to be associated to the two external holes (or tips of the GKP string). Together with the change of length, these are somehow unprecedented features in the theory of (quantum) integrable systems though their common origin can be traced back to the $sl(2)$ spin chain (describing the one-scalar sector (1.2) at one-loop): nevertheless, we are used to insert the defect *ab initio* in the theory on the ferromagnetic vacuum and then finding the anti-ferromagnetic dynamics with defect (possibly characterised by a different scattering factor) and the same length.

As a consequence of this new ABA, also the exact Thermodynamic Bethe Ansatz (TBA) [28, 29] for the spectrum of anomalous dimensions as derived from its mirror version should osten-

sibly look very differently[6] than the usual one on the BMN vacuum [30], although they should give the same spectrum after all. Even more interestingly, a recent series of papers by Basso, Sever and Vieira (BSV) extended to all terms the operator product expansion of [41] and thus proposed a non-perturbative approach to 4D gluon scattering amplitudes/null polygonal Wilson loops (which are allegedly the same [31–33]) in $\mathcal{N} = 4$ SYM, which relies on these 2D scattering factors as input data or building blocks [34–38]. In this perspective, light-like polygonal Wilson loops (WLs) can thought of as an infinite sum over more fundamental polygons, namely square and pentagonal WLs, whose knowledge relies on the GKP scattering factors. By virtue of the AdS/CFT strong/weak duality, this superposition of pentagons and squares should lead, at large coupling g, to the classical string regime, namely the minimisation of the supersymmetric string action [31]. In general, this is a complicated problem of minimal area (string action) subtending a polygon living on the boundary of AdS_5, and results in a set of non-linear coupled integral equations [39,41]. For some still hidden reason, their form resembles that of a relativistic Thermodynamic Bethe (or Bubble, in this case!) Ansatz system whose free energy yields the area [29,28].[7] Instead, we wish in this paper construct this TBA set-up by summing the infinite BSV series and performing a saddle point evaluation. For this aim, we will perform a propaedeutic analysis of all the different strong coupling regimes.

The article is organised according to the following plan. In Section 2 we derive the ABA equations, first at one-loop as exemplifying case so to highlight all the relevant features, then for any value of the coupling. In Section 3 the conserved charges of the excitations (on the GKP vacuum) are computed. In Section 4 the strong coupling limit of the scattering factors is considered, in the different dynamical regimes, *i.e.* non-perturbative, perturbative and giant hole regimes. Section 5 contains equivalent forms for the momentum associated to any elementary particle excitation, in particular that elaborated in [20]. Section 6 is a study of the strong coupling behaviour of (the scattering factor for) the spin chain defects. In Section 7 the properties of the different kinds of particles under the SU(4) symmetry are taken into exam, so that in Section 8 we are able to describe the structure of the overall S-matrix. In Section 9 the so-called *string hypothesis* is used on the GKP ABA, in order to survey the on-shell states and, in particular, the bound states of elementary particles; for later purposes, an accurate study is devoted to the behaviour of bound states of gluons at any coupling and fermion – antifermion = meson and its bound states which, instead, do appear only at the leading order of perturbative strong coupling regime, *i.e.* in the classical string theory. Section 10 computes all the string perturbative expressions of the pentagonal amplitudes (contributing to the BSV series). Finally, these infinite contributions are summed up exactly in Section 11 as for the hexagonal Wilson loop; the result is remarkably coinciding with the Yang–Yang functional and Thermodynamic Bubble Ansatz (TBA) equations for minimal area solution [39,41,40]. After the conclusions (Section 12), several appendices follow. In Appendices A and B the definition of the functions employed throughout the text, as well as some useful formulae and integrals are listed. Appendix C is a synopsis of scattering factors, displayed at arbitrary coupling, one-loop, strong coupling (mirror transformed too), while Appendix D gives some details about their derivation. Finally, in Appendix E all the ABA equations are listed.

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[6] The attentive reader may guess many aspects of it from the form of *string/stack solutions* as reported in Section 9.

[7] In the particular case of the hexagon WL [39] the system does coincide with the usual TBA one [42].

2. General equations

2.1. Excitations

The main aim of this section is to write Bethe equations describing 'elementary' excitations over the long GKP string, in the more general case when H scalars (u_h, $h = 1, \ldots, H$), N_F large fermions, $N_{\bar{F}}$ large antifermions, N_f small fermions, $N_{\bar{f}}$ small antifermions, N_g gauge fields $F_{+\perp}$ and $N_{\bar{g}}$ gauge fields $\bar{F}_{+\perp}$ are present.

In the notation of Beisert–Staudacher equations [2] the GKP vacuum is described by a large even number s of type-4 roots filling the interval $[-b, b]$ of the real axis, together with two external holes [43,7]. In the large s limit b is approximated with $s/2$ and the positions of the two external holes with $\pm s/\sqrt{2}$: corrections to those estimates give rise to $O(1/s^2)$ terms in the final Bethe equations, that we will neglect.

It is a general fact that, in order to deal with a large number of Bethe roots, it is convenient to use their counting function $Z_4(v)$, which satisfies a nonlinear integral equation [25,26]. We found then natural to apply that strategy to the study of GKP vacuum and its excitations. In this approach scalar excitations, which are represented by holes in the distribution of type-4 roots in $[-b, b]$, are classified by quantisation conditions for $Z_4(v)$. The same function $Z_4(v)$ governs the interaction between roots with different flavour and scalars.

Coming in specific to the classification of the various excitations [20], we have already said that scalars are represented by holes in the distribution of type-4 roots. Large (small) fermions are described by u_3-type (u_1-type) roots and large (small) antifermions are described by u_5-type (u_7-type) roots. Rapidity of large fermions is the function $x_F(u_3) = x(u_3)$, where (g is related to the 't Hooft coupling λ by $\lambda = 8\pi^2 g^2$)

$$x(u) = \frac{u}{2} \left[1 + \sqrt{1 - \frac{2g^2}{u^2}} \right], \quad u^2 \geq 2g^2, \tag{2.1}$$

with the arithmetic definition of the square root. Therefore, rapidity of large fermions satisfies the inequality $|x_F| \geq g/\sqrt{2}$. On the other hand rapidity of small fermions is the function $x_f(u_1) = \frac{g^2}{2x(u_1)}$, with definition (2.1) for $x(u)$ and, consequently, it is constrained by the inequality $|x_f| \leq g/\sqrt{2}$. Changing $u_3 \to u_5$ and $u_1 \to u_7$ allows to describe large and small antifermions, respectively.

Gauge fields $F_{+\perp}$ with rapidity u_j^g correspond to stacks,

$$u_{2,j} = u_j^g, \quad u_{3,j} = u_j^g \pm i/2, \quad j = 1, \ldots, N_g, \tag{2.2}$$

with real centres u_j^g, while gauge fields $\bar{F}_{+\perp}$ with rapidity $u_j^{\bar{g}}$ are described by stacks,

$$u_{6,j} = u_j^{\bar{g}}, \quad u_{5,j} = u_j^{\bar{g}} \pm i/2, \quad j = 1, \ldots, N_{\bar{g}}, \tag{2.3}$$

with real centres $u_j^{\bar{g}}$.

We consider also the presence of isotopic roots, which do not carry momentum and energy, but take into account internal degrees of freedom, *i.e.* the residual $SU(4)$ symmetry of the GKP vacuum. In specific, we have K_a roots $u_{a,j}$ of type u_2,

$$u_{a,j} = u_{2,j}, \quad j = 1, \ldots, K_a, \tag{2.4}$$

K_c roots $u_{c,j}$ of type u_6

$$u_{c,j} = u_{6,j}, \quad j = 1, \ldots, K_c, \tag{2.5}$$

and K_b stacks,

$$u_{4,j} = u_{b,j} \pm \frac{i}{2}, \quad u_{b,j} = u_{3,j} = u_{5,j}, \quad j = 1, \ldots, K_b, \tag{2.6}$$

with centers $u_{b,j}$.

We are now going to present our derivation of the full set of Bethe–Yang equations for excitations on the GKP vacuum. For excitations with rapidity u_m belonging to the representation ρ of the symmetry group $SU(4)$ of the GKP vacuum such equations will appear in the following form

$$\prod_m \frac{u_{q,k} - u_m + i\vec{\alpha}_q \cdot \vec{w}_\rho}{u_{q,k} - u_m - i\vec{\alpha}_q \cdot \vec{w}_\rho}$$

$$= \prod_{j \neq k}^{K_q} \frac{u_{q,k} - u_{q,j} + i\vec{\alpha}_q \cdot \vec{\alpha}_q}{u_{q,k} - u_{q,j} - i\vec{\alpha}_q \cdot \vec{\alpha}_q} \prod_{q' \neq q} \prod_{j=1}^{K_{q'}} \frac{u_{q,k} - u_{q',j} + i\vec{\alpha}_q \cdot \vec{\alpha}_{q'}}{u_{q,k} - u_{q',j} - i\vec{\alpha}_q \cdot \vec{\alpha}_{q'}}, \tag{2.7}$$

$$1 = e^{iRP(u_m) + 2iD(u_m)} \prod_q \prod_{k=1}^{K_q} \frac{u_m - u_{q,k} + i\vec{\alpha}_q \cdot \vec{w}_\rho}{u_m - u_{q,k} - i\vec{\alpha}_q \cdot \vec{w}_\rho} \prod_{m' \neq m} S(u_m, u_{m'}), \tag{2.8}$$

where $\{\alpha_q\}$ are the set of simple roots of $SU(4)$, $u_{q,k}$ are the isotopic roots associated and $\vec{w}_\rho$ the highest weight of the representation ρ. The structure of these equations agrees with the general pattern shown in [44]. While the first equation (2.7) comes from the symmetry properties of the vacuum, the second one (2.8) is a quantisation condition for the rapidity u_m of an excitation moving in a one dimensional chain. Within this interpretation, R is given the meaning of the physical length of the chain, $P(u_m)$ that of the momentum of an excitation with rapidity u_m. The extra term $2D(u_m)$ in the exponent is interpreted as the effect of two *purely transmitting* (*i.e.* without reflection [45]) defects related to the tips of the GKP string. The rational factor in the right hand side of (2.8) takes into account the internal degrees of freedom: solving (2.7) one obtains $u_{q,k}$ in terms of u_m: plugging this result in the rational term in (2.8) one obtains, together with the products over the various $S(u_m, u_{m'})$, the phase change due to the scattering between an excitation with rapidity u_m and the other excitations with rapidity $u_{m'}$.

We will start with the one loop case, where all factors entering equations (2.8) are written in an explicit form, specifically in terms of products of Euler Gamma functions. The general all loops case will appear as a technical complication of the one loop, since the building blocks of the various equations (2.8) will be obtained after solving a linear integral equation.

2.2. Equations at one loop

Scalars In order to show how our strategy works, we first concentrate on the one loop case. We start from the fourth of the Beisert–Staudacher equations, in the presence of a large number s of real type-4 roots, together with the general pattern of excitations and isotopic roots described in previous section. We remark that in the one loop case only large fermions and large antifermions are present: for uniformity of notations in this subsection we will denote large fermions rapidity x_F with u_F and large antifermions rapidity $x_{\bar{F}}$ with $u_{\bar{F}}$.

We introduce the counting function

$$
\begin{aligned}
Z_4(v) = {}& i L \ln \frac{\frac{i}{2} - v}{\frac{i}{2} + v} + i \sum_{j=1}^{K_4} \ln \frac{i - v + u_{4,j}}{i + v - u_{4,j}} + 2i \sum_{j=1}^{K_b} \ln \frac{\frac{i}{2} + v - u_{b,j}}{\frac{i}{2} - v + u_{b,j}} \\
& + i \sum_{j=1}^{N_F} \ln \frac{\frac{i}{2} + v - u_{F,j}}{\frac{i}{2} - v + u_{F,j}} + i \sum_{j=1}^{N_{\bar{F}}} \ln \frac{\frac{i}{2} + v - u_{\bar{F},j}}{\frac{i}{2} - v + u_{\bar{F},j}} + i \sum_{j=1}^{N_g} \ln \frac{i + v - u_j^g}{i - v + u_j^g} \\
& + i \sum_{j=1}^{N_{\bar{g}}} \ln \frac{i + v - u_j^{\bar{g}}}{i - v + u_j^{\bar{g}}},
\end{aligned}
\tag{2.9}
$$

where the sum up to $K_4 = s + 2K_b$ is a sum over s real type-4 roots and over the $2K_b$ complex type-4 roots contained in the stack (2.6). In terms of Z_4 the fourth of Beisert–Staudacher equations reads as

$$
e^{-i Z_4(u_{4,k})} = (-1)^{L + K_4 + 2K_b + N_F + N_{\bar{F}} + N_g + N_{\bar{g}} - 1} .
\tag{2.10}
$$

In addition, in the large s limit the behaviour of $Z_4(v)$ is dominated by the second term in the right hand side, which implies that for v real $Z_4'(v) < 0$. With this information we can prove that the length L is not independent of the total number of excitations. Indeed, it is widely known [43,7] that condition (2.10) is satisfied on the real axis not only by the type-4 real roots, but also by $H + 2$ real numbers, called holes. H holes are inside the interval $[-b, b]$ ($b = s/2 + O(1/s^2)$) and, consequently, are often called 'internal' or 'small'; the remaining two holes have positions $\pm s/\sqrt{2} + O(1/s^2)$ and then are dubbed 'external' or 'large'. Since $Z_4'(v) < 0$ for v real, the difference between the extremal values on the real axis $Z_4(+\infty) - Z_4(-\infty)$ has to count the total number of real roots and holes, *i.e.*

$$
Z_4(+\infty) - Z_4(-\infty) = -2\pi (s + H + 2).
\tag{2.11}
$$

On the other hand, the definition (2.9) implies the asymptotic behaviours

$$
\begin{aligned}
Z_4(\pm\infty) &= \mp\pi (L + K_4 - 2K_b - N_F - N_{\bar{F}} - N_g - N_{\bar{g}}) \\
&= \mp\pi (L + s - N_F - N_{\bar{F}} - N_g - N_{\bar{g}}) .
\end{aligned}
\tag{2.12}
$$

Comparison between (2.11) and (2.12) gives the desired connexion

$$
L = H + 2 + N_F + N_{\bar{F}} + N_g + N_{\bar{g}},
\tag{2.13}
$$

between the length L and the total number of excitations.

Relation (2.13) once plugged in the exponent of the right hand side of (2.10), provides a simplification of quantisation condition for type-4 roots and real holes. Restricting to holes, whose position we call u_h, $h = 1, \ldots, H$, we get the compact formula (remember that s is always even)

$$
e^{-i Z_4(u_h)} = (-1)^{H-1} .
\tag{2.14}
$$

After fixing these preliminary aspects, we come back to equation (2.9). In order to get manageable expressions, we convert the sum over real type-4 roots into an integral by means of the master equation [26]:

$$u_{4,j} \quad \text{real} \quad \Rightarrow \quad \sum_{j=1}^{s} f(u_{4,j}) = -f\left(\frac{s}{\sqrt{2}}\right) - f\left(-\frac{s}{\sqrt{2}}\right) - \sum_{h=1}^{H} f(u_h)$$

$$- \int_{-\infty}^{+\infty} \frac{dv}{2\pi} f(v) \frac{d}{dv} [Z_4(v) - 2L_4(v)], \tag{2.15}$$

where $L_4(v) = \text{Im} \ln[1 + (-1)^H e^{iZ_4(v-i0^+)}]$. We specialise formula (2.15) to our case and get

$$\sum_{j=1}^{K_4} i \ln \frac{i - v + u_{4,j}}{i + v - u_{4,j}}$$

$$= \sum_{j=1}^{K_b} i \ln \frac{(\frac{i}{2} - v + u_{b,j})(\frac{3i}{2} - v + u_{b,j})}{(\frac{i}{2} + v - u_{b,j})(\frac{3i}{2} + v - u_{b,j})} - i \ln \frac{(i - v + \frac{s}{\sqrt{2}})(i - v - \frac{s}{\sqrt{2}})}{(i + v - \frac{s}{\sqrt{2}})(i + v + \frac{s}{\sqrt{2}})}$$

$$- i \sum_{h=1}^{H} \ln \frac{i - v + u_h}{i + v - u_h} + \int_{-\infty}^{+\infty} \frac{dw}{\pi} \frac{1}{1 + (v - w)^2} [Z_4(w) - 2L_4(w)]. \tag{2.16}$$

Eventually, putting together (2.9) and (2.16), we find that

$$Z_4(v) = F(v) + 2 \int_{-\infty}^{+\infty} dw \, G(v - w) \, L_4(w), \tag{2.17}$$

where $F(v)$, $G(v)$ satisfy the linear integral equations

$$F(v) = iL \ln \frac{\frac{i}{2} - v}{\frac{i}{2} + v} + i \sum_{j=1}^{K_b} \ln \frac{\frac{i}{2} + v - u_{b,j}}{\frac{i}{2} - v + u_{b,j}} \frac{\frac{3i}{2} - v + u_{b,j}}{\frac{3i}{2} + v - u_{b,j}} + i \sum_{j=1}^{N_g} \ln \frac{i + v - u_j^g}{i - v + u_j^g}$$

$$+ i \sum_{j=1}^{N_{\bar{g}}} \ln \frac{i + v - u_j^{\bar{g}}}{i - v + u_j^{\bar{g}}} + i \sum_{j=1}^{N_F} \ln \frac{\frac{i}{2} + v - u_{F,j}}{\frac{i}{2} - v + u_{F,j}} + i \sum_{j=1}^{N_{\bar{F}}} \ln \frac{\frac{i}{2} + v - u_{\bar{F},j}}{\frac{i}{2} - v + u_{\bar{F},j}}$$

$$+ i \sum_{h=1}^{H} \ln \frac{i + v - u_h}{i - v + u_h} + i \ln \frac{i + v - \frac{s}{\sqrt{2}}}{i - v + \frac{s}{\sqrt{2}}} \frac{i + v + \frac{s}{\sqrt{2}}}{i - v - \frac{s}{\sqrt{2}}}$$

$$+ \int_{-\infty}^{+\infty} \frac{dw}{\pi} \frac{1}{1 + (v - w)^2} F(w), \tag{2.18}$$

$$G(u - v) = -\frac{1}{\pi} \frac{1}{1 + (u - v)^2} + \int_{-\infty}^{+\infty} \frac{dw}{\pi} \frac{1}{1 + (u - w)^2} G(w - v). \tag{2.19}$$

Equation (2.18) is solved exactly; however, we remember that the pattern of excitations discussed before holds only in the large spin limit. To be consistent with that, we have to use the large s asymptotic behaviour

$$i \ln \frac{\Gamma\left(1 + iv - i\frac{s}{\sqrt{2}}\right)}{\Gamma\left(1 - iv + i\frac{s}{\sqrt{2}}\right)} + i \ln \frac{\Gamma\left(1 + iv + i\frac{s}{\sqrt{2}}\right)}{\Gamma\left(1 - iv - i\frac{s}{\sqrt{2}}\right)} \to -4v \ln \frac{s}{\sqrt{2}} + O(1/s^2), \qquad (2.20)$$

and write the final result for $F(v)$ as

$$F(v) = -iL \ln \frac{\Gamma\left(\frac{1}{2} + iv\right)}{\Gamma\left(\frac{1}{2} - iv\right)} - 4v \ln \frac{s}{\sqrt{2}} + i \sum_{h=1}^{H} \ln \frac{\Gamma(1 + iv - iu_h)}{\Gamma(1 - iv + iu_h)}$$

$$+ i \sum_{j=1}^{K_b} \ln \frac{\frac{i}{2} + v - u_{b,j}}{\frac{i}{2} - v + u_{b,j}} + i \sum_{j=1}^{N_g} \ln \frac{\Gamma(1 + iv - iu_j^g)}{\Gamma(1 - iv + iu_j^g)} + i \sum_{j=1}^{N_{\bar{g}}} \ln \frac{\Gamma(1 + iv - iu_j^{\bar{g}})}{\Gamma(1 - iv + iu_j^{\bar{g}})}$$

$$+ i \sum_{j=1}^{N_F} \ln \frac{\Gamma(\frac{1}{2} + iv - iu_{F,j})}{\Gamma(\frac{1}{2} - iv + iu_{F,j})} + i \sum_{j=1}^{N_{\bar{F}}} \ln \frac{\Gamma(\frac{1}{2} + iv - iu_{\bar{F},j})}{\Gamma(\frac{1}{2} - iv + iu_{\bar{F},j})} + O(1/s^2). \qquad (2.21)$$

On the other hand, the solution of (2.19) for G reads as

$$G(v - w) = \frac{1}{2\pi} [\psi(1 + iv - iw) + \psi(1 - iv + iw)]. \qquad (2.22)$$

Then, we notice that in the large s limit $Z_4(v) \sim F(v) \sim -4v \ln(s/\sqrt{2}) + O(s^0)$. This means that the leading behaviour of the nonlinear term in (2.17) is the same as that of the analogous term for the GKP vacuum: therefore, we can use results of [12] and approximate the non-linear term in (2.17) as

$$2 \int_{-\infty}^{+\infty} dw\, G(v - w)\, L_4(w) = -2v \ln 2 + O(1/s^2). \qquad (2.23)$$

Plugging (2.21) and (2.23) into (2.17), we eventually get

$$Z_4(v) = -iL \ln \frac{\Gamma\left(\frac{1}{2} + iv\right)}{\Gamma\left(\frac{1}{2} - iv\right)} - 4v \ln s + i \sum_{h=1}^{H} \ln \frac{\Gamma(1 + iv - iu_h)}{\Gamma(1 - iv + iu_h)}$$

$$+ i \sum_{j=1}^{K_b} \ln \frac{\frac{i}{2} + v - u_{b,j}}{\frac{i}{2} - v + u_{b,j}} + i \sum_{j=1}^{N_g} \ln \frac{\Gamma(1 + iv - iu_j^g)}{\Gamma(1 - iv + iu_j^g)} + i \sum_{j=1}^{N_{\bar{g}}} \ln \frac{\Gamma(1 + iv - iu_j^{\bar{g}})}{\Gamma(1 - iv + iu_j^{\bar{g}})}$$

$$+ i \sum_{j=1}^{N_F} \ln \frac{\Gamma(\frac{1}{2} + iv - iu_{F,j})}{\Gamma(\frac{1}{2} - iv + iu_{F,j})} + i \sum_{j=1}^{N_{\bar{F}}} \ln \frac{\Gamma(\frac{1}{2} + iv - iu_{\bar{F},j})}{\Gamma(\frac{1}{2} - iv + iu_{\bar{F},j})} + O(1/s^2). \qquad (2.24)$$

Now, it is clear that imposing quantisation condition (2.14) on (2.24) provides a constraint between the rapidity u_h of a scalar and the rapidities of all the other excitations. As in all integrable models, this constraint has the general form (2.8): therefore we could be tempted to use (2.14) to define momenta and scattering factors of excitations, as well as the effective length of the chain. Such procedure, however, will provide scattering factors $i \ln S$ which diverge as $u_* \ln u_*$ when the rapidity u_* of a generic excitation becomes very large. Fortunately, it happens that this problem can be avoided if we make use of the zero momentum condition, which is a selection rule to

extract physical states out of the Beisert–Staudacher equations. To be specific, all physical states have to satisfy the condition $e^{iP} = 1$, where

$$P = i \sum_{j=1}^{K_4} \ln \frac{u_{4,j} + \frac{i}{2}}{u_{4,j} - \frac{i}{2}} . \tag{2.25}$$

Since $K_4 = s + 2K_b$ is even, we can also write

$$P = i \sum_{j=1}^{K_4} \ln \frac{\frac{i}{2} + u_{4,j}}{\frac{i}{2} - u_{4,j}} + 2\pi \mathbb{Z} . \tag{2.26}$$

This expression is regular for $u_{4,j} = 0$ and, therefore, it is more convenient for our calculations:

$$P = i \sum_{j=1}^{K_4} \ln \frac{\frac{i}{2} + u_{4,j}}{\frac{i}{2} - u_{4,j}} = i \sum_{h=1}^{H} \ln \frac{\frac{i}{2} - u_h}{\frac{i}{2} + u_h} + i \sum_{j=1}^{K_b} \ln \frac{i + u_{b,j}}{i - u_{b,j}}$$

$$+ \pi K_b - i \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{\frac{i}{2} + v}{\frac{i}{2} - v} \frac{d}{dv} [Z_4(v) - 2L_4(v)] = \pi K_b - i \sum_{h=1}^{H} \ln \frac{\Gamma\left(\frac{1}{2} + iu_h\right)}{\Gamma\left(\frac{1}{2} - iu_h\right)}$$

$$- i \sum_{j=1}^{N_g} \ln \frac{\Gamma\left(\frac{3}{2} + iu_j^g\right)}{\Gamma\left(\frac{3}{2} - iu_j^g\right)} - i \sum_{j=1}^{N_{\bar{g}}} \ln \frac{\Gamma\left(\frac{3}{2} + iu_j^{\bar{g}}\right)}{\Gamma\left(\frac{3}{2} - iu_j^{\bar{g}}\right)}$$

$$- i \sum_{j=1}^{N_F} \ln \frac{\Gamma\left(1 + iu_j^F\right)}{\Gamma\left(1 - iu_j^F\right)} - i \sum_{j=1}^{N_{\bar{F}}} \ln \frac{\Gamma\left(1 + iu_j^{\bar{F}}\right)}{\Gamma\left(1 - iu_j^{\bar{F}}\right)} + O(1/s^2) . \tag{2.27}$$

As a technical remark, we notice that nonlinear terms give no contributions at the orders $\ln s$ and $(\ln s)^0$.

Putting together (2.24) and (2.27) we obtain the equality

$$Z_4(v) - P = i \sum_{j=1}^{K_b} \ln \frac{v - u_{b,j} + \frac{i}{2}}{v - u_{b,j} - \frac{i}{2}}$$

$$+ i \sum_{h=1}^{H} \ln \frac{\Gamma\left(\frac{1}{2} - iv\right) \Gamma\left(\frac{1}{2} + iu_h\right) \Gamma(1 + iv - iu_h)}{\Gamma\left(\frac{1}{2} + iv\right) \Gamma\left(\frac{1}{2} - iu_h\right) \Gamma(1 - iv + iu_h)}$$

$$+ i \sum_{j=1}^{N_g} \ln \frac{\Gamma\left(\frac{1}{2} - iv\right) \Gamma\left(\frac{3}{2} + iu_j^g\right) \Gamma(1 + iv - iu_j^g)}{\Gamma\left(\frac{1}{2} + iv\right) \Gamma\left(\frac{3}{2} - iu_j^g\right) \Gamma(1 - iv + iu_j^g)} + (g \to \bar{g})$$

$$+ i \sum_{j=1}^{N_F} \ln \frac{\Gamma\left(\frac{1}{2} - iv\right) \Gamma(1 + iu_j^F) \Gamma\left(\frac{1}{2} + iv - iu_j^F\right)}{\Gamma\left(\frac{1}{2} + iv\right) \Gamma(1 - iu_j^F) \Gamma\left(\frac{1}{2} - iv + iu_j^F\right)} + (F \to \bar{F})$$

$$- 2i \ln \frac{\Gamma\left(\frac{1}{2} + iv\right)}{\Gamma\left(\frac{1}{2} - iv\right)} - 4v \ln s + O(1/s^2) . \tag{2.28}$$

Therefore, we have gained the possibility to write the condition $e^{-i[Z_4(u_h)-P]} = (-1)^{H-1}$ as a convenient alternative to (2.14):

$$
1 = e^{4iu_h \ln s} \left(\frac{\Gamma(\frac{1}{2} - iu_h)}{\Gamma(\frac{1}{2} + iu_h)} \right)^2 \prod_{\substack{h'=1 \\ h' \neq h}}^{H} (-) \frac{\Gamma\left(\frac{1}{2} - iu_h\right) \Gamma\left(\frac{1}{2} + iu_{h'}\right) \Gamma(1 + iu_h - iu_{h'})}{\Gamma\left(\frac{1}{2} + iu_h\right) \Gamma\left(\frac{1}{2} - iu_{h'}\right) \Gamma(1 - iu_h + iu_{h'})}
$$

$$
\cdot \prod_{j=1}^{K_b} \frac{u_h - u_{b,j} + \frac{i}{2}}{u_h - u_{b,j} - \frac{i}{2}} \prod_{j=1}^{N_g} \frac{\Gamma\left(1 + i(u_h - u_j^g)\right) \Gamma\left(\frac{1}{2} - iu_h\right) \Gamma\left(\frac{3}{2} + iu_j^g\right)}{\Gamma\left(1 - i(u_h - u_j^g)\right) \Gamma\left(\frac{1}{2} + iu_h\right) \Gamma\left(\frac{3}{2} - iu_j^g\right)}
$$

$$
\cdot \prod_{j=1}^{N_{\bar{g}}} \frac{\Gamma\left(1 + i(u_h - u_j^{\bar{g}})\right) \Gamma\left(\frac{1}{2} - iu_h\right) \Gamma\left(\frac{3}{2} + iu_j^{\bar{g}}\right)}{\Gamma\left(1 - i(u_h - u_j^{\bar{g}})\right) \Gamma\left(\frac{1}{2} + iu_h\right) \Gamma\left(\frac{3}{2} - iu_j^{\bar{g}}\right)}
$$

$$
\cdot \prod_{j=1}^{N_F} \frac{\Gamma(\frac{1}{2} + i(u_h - u_{F,j}))\Gamma(1 + iu_{F,j})\Gamma(\frac{1}{2} - iu_h)}{\Gamma(\frac{1}{2} - i(u_h - u_{F,j}))\Gamma(1 - iu_{F,j})\Gamma(\frac{1}{2} + iu_h)}
$$

$$
\cdot \prod_{j=1}^{N_{\bar{F}}} \frac{\Gamma(\frac{1}{2} + i(u_h - u_{\bar{F},j}))\Gamma(1 + iu_{\bar{F},j})\Gamma(\frac{1}{2} - iu_h)}{\Gamma(\frac{1}{2} - i(u_h - u_{\bar{F},j}))\Gamma(1 - iu_{\bar{F},j})\Gamma(\frac{1}{2} + iu_h)}
$$

$$(2.29)$$

We take (2.29) as Bethe–Yang equations for scalars. In the spirit of (2.8) we make the following identifications:

- Length of the chain $R = 2 \ln s$
- Momentum of a scalar $P_0^{(s)}(u_h) = 2u_h$

The terms in (2.29) depending on two rapidities have the natural interpretation of scattering factors between scalars and other excitations. Using notations given in Appendix C, we write

$$
1 = e^{iRP_0^{(s)}(u_h)} \left[\frac{\Gamma(\frac{1}{2} - iu_h)}{\Gamma(\frac{1}{2} + iu_h)} \right]^2 \prod_{j=1}^{K_b} \frac{u_h - u_{b,j} + \frac{i}{2}}{u_h - u_{b,j} - \frac{i}{2}} \prod_{\substack{h'=1 \\ h' \neq h}}^{H} S_0^{(ss)}(u_h, u_{h'})
$$

$$
\cdot \prod_{j=1}^{N_g} S_0^{(sg)}(u_h, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S_0^{(s\bar{g})}(u_h, u_j^{\bar{g}}) \prod_{j=1}^{N_F} S_0^{(sF)}(u_h, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S_0^{(s\bar{F})}(u_h, u_{\bar{F},j}), \quad (2.30)
$$

where $S_0^{(s*)}$ denotes the scattering factors between a scalar and a generic excitation. We remark that $i \ln S_0^{(s*)}$ behaves like $\ln u_*$ when the rapidity of an excitation becomes large. Eventually, the last term

$$
\left[\frac{\Gamma(\frac{1}{2} - iu_h)}{\Gamma(\frac{1}{2} + iu_h)} \right]^2
\tag{2.31}
$$

has the form of the phase delay due to two purely transmitting defects.

Finally, in view of generalisations to all loops we find convenient to identify the various pieces entering the function $Z_4(v)$

$$Z_4(v) = \Theta_0'(v, s/\sqrt{2}) + \Theta_0'(v, -s/\sqrt{2}) + \sum_{h=1}^{H} \Theta_0'(v, u_h) + i \sum_{j=1}^{K_b} \ln \frac{\frac{i}{2} + v - u_{b,j}}{\frac{i}{2} - v + u_{b,j}}$$

$$+ \sum_{j=1}^{N_g} F_0^G(v, u_j^g) + \sum_{j=1}^{N_{\bar{g}}} F_0^G(v, u_j^{\bar{g}}) + \sum_{j=1}^{N_F} F_0^F(v, u_{F,j}) + \sum_{j=1}^{N_{\bar{F}}} F_0^F(v, u_{\bar{F},j})$$

$$- 2v \ln 2, \tag{2.32}$$

as solutions of integral equations

$$\Theta_0'(v, u) = \phi_0(v - u) + \Phi_0(v) - \int_{-\infty}^{+\infty} dw\, \varphi_0(v - w) \Theta_0'(w, u)$$

$$= i \ln \frac{\Gamma(1 + iv - iu)\,\Gamma(1/2 - iv)}{\Gamma(1 - iv + iu)\,\Gamma(1/2 + iv)}, \tag{2.33}$$

$$F_0^F(v, u) = \chi_0(v - u|1) + \Phi_0(v) - \int_{-\infty}^{+\infty} dw\, \varphi_0(v - w) F_0^F(w, u)$$

$$= i \ln \frac{\Gamma(1/2 + iv - iu)\,\Gamma(1/2 - iv)}{\Gamma(1/2 - iv + iu)\,\Gamma(1/2 + iv)}, \tag{2.34}$$

$$F_0^G(v) = \chi_0(v - u|2) + \Phi_0(v) - \int_{-\infty}^{+\infty} dw\, \varphi_0(v - w) F_0^G(w, u)$$

$$= i \ln \frac{\Gamma(1 + iv - iu)\,\Gamma(1/2 - iv)}{\Gamma(1 - iv + iu)\,\Gamma(1/2 + iv)}, \tag{2.35}$$

where Φ_0, ϕ_0, χ_0 are defined in Appendix A. In (2.32) the large s limit has to be taken in the first two terms in the right hand side. This limit gives

$$\Theta_0'(v, s/\sqrt{2}) + \Theta_0'(v, -s/\sqrt{2}) \rightarrow -4v \ln \frac{s}{\sqrt{2}} - 2i \ln \frac{\Gamma\left(\frac{1}{2} + iv\right)}{\Gamma\left(\frac{1}{2} - iv\right)}. \tag{2.36}$$

Fermions The equations for (large) fermions with rapidity $x_{F,k} = u_{F,k}$ come from the (inverse of the) third of the Beisert–Staudacher equations. We have

$$1 = \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + \frac{i}{2}}{u_{F,k} - u_{a,j} - \frac{i}{2}} \prod_{j=1}^{N_g} \frac{u_{F,k} - u_j^g + \frac{i}{2}}{u_{F,k} - u_j^g - \frac{i}{2}} (-1)^{K_4} \prod_{j=1}^{K_4} \frac{\frac{i}{2} + u_{4,j} - u_{F,k}}{\frac{i}{2} + u_{F,k} - u_{4,j}}$$

$$= \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + \frac{i}{2}}{u_{F,k} - u_{a,j} - \frac{i}{2}} \prod_{j=1}^{N_g} \frac{u_{F,k} - u_j^g + \frac{i}{2}}{u_{F,k} - u_j^g - \frac{i}{2}} (-1)^{K_b} \prod_{j=1}^{K_b} \frac{i - u_{F,k} + u_{b,j}}{i + u_{F,k} - u_{b,j}}$$

$$\cdot \prod_{h=1}^{H} \frac{\frac{i}{2} + u_{F,k} - u_h}{\frac{i}{2} + u_h - u_{F,k}} \exp\left[-\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{\frac{i}{2} + v - u_{F,k}}{\frac{i}{2} - v + u_{F,k}} \frac{d}{dv} (Z_4(v) - 2L_4(v)) \right], \tag{2.37}$$

since $K_4 = s + 2K_b$ is even. Then, we evaluate the integral (L_4 contributes with subleading $O(1/s^2)$ terms)

$$
-\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{\frac{i}{2} + v - u_{F,k}}{\frac{i}{2} - v + u_{F,k}} \frac{d}{dv}(Z_4(v) - 2L_4(v))
$$

$$
= L \ln \frac{\Gamma(1 - iu_{F,k})}{\Gamma(1 + iu_{F,k})} + 4iu_{F,k} \ln s - \sum_{h=1}^{H} \ln \frac{\Gamma(\frac{3}{2} + iu_h - iu_{F,k})}{\Gamma(\frac{3}{2} - iu_h + iu_{F,k})}
$$

$$
- \sum_{j=1}^{N_g} \ln \frac{\Gamma\left(\frac{3}{2} + iu_j^g - iu_{F,k}\right)}{\Gamma\left(\frac{3}{2} - iu_j^g + iu_{F,k}\right)} - \sum_{j=1}^{N_{\bar{g}}} \ln \frac{\Gamma\left(\frac{3}{2} + iu_j^{\bar{g}} - iu_{F,k}\right)}{\Gamma\left(\frac{3}{2} - iu_j^{\bar{g}} + iu_{F,k}\right)}
$$

$$
- \sum_{j=1}^{N_F} \ln \frac{\Gamma(1 + iu_{F,j} - iu_{F,k})}{\Gamma(1 - iu_{F,j} + iu_{F,k})} - \sum_{j=1}^{N_{\bar{F}}} \ln \frac{\Gamma(1 + iu_{\bar{F},j} - iu_{F,k})}{\Gamma(1 - iu_{\bar{F},j} + iu_{F,k})}
$$

$$
+ \sum_{j=1}^{K_b} \ln \frac{i + u_{F,k} - u_{b,j}}{i - u_{F,k} + u_{b,j}}.
\tag{2.38}
$$

Now, in order to reproduce scattering factors already appearing in equations for scalars (2.29), (2.30), we use the zero momentum condition: we multiply (2.37) with $1 = e^{iP}$, with P given by (2.27). Using notations defined in Appendix C we write the final Bethe equations for fermionic excitations as

$$
1 = e^{iRP_0^{(F)}(u_{F,k})} \left[\frac{\Gamma(1 - iu_{F,k})}{\Gamma(1 + iu_{F,k})}\right]^2 \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + i/2}{u_{F,k} - u_{a,j} - i/2} \prod_{h=1}^{H} S_0^{(Fs)}(u_{F,k}, u_h)
$$

$$
\cdot \prod_{j=1}^{N_g} S_0^{(Fg)}(u_{F,k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S_0^{(F\bar{g})}(u_{F,k}, u_j^{\bar{g}}) \prod_{j=1}^{N_F} S_0^{(FF)}(u_{F,k}, u_{F,j})
$$

$$
\cdot \prod_{j=1}^{N_{\bar{F}}} S_0^{(F\bar{F})}(u_{F,k}, u_{\bar{F},j}),
\tag{2.39}
$$

where we introduced the length $R = 2 \ln s$ of the chain and the momentum $P_0^{(F)}(u_{F,k}) = 2u_{F,k}$ of a fermionic excitation. As for scalars, the term

$$
\left[\frac{\Gamma(1 - iu_{F,k})}{\Gamma(1 + iu_{F,k})}\right]^2
\tag{2.40}
$$

stands for phase delay due to purely transmitting defects.

Equations for large antifermions come from the (inverse of the) fifth of the Beisert–Staudacher equations and are obtained in a completely similar way as in the fermions case. The final result is:

$$
1 = e^{iRP_0^{(F)}(u_{\bar{F},k})} \left[\frac{\Gamma(1 - iu_{\bar{F},k})}{\Gamma(1 + iu_{\bar{F},k})}\right]^2 \prod_{j=1}^{K_c} \frac{u_{\bar{F},k} - u_{c,j} + \frac{i}{2}}{u_{\bar{F},k} - u_{c,j} - \frac{i}{2}}
$$

$$\cdot \prod_{j=1}^{N_g} S_0^{(\bar{F}g)}(u_{\bar{F},k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S_0^{(\bar{F}\bar{g})}(u_{\bar{F},k}, u_j^{\bar{g}}) \prod_{j=1}^{N_F} S_0^{(\bar{F}F)}(u_{\bar{F},k}, u_{F,j})$$

$$\cdot \prod_{j=1}^{N_{\bar{F}}} S_0^{(\bar{F}\bar{F})}(u_{\bar{F},k}, u_{\bar{F},j}), \tag{2.41}$$

where we introduced the length $R = 2 \ln s$ of the chain, the momentum $P_0^{(\bar{F})}(u_{\bar{F},k}) = 2u_{\bar{F},k}$ of an antifermionic excitation and the 'defect' term

$$\left[\frac{\Gamma(1 - iu_{\bar{F},k})}{\Gamma(1 + iu_{\bar{F},k})} \right]^2. \tag{2.42}$$

Gluons   In the presence of a large number s of real type-4 roots, a gluon with rapidity u_k^g is described by a stack composed of a single type-2 root $u_{2,k} = u_k^g$ and a two-string formed by two type-3 roots $u_{3,k} = u_k^g \pm i/2$. Rapidity u_k^g is then constrained by the equation obtained by multiplying together the (inverse of the) second of the Beisert–Staudacher equations with $u_{2,k} = u_k^g$ with the (inverse of the) third for $u_{3,k} = u_k^g \pm i/2$, *i.e.*

$$1 = \prod_{\substack{j=1 \\ j \neq k}}^{N_g} \frac{u_k^g - u_j^g + i}{u_k^g - u_j^g - i} \prod_{j=1}^{K_b} \frac{u_k^g - u_{b,j} + \frac{i}{2}}{u_k^g - u_{b,j} - \frac{i}{2}} \prod_{j=1}^{N_F} \frac{u_k^g - u_{F,j} + \frac{i}{2}}{u_k^g - u_{F,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{u_k^g - u_{4,j} - i}{u_k^g - u_{4,j} + i}. \tag{2.43}$$

We concentrate on the last term in (2.43), which we rewrite as (remember that K_4 is even)

$$\prod_{j=1}^{K_4} \frac{u_k^g - u_{4,j} - i}{u_k^g - u_{4,j} + i} = \prod_{j=1}^{K_4} \frac{i - u_k^g + u_{4,j}}{i + u_k^g - u_{4,j}} = \prod_{j=1}^{K_b} \left[\frac{\frac{3i}{2} - u_k^g + u_{b,j}}{\frac{3i}{2} + u_k^g - u_{b,j}} \frac{\frac{i}{2} - u_k^g + u_{b,j}}{\frac{i}{2} + u_k^g - u_{b,j}} \right]$$

$$\cdot \prod_{h=1}^{H} \frac{i + u_k^g - u_h}{i - u_k^g + u_h} \exp \left[-\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{i - u_k^g + v}{i + u_k^g - v} \frac{d}{dv} (Z_4(v) - 2L_4(v)) \right]. \tag{2.44}$$

The integral term equals

$$-\int \frac{dv}{2\pi} \ln \frac{i - u_k^g + v}{i + u_k^g - v} \frac{d}{dv} (Z_4(v) - 2L_4(v))$$

$$= 4iu_k^g \ln s - L \ln \frac{\Gamma\left(\frac{3}{2} + iu_k^g\right)}{\Gamma\left(\frac{3}{2} - iu_k^g\right)} + \sum_{h=1}^{H} \ln \frac{\Gamma(2 - iu_h + iu_k^g)}{\Gamma(2 + iu_h - iu_k^g)} + \sum_{j=1}^{N_g} \ln \frac{\Gamma(2 - iu_j^g + iu_k^g)}{\Gamma(2 + iu_j^g - iu_k^g)}$$

$$+ \sum_{j=1}^{N_{\bar{g}}} \ln \frac{\Gamma(2 - iu_j^{\bar{g}} + iu_k^g)}{\Gamma(2 + iu_j^{\bar{g}} - iu_k^g)} + \sum_{j=1}^{N_F} \ln \frac{\Gamma\left(\frac{3}{2} - iu_{F,j} + iu_k^g\right)}{\Gamma\left(\frac{3}{2} + iu_{F,j} - iu_k^g\right)}$$

$$+ \sum_{j=1}^{N_{\bar{F}}} \ln \frac{\Gamma\left(\frac{3}{2} - iu_{\bar{F},j} + iu_k^g\right)}{\Gamma\left(\frac{3}{2} + iu_{\bar{F},j} - iu_k^g\right)} - \sum_{j=1}^{K_b} \ln \frac{\frac{3i}{2} - u_k^g + u_{b,j}}{\frac{3i}{2} + u_k^g - u_{b,j}}. \tag{2.45}$$

Putting the last two formulæ together, we have

$$
\prod_{j=1}^{K_4} \frac{u_k^g - u_{4,j} - i}{u_k^g - u_{4,j} + i} = e^{4iu_k^g \ln s} \left[\frac{\Gamma\left(\frac{3}{2} - iu_k^g\right)}{\Gamma\left(\frac{3}{2} + iu_k^g\right)} \right]^L \prod_{j=1}^{K_b} \frac{\frac{i}{2} - u_k^g + u_{b,j}}{\frac{i}{2} + u_k^g - u_{b,j}} \prod_{h=1}^{H} \frac{\Gamma(1 - iu_h + iu_k^g)}{\Gamma(1 + iu_h - iu_k^g)}
$$

$$
\cdot \prod_{j=1}^{N_g} \frac{\Gamma(2 - iu_j^g + iu_k^g)}{\Gamma(2 + iu_j^g - iu_k^g)} \prod_{j=1}^{N_{\bar{g}}} \frac{\Gamma(2 - iu_j^{\bar{g}} + iu_k^g)}{\Gamma(2 + iu_j^{\bar{g}} - iu_k^g)}
$$

$$
\cdot \prod_{j=1}^{N_F} \frac{\Gamma\left(\frac{3}{2} - iu_{F,j} + iu_k^g\right)}{\Gamma\left(\frac{3}{2} + iu_{F,j} - iu_k^g\right)} \prod_{j=1}^{N_{\bar{F}}} \frac{\Gamma\left(\frac{3}{2} - iu_{\bar{F},j} + iu_k^g\right)}{\Gamma\left(\frac{3}{2} + iu_{\bar{F},j} - iu_k^g\right)} .
$$

We now plug such expression in (2.43) and multiply the resulting expression by $1 = e^{iP}$, where P is given by (2.27). We observe the exact cancellation of the term depending on type-b isotopic roots and get the final set of equations, written in terms of scattering factors listed in Appendix C:

$$
1 = e^{iR P^{(g)}(u_k^g)} \left[\frac{\Gamma\left(\frac{3}{2} - iu_k^g\right)}{\Gamma\left(\frac{3}{2} + iu_k^g\right)} \right]^2 \prod_{h=1}^{H} S_0^{(gs)}(u_k^g, u_h)
$$

$$
\cdot \prod_{\substack{j=1 \\ j \neq k}}^{N_g} S_0^{(gg)}(u_k^g, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S_0^{(g\bar{g})}(u_k^g, u_j^{\bar{g}}) \prod_{j=1}^{N_F} S_0^{(gF)}(u_k^g, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S_0^{(g\bar{F})}(u_k^g, u_{\bar{F},j}), \quad (2.46)
$$

where we introduced the length $R = 2 \ln s$ of the chain and the momentum $P_0^{(g)}(u_k^g) = 2u_k^g$ of a gluon $F_{+\perp}$. In this case, the effect of the two transmitting defects on gluons is

$$
\left[\frac{\Gamma\left(\frac{3}{2} - iu_k^g\right)}{\Gamma\left(\frac{3}{2} + iu_k^g\right)} \right]^2 . \tag{2.47}
$$

In analogous fashion, we obtain the equation for the gluon field $\bar{F}_{+\perp}$:

$$
1 = e^{iR P^{(g)}(u_k^{\bar{g}})} \left[\frac{\Gamma\left(\frac{3}{2} - iu_k^{\bar{g}}\right)}{\Gamma\left(\frac{3}{2} + iu_k^{\bar{g}}\right)} \right]^2 \prod_{h=1}^{H} S_0^{(\bar{g}s)}(u_k^{\bar{g}}, u_h)
$$

$$
\cdot \prod_{\substack{j=1 \\ j \neq k}}^{N_g} S_0^{(\bar{g}g)}(u_k^{\bar{g}}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S_0^{(\bar{g}\bar{g})}(u_k^{\bar{g}}, u_j^{\bar{g}}) \prod_{j=1}^{N_F} S_0^{(\bar{g}F)}(u_k^{\bar{g}}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S_0^{(\bar{g}\bar{F})}(u_k^{\bar{g}}, u_{\bar{F},j}), \quad (2.48)
$$

where again $R = 2 \ln s$ is the length of the chain and $P_0^{(g)}(u_k^{\bar{g}}) = 2u_k^{\bar{g}}$ is the momentum of the gluon excitation $\bar{F}_{+\perp}$.

Isotopic roots　　We remember (see (2.4), (2.5), (2.6)) the definition of the three sets of isotopic roots, which do not carry momentum and energy, but take into account the $su(4)$ symmetry of the GKP vacuum.

We have K_a roots $u_{a,j}$ of type u_2, K_c roots $u_{c,j}$ of type u_6 and K_b stacks, $u_{4,j} = u_{b,j} \pm \frac{i}{2}$, $u_{3,j} = u_{5,j} = u_{b,j}$ with centers $u_{b,j}$.

The equations for the isotopic roots u_a and u_c come directly from the second and the sixth of the Beisert–Staudacher equations: we observe the cancellation of the contributions coming from gauge field stacks:

$$1 = \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{N_F} \frac{u_{a,k} - u_{F,j} - \frac{i}{2}}{u_{a,k} - u_{F,j} + \frac{i}{2}} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}} \tag{2.49}$$

$$1 = \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{N_{\bar{F}}} \frac{u_{c,k} - u_{\bar{F},j} - \frac{i}{2}}{u_{c,k} - u_{\bar{F},j} + \frac{i}{2}} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}}. \tag{2.50}$$

For what concerns the isotopic roots u_b, we consider the product of the third Beisert–Staudacher equation for $u_{3,k} = u_{b,k}$ with the fifth for $u_{5,k} = u_{b,k}$, the fourth for $u_{4,k} = u_{b,k} + i/2$ and the fourth for $u_{4,k} = u_{b,k} - i/2$. We arrive at the following equation,

$$1 = \left(\frac{u_{b,k} - i}{u_{b,k} + i} \right)^L \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}}$$

$$\cdot \prod_{j=1}^{K_4} \frac{u_{b,k} - u_{4,j} + \frac{i}{2}}{u_{b,k} - u_{4,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{u_{b,k} - u_{4,j} - \frac{3i}{2}}{u_{b,k} - u_{4,j} + \frac{3i}{2}} \prod_{j=1}^{K_b} \left(\frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \right)^2$$

$$\cdot \prod_{j=1}^{N_g} \frac{u_{b,k} - u_j^g + \frac{3i}{2}}{u_{b,k} - u_j^g - \frac{3i}{2}} \prod_{j=1}^{N_{\bar{g}}} \frac{u_{b,k} - u_j^{\bar{g}} + \frac{3i}{2}}{u_{b,k} - u_j^{\bar{g}} - \frac{3i}{2}} \prod_{j=1}^{N_F} \frac{u_{b,k} - u_{F,j} + i}{u_{b,k} - u_{F,j} - i} \prod_{j=1}^{N_{\bar{F}}} \frac{u_{b,k} - u_{\bar{F},j} + i}{u_{b,k} - u_{\bar{F},j} - i}. \tag{2.51}$$

We have

$$\prod_{j=1}^{K_4} \frac{u_{b,k} - u_{4,j} + \frac{i}{2}}{u_{b,k} - u_{4,j} - \frac{i}{2}} = \prod_{j=1}^{K_4} \frac{\frac{i}{2} + u_{b,k} - u_{4,j}}{\frac{i}{2} - u_{b,k} + u_{4,j}} = \prod_{\substack{j=1 \\ j \neq k}}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{h=1}^{H} \frac{\frac{i}{2} - u_{b,k} + u_h}{\frac{i}{2} + u_{b,k} - u_h}$$

$$\cdot \exp\left[-\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{\frac{i}{2} + u_{b,k} - v}{\frac{i}{2} - u_{b,k} + v} (Z_4'(v) - 2L_4'(v)) \right] \left[1 + O(1/s^2) \right]. \tag{2.52}$$

Plugging (2.9) into the integral in the last term of (2.52), we find that

$$\exp\left[-\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{\frac{i}{2} + u_{b,k} - v}{\frac{i}{2} - u_{b,k} + v} (Z_4'(v) - 2L_4'(v)) \right]$$

$$= \left(\frac{i + u_{b,k}}{i - u_{b,k}} \right)^L \prod_{j=1}^{K_b} \left(\frac{i - u_{b,k} + u_{b,j}}{i + u_{b,k} - u_{b,j}} \right)^2 \prod_{j=1}^{K_4} \frac{\frac{3i}{2} + u_{b,k} - u_{4,j}}{\frac{3i}{2} - u_{b,k} + u_{4,j}}$$

$$\cdot \prod_{j=1}^{N_g} \frac{\frac{3i}{2} - u_{b,k} + u_j^g}{\frac{3i}{2} + u_{b,k} - u_j^g} \prod_{j=1}^{N_{\bar{g}}} \frac{\frac{3i}{2} - u_{b,k} + u_j^{\bar{g}}}{\frac{3i}{2} + u_{b,k} - u_j^{\bar{g}}} \prod_{j=1}^{N_F} \frac{i - u_{b,k} + u_{F,j}}{i + u_{b,k} - u_{F,j}} \prod_{j=1}^{N_{\bar{F}}} \frac{i - u_{b,k} + u_{\bar{F},j}}{i + u_{b,k} - u_{\bar{F},j}}.$$

Putting all together, we eventually get the following equation, for the third isotopic root u_b:

$$1 = \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}} \prod_{\substack{j=1 \\ j \neq k}}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{h=1}^{H} \frac{u_{b,k} - u_h - \frac{i}{2}}{u_{b,k} - u_h + \frac{i}{2}},$$

(2.53)

which does not depend on the roots associated to gluons.

2.3. The general (all loops) case

We now generalise all the results discussed in the one loop case to the most general all loops case. For the sake of clarity, the complete set of equations is summarised in Appendix E.

As we did in the one loop case, we start from scalar excitations.

Scalars   Let us introduce the counting function for the type-4 roots

$$Z_4(v) = iL \ln\left(-\frac{x^-(v)}{x^+(v)}\right) + i \sum_{j \neq k}^{K_4} \ln\left(-\frac{x^-(v) - x_{4,j}^+}{x^+(v) - x_{4,j}^-} \frac{1 - \frac{g^2}{2x^+(v)x_{4,j}^-}}{1 - \frac{g^2}{2x^-(v)x_{4,j}^+}} \sigma^2(v, u_{4,j})\right)$$

$$+ 2i \sum_{j=1}^{K_b} \ln\frac{x^+(v) - x_{b,j}}{x_{b,j} - x^-(v)} + i \sum_{j=1}^{N_F} \ln\frac{x^+(v) - x_{F,j}}{x_{F,j} - x^-(v)} + i \sum_{j=1}^{N_{\bar{F}}} \ln\frac{x^+(v) - x_{\bar{F},j}}{x_{\bar{F},j} - x^-(v)}$$

$$+ i \sum_{j=1}^{N_f} \ln\frac{1 - \frac{x_{f,j}}{x^+(v)}}{1 - \frac{x_{f,j}}{x^-(v)}} + i \sum_{j=1}^{N_{\bar{f}}} \ln\frac{1 - \frac{x_{\bar{f},j}}{x^+(v)}}{1 - \frac{x_{\bar{f},j}}{x^-(v)}}$$

$$+ i \sum_{j=1}^{N_g} \ln\frac{x^+(v) - x_j^{g+}}{x^-(v) - x_j^{g+}} \frac{x_j^{g-} - x^+(v)}{x^-(v) - x_j^{g-}} + i \sum_{j=1}^{N_{\bar{g}}} \ln\frac{x^+(v) - x_j^{\bar{g}+}}{x^-(v) - x_j^{\bar{g}+}} \frac{x_j^{\bar{g}-} - x^+(v)}{x^-(v) - x_j^{\bar{g}-}},$$

(2.54)

where $\sigma^2(v, u)$ is the so-called dressing factor [46,10], $x^\pm(v) = x(v \pm i/2)$ and we are using the notations

$$x_j^{g\pm} = x\left(u_j^g \pm \frac{i}{2}\right), \quad x_j^{\bar{g}\pm} = x\left(u_j^{\bar{g}} \pm \frac{i}{2}\right)$$

$$x_{4,j}^\pm = x\left(u_{4,j} \pm \frac{i}{2}\right), \quad x_{b,j} = x(u_{b,j}).$$

(2.55)

The property $e^{iZ_4(u_{4,k})} = (-1)^{H-1}$ follows from the definition (2.54) and from the relation (2.13) between L and the number of the various excitations: the condition $e^{iZ_4(u_h)} = (-1)^{H-1}$ identifies the H internal holes, *i.e.* the scalar excitations.

It is convenient to write (2.54) in terms of functions Φ, ϕ, χ, introduced in Appendix A

$$Z_4(v) = L\Phi(v) - \sum_{j=1}^{K_4} \phi(v, u_j) + 2i \sum_{j=1}^{K_b} \ln\frac{x^+(v) - x_{b,j}}{x_{b,j} - x^-(v)}$$

$$+ \sum_{j=1}^{N_g} \chi(v, u_j^g | 1) + \sum_{j=1}^{N_{\bar{g}}} \chi(v, u_j^{\bar{g}} | 1) + \sum_{j=1}^{N_F} \chi_F(v, u_{F,j}) + \sum_{j=1}^{N_{\bar{F}}} \chi_F(v, u_{\bar{F},j})$$

$$- \sum_{j=1}^{N_f} \chi_H(v, u_{f,j}) - \sum_{j=1}^{N_{\bar{f}}} \chi_H(v, u_{\bar{f},j}), \tag{2.56}$$

where s of the type-4 roots involved in the sum are real, while $2K_b$ are part of the stack defining the isotopic root u_b. We concentrate on the real type-4 roots and write the sum over them as an integral by means of (2.15), getting

$$Z_4(v) = L\Phi(v) + 2i \sum_{j=1}^{K_b} \ln \frac{x^+(v) - x_{b,j}}{x_{b,j} - x^-(v)} - \sum_{j=1}^{K_b} [\phi(v, u_{b,j} + i/2) + \phi(v, u_{b,j} - i/2)]$$

$$+ \sum_{j=1}^{N_g} \chi(v, u_j^g | 1) + \sum_{j=1}^{N_{\bar{g}}} \chi(v, u_j^{\bar{g}} | 1) + \sum_{j=1}^{N_F} \chi_F(v, u_{F,j}) + \sum_{j=1}^{N_{\bar{F}}} \chi_F(v, u_{\bar{F},j})$$

$$- \sum_{j=1}^{N_f} \chi_H(v, u_{f,j}) - \sum_{j=1}^{N_{\bar{f}}} \chi_H(v, u_{\bar{f},j}) + \sum_{h=1}^{H} \phi(v, u_h) + \phi(v, s/\sqrt{2})$$

$$+ \phi(v, -s/\sqrt{2}) - \int_{-\infty}^{+\infty} dw \varphi(v, w) [Z_4(w) - 2L_4(w)] + O(1/s^2), \tag{2.57}$$

where φ is defined in (A.4). Then, we can write

$$Z_4(v) = F(v) + 2 \int_{-\infty}^{+\infty} dw\, G(v, w) L_4(w), \tag{2.58}$$

where $F(v)$ satisfies the linear integral equation

$$F(v) = L\Phi(v) + 2i \sum_{j=1}^{K_b} \ln \frac{x^+(v) - x_{b,j}}{x_{b,j} - x^-(v)} - \sum_{j=1}^{K_b} [\phi(v, u_{b,j} + i/2) + \phi(v, u_{b,j} - i/2)]$$

$$+ \sum_{j=1}^{N_g} \chi(v, u_j^g | 1) + \sum_{j=1}^{N_{\bar{g}}} \chi(v, u_j^{\bar{g}} | 1) + \sum_{j=1}^{N_F} \chi_F(v, u_{F,j}) + \sum_{j=1}^{N_{\bar{F}}} \chi_F(v, u_{\bar{F},j})$$

$$- \sum_{j=1}^{N_f} \chi_H(v, u_{f,j}) - \sum_{j=1}^{N_{\bar{f}}} \chi_H(v, u_{\bar{f},j}) + \sum_{h=1}^{H} \phi(v, u_h)$$

$$+ \phi(v, s/\sqrt{2}) + \phi(v, -s/\sqrt{2}) - \int_{-\infty}^{+\infty} dw \varphi(v, w) F(w) + O(1/s^2), \tag{2.59}$$

and

$$G(v, w) = \varphi(v, w) - \int_{-\infty}^{+\infty} dw' \varphi(v, w') G(w', w). \tag{2.60}$$

We now work out the solution to (2.59). The part depending on the isotopic roots is written in an explicit form. For the remaining parts we remember that $L = H + 2 + N_g + N_{\bar{g}} + N_F + N_{\bar{F}}$: this allows to put $L\Phi(v)$ together with the other functions in the right hand side of (2.59). Eventually, the solution to (2.59) is written in terms of solutions of linear integral equations. In specific, we have

$$F(v) = \Theta'(v, s/\sqrt{2}) + \Theta'(v, -s/\sqrt{2}) + \sum_{h=1}^{H} \Theta'(v, u_h) + i \sum_{j=1}^{K_b} \ln \frac{i/2 + v - u_{b,j}}{i/2 - v + u_{b,j}}$$

$$+ \sum_{j=1}^{N_g} F^G(v, u_j^g) + \sum_{j=1}^{N_{\bar{g}}} F^G(v, u_j^{\bar{g}}) + \sum_{j=1}^{N_F} F^F(v, u_{F,j}) + \sum_{j=1}^{N_{\bar{F}}} F^F(v, u_{\bar{F},j})$$

$$+ \sum_{j=1}^{N_f} F^f(v, u_{f,j}) + \sum_{j=1}^{N_{\bar{f}}} F^f(v, u_{\bar{f},j}) + O(1/s^2), \tag{2.61}$$

where

$$\Theta'(v, u) = \phi(v, u) + \Phi(v) - \int_{-\infty}^{+\infty} dw\, \varphi(v, w)\Theta'(w, u), \tag{2.62}$$

$$F^F(v, u) = \chi_F(v, u) + \Phi(v) - \int_{-\infty}^{+\infty} dw\, \varphi(v, w) F^F(w, u), \tag{2.63}$$

$$F^f(v, u) = -\chi_H(v, u) - \int_{-\infty}^{+\infty} dw\, \varphi(v, w) F^f(w, u), \tag{2.64}$$

$$F^G(v, u) = \chi(v, u|1) + \Phi(v) - \int_{-\infty}^{+\infty} dw\, \varphi(v, w) F^G(w, u). \tag{2.65}$$

We first analyse the s-depending terms. A tedious calculation[8] shows that in the large s limit

$$\Theta'(v, s/\sqrt{2}) + \Theta'(v, -s/\sqrt{2}) = \ln \frac{s}{\sqrt{2}} [-4v + Z_{BES}(v)] - 2\tilde{P}(v) + O(1/s^2), \tag{2.66}$$

where $Z_{BES}(v) = -Z_{BES}(-v)$ and $\frac{d}{dv} Z_{BES}(v) = \sigma_{BES}(v)$, $\sigma_{BES}(v)$ is the famous BES density [10] and $\tilde{P}(v)$ is the solution of the integral equation[9]

$$\tilde{P}(v) = -\Phi(v) - \int_{-\infty}^{+\infty} \frac{dw}{2} [\varphi(v, w) - \varphi(v, -w)] \tilde{P}(w). \tag{2.67}$$

Then, we pass to study the nonlinear term $NL(v) = 2 \int_{-\infty}^{+\infty} dw\, G(v, w) L_4(w)$. The same term was computed in [21], where only real type-4 roots were present. Here we can use the same

---

[8] Formula (2.66) clarifies the origin of the length $2\ln s$ and the two 'defects': they are both due to the interaction with the two heavy large holes.

[9] We write the kernel of equation (2.67) in an explicitly antisymmetric form in order to avoid one loop divergencies.

results, since in the large s limit the leading behaviour of $Z_4(v)$ does not depend on the presence of excitations. We have

$$NL(v) = -2v \ln 2 + \frac{\ln 2}{2} Z_{BES}(v) + O(1/s^2). \tag{2.68}$$

Putting everything together, we arrive at

$$Z_4(v) = \ln s \left[-4v + Z_{BES}(v) \right] - 2\tilde{P}(v) + \sum_{h=1}^{H} \Theta'(v, u_h) + i \sum_{j=1}^{K_b} \ln \frac{i/2 + v - u_{b,j}}{i/2 - v + u_{b,j}}$$

$$+ \sum_{j=1}^{N_g} F^G(v, u_j^g) + \sum_{j=1}^{N_{\bar{g}}} F^G(v, u_j^{\bar{g}}) + \sum_{j=1}^{N_F} F^F(v, u_{F,j}) + \sum_{j=1}^{N_{\bar{F}}} F^F(v, u_{\bar{F},j})$$

$$+ \sum_{j=1}^{N_f} F^f(v, u_{f,j}) + \sum_{j=1}^{N_{\bar{f}}} F^f(v, u_{\bar{f},j}) + O(1/s^2). \tag{2.69}$$

Now, as in the one loop case, before imposing the quantisation condition for holes, we introduce the momentum of the chain

$$P = i \sum_{j=1}^{K_4} \ln \frac{x_{4,j}^+}{x_{4,j}^-} = -\sum_{j=1}^{K_4} \Phi(u_{4,j}) = \sum_{h=1}^{H} \Phi(u_h) + i \sum_{j=1}^{K_b} \ln \left(-\frac{x_{b,j}^{++}}{x_{b,j}^{--}} \right)$$

$$+ \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \Phi(v) \frac{d}{dv} [Z_4(v) - 2L_4(v)], \tag{2.70}$$

where $x_{b,j}^{\pm\pm} = x(u_{b,j} \pm i)$. Terms containing L_4 give no contributions at the orders $\ln s$ and $(\ln s)^0$. Terms containing u_b produce only a term πK_b. The dependence on excitations is worked out after inserting for $Z_4(v)$ expression (2.69). However, for our convenience we prefer to work directly on the expression $Z_4(v) - P$: after some calculation (see Appendix D for details) we arrive at the expression

$$Z_4(v) - P = \ln s \left[-4v + Z_{BES}(v) \right] - 2\tilde{P}(v) + i \sum_{j=1}^{K_b} \ln \frac{v - u_{b,j} + \frac{i}{2}}{v - u_{b,j} - \frac{i}{2}} + \sum_{h=1}^{H} \Theta(v, u_h)$$

$$+ \sum_{j=1}^{N_F} i \ln S^{(sF)}(v, u_{F,j}) + \sum_{j=1}^{N_{\bar{F}}} i \ln S^{(s\bar{F})}(v, u_{\bar{F},j}) + \sum_{j=1}^{N_f} i \ln S^{(sf)}(v, u_{f,j})$$

$$+ \sum_{j=1}^{N_{\bar{f}}} i \ln S^{(s\bar{f})}(v, u_{\bar{f},j}) + \sum_{j=1}^{N_g} i \ln S^{(sg)}(v, u_j^g) + \sum_{j=1}^{N_{\bar{g}}} i \ln S^{(s\bar{g})}(v, u_j^{\bar{g}}), \tag{2.71}$$

where we introduced the scalar–scalar phase

$$\Theta(v, u) = \Theta'(v, u) + \tilde{P}(u) = i \ln[-S^{(ss)}(v, u)] \tag{2.72}$$

and the scattering factors $S^{(s*)}(v, u_j^*)$ between scalars and other excitations, which are listed in Appendix C. Imposing the quantisation condition $e^{-i[Z_4(u_h) - P]} = (-1)^{H-1}$, we get the final

Bethe equation for scalars

$$1 = e^{iRP^{(s)}(u_h) + 2iD^{(s)}(u_h)} \prod_{j=1}^{K_b} \frac{u_h - u_{b,j} + \frac{i}{2}}{u_h - u_{b,j} - \frac{i}{2}} \prod_{\substack{h'=1 \\ h' \neq h}}^{H} S^{(ss)}(u_h, u_{h'}) \prod_{j=1}^{N_g} S^{(sg)}(u_h, u_j^g)$$

$$\cdot \prod_{j=1}^{N_{\bar{g}}} S^{(s\bar{g})}(u_h, u_j^{\bar{g}}) \prod_{j=1}^{N_F} S^{(sF)}(u_h, u_{F,j})$$

$$\cdot \prod_{j=1}^{N_{\bar{F}}} S^{(s\bar{F})}(u_h, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(sf)}(u_h, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(s\bar{f})}(u_h, u_{\bar{f},j}), \tag{2.73}$$

where we introduced the length of the chain $R = 2 \ln s$, the momentum of a scalar excitation with rapidity u

$$P^{(s)}(u) = 2u - \frac{1}{2} Z_{BES}(u) \tag{2.74}$$

and the effect of the two purely transmitting defects

$$2D^{(s)}(u) = 2\tilde{P}(u). \tag{2.75}$$

Important properties of the scalar–scalar phase (2.72), which can be proven using equations (2.62), (2.67) are

$$\Theta(u, v) = -\Theta(-u, -v), \quad \Theta(u, v) = -\Theta(v, u). \tag{2.76}$$

Eventually, we remember an efficient way proposed in [23] to compute the scalar–scalar phase. We found that

$$\Theta(u, v) = M(u, v) - M(v, u), \tag{2.77}$$

where $M(u, v) = Z^{(1)}(u) + Z(u; v)$ and $Z^{(1)}(u), Z(u; v)$ are univocally defined by the conditions

$$\frac{d}{du} Z^{(1)}(u) = \sigma^{(1)}(u), \quad \frac{d}{du} Z(u; v) = \sigma(u; v),$$

$$Z^{(1)}(u) = -Z^{(1)}(-u), \quad Z(u; v) = -Z(-u; v), \tag{2.78}$$

with the functions $\sigma^{(1)}(u), \sigma(u; v)$ solutions of equations (B.22), (B.23), respectively.

This procedure provides an alternative (with respect to solving equation (2.67)) way to determine the function $\tilde{P}(u)$, once $Z_{BES}(u), Z^{(1)}(u)$ and $Z(u; v)$ are known. Indeed using (2.66) we have

$$2\tilde{P}(v) = \lim_{s \to +\infty} \left[\ln \frac{s}{\sqrt{2}} [-4v + Z_{BES}(v)] - \Theta(v, s/\sqrt{2}) - \Theta(v, -s/\sqrt{2}) \right]$$

$$= \lim_{s \to +\infty} \left[\ln \frac{s}{\sqrt{2}} [-4v + Z_{BES}(v)] - 2Z^{(1)}(v) - Z(v; s/\sqrt{2}) - Z(v; -s/\sqrt{2}) \right]. \tag{2.79}$$

A final alternative to compute $\tilde{P}(u)$ is to look at equation (2.69) when no excitations nor isotopic roots are present. Then we see that $-2\tilde{P}(u)$ represent the contribution $O(\ln s^0)$ to the twist

two counting function of the pure $sl(2)$ sector. This function has been analysed in [13,14] (in notations of the second of [14] it is connected to the function S^{extra}).

Fermions   The equations for large fermions come from the (inverse) of the third of the Beisert–Staudacher equations. We have

$$
1 = \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + i/2}{u_{F,k} - u_{a,j} - i/2} \prod_{j=1}^{N_g} \frac{u_{F,k} - u_j^g + i/2}{u_{F,k} - u_j^g - i/2} (-1)^{K_4} \prod_{j=1}^{K_4} \frac{x_{4,j}^+ - x_{F,k}}{x_{F,k} - x_{4,j}^-}
$$

$$
= \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + i/2}{u_{F,k} - u_{a,j} - i/2} \prod_{j=1}^{N_g} \frac{u_{F,k} - u_j^g + i/2}{u_{F,k} - u_j^g - i/2} \prod_{j=1}^{K_b} \frac{x_{b,j}^{++} - x_{F,k}}{x_{F,k} - x_{b,j}^{--}} \prod_{h=1}^{H} \frac{x_{F,k} - x_h^-}{x_h^+ - x_{F,k}}
$$

$$
\cdot \exp\left[i \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \chi_F(v, u_{F,k}) \frac{d}{dv} \Big(Z_4(v) - 2L_4(v) \Big) \right], \tag{2.80}
$$

since $K_4 = s + 2K_b$ is even, where $x_h^\pm = x\left(u_h \pm \frac{i}{2}\right)$. As in the one loop case, we multiply such expression by $1 = e^{iP}$. Then we use expression (2.69) for $Z_4(v)$ and remember that the term containing $L_4(v)$ gives subleading $O(1/s^2)$ contributions. We get

$$
1 = e^{iRP^{(F)}(u_{F,k}) + 2iD^{(F)}(u_{F,k})} \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + i/2}{u_{F,k} - u_{a,j} - i/2} \prod_{h=1}^{H} S^{(Fs)}(u_{F,k}, u_h)
$$

$$
\cdot \prod_{j=1}^{N_F} S^{(FF)}(u_{F,k}, u_{F,j}) \prod_{j=1}^{N_{\bar F}} S^{(F\bar F)}(u_{F,k}, u_{\bar F,j}) \prod_{j=1}^{N_f} S^{(Ff)}(u_{F,k}, u_{f,j})
$$

$$
\cdot \prod_{j=1}^{N_{\bar f}} S^{(F\bar f)}(u_{F,k}, u_{\bar f,j}) \prod_{j=1}^{N_g} S^{(Fg)}(u_{F,k}, u_j^g) \prod_{j=1}^{N_{\bar g}} S^{(F\bar g)}(u_{F,k}, u_j^{\bar g}), \tag{2.81}
$$

where $R = 2\ln s$ is the length of the chain and

$$
P^{(F)}(u) = - \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\chi_F(v, u) + \chi_F(-v, u) \right] \left[1 - \frac{\sigma_{BES}(v)}{4} \right], \tag{2.82}
$$

$$
2D^{(F)}(u) = - \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\chi_F(v, u) + \chi_F(-v, u) \right] \frac{d}{dv} \tilde{P}(v) \tag{2.83}
$$

are the momentum of a fermion and the effect on it of the two defects.

   The equations for large antifermions come from the (inverse of the) fifth of the Beisert–Staudacher equations. Their derivation is analogous to the fermionic case:

$$
1 = e^{iRP^{(F)}(u_{\bar F,k}) + 2iD^{(F)}(u_{\bar F,k})} \prod_{j=1}^{K_c} \frac{u_{\bar F,k} - u_{c,j} + i/2}{u_{\bar F,k} - u_{c,j} - i/2} \prod_{h=1}^{H} S^{(\bar Fs)}(u_{\bar F,k}, u_h)
$$

$$\cdot \prod_{j=1}^{N_F} S^{(\bar{F}F)}(u_{\bar{F},k},u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{F}\bar{F})}(u_{\bar{F},k},u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(\bar{F}f)}(u_{\bar{F},k},u_{f,j})$$

$$\cdot \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{F}\bar{f})}(u_{\bar{F},k},u_{\bar{f},j}) \prod_{j=1}^{N_g} S^{(\bar{F}g)}(u_{\bar{F},k},u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(\bar{F}\bar{g})}(u_{\bar{F},k},u_j^{\bar{g}}). \tag{2.84}$$

Equations for small fermions are obtained starting from the (inverse of the) first of the Beisert–Staudacher equations. We have

$$1 = \prod_{j=1}^{K_2} \frac{u_{f,k}-u_{2,j}+i/2}{u_{f,k}-u_{2,j}-i/2} \prod_{j=1}^{K_4} \frac{1-\frac{x_{f,k}}{x_{4,j}^+}}{1-\frac{x_{f,k}}{x_{4,j}^-}} = \prod_{j=1}^{K_a} \frac{u_{f,k}-u_{a,j}+i/2}{u_{f,k}-u_{a,j}-i/2} \prod_{j=1}^{N_g} \frac{u_{f,k}-u_j^g+i/2}{u_{f,k}-u_j^g-i/2}$$

$$\cdot \prod_{j=1}^{K_b} \frac{1-\frac{x_{f,k}}{x_{b,j}^{++}}}{1-\frac{x_{f,k}}{x_{b,j}^{--}}} \prod_{h=1}^{H} \frac{1-\frac{x_{f,k}}{x_h^-}}{1-\frac{x_{f,k}}{x_h^+}} \exp\left[-i\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \chi_H(v,u_{f,k}) \frac{d}{dv}\Big(Z_4(v)-2L_4(v)\Big)\right]. \tag{2.85}$$

In contrast with the fermionic case, we do not multiply this equality by $1=e^{iP}$. We use expression (2.69) for $Z_4(v)$. Working out the various terms we get

$$1 = e^{iRP^{(f)}(u_{f,k})+2iD^{(f)}(u_{f,k})} \prod_{j=1}^{K_a} \frac{u_{f,k}-u_{a,j}+i/2}{u_{f,k}-u_{a,j}-i/2} \prod_{h=1}^{H} S^{(fs)}(u_{f,k},u_h)$$

$$\cdot \prod_{j=1}^{N_F} S^{(fF)}(u_{f,k},u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(f\bar{F})}(u_{f,k},u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(ff)}(u_{f,k},u_{f,j})$$

$$\cdot \prod_{j=1}^{N_{\bar{f}}} S^{(f\bar{f})}(u_{f,k},u_{\bar{f},j}) \prod_{j=1}^{N_g} S^{(fg)}(u_{f,k},u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(f\bar{g})}(u_{f,k},u_j^{\bar{g}}), \tag{2.86}$$

where

$$P^{(f)}(u) = \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\chi_H(v,u)+\chi_H(-v,u)\right]\left[1-\frac{\sigma_{BES}(v)}{4}\right], \tag{2.87}$$

$$2D^{(f)}(u) = \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\chi_H(v,u)+\chi_H(-v,u)\right] \frac{d}{dv}\tilde{P}(v). \tag{2.88}$$

In a completely analogous way we work on the (inverse of the) seventh of the Beisert–Staudacher equations, which gives the quantisation condition for small antifermions:

$$1 = e^{iRP^{(f)}(u_{\bar{f},k})+2iD^{(f)}(u_{\bar{f},k})} \prod_{j=1}^{K_c} \frac{u_{\bar{f},k}-u_{c,j}+i/2}{u_{\bar{f},k}-u_{c,j}-i/2} \prod_{h=1}^{H} S^{(\bar{f}s)}(u_{\bar{f},k},u_h)$$

$$\cdot \prod_{j=1}^{N_F} S^{(\bar{f}F)}(u_{\bar{f},k},u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{f}\bar{F})}(u_{\bar{f},k},u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(\bar{f}f)}(u_{\bar{f},k},u_{f,j})$$

$$\cdot \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{f}\bar{f})}(u_{\bar{f},k},u_{\bar{f},j}) \prod_{j=1}^{N_g} S^{(\bar{f}g)}(u_{\bar{f},k},u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(\bar{f}\bar{g})}(u_{\bar{f},k},u_j^{\bar{g}}). \tag{2.89}$$

Gluons   As in the one loop case, we multiply (the inverse of) the second of the Beisert–Staudacher equations for $u_{2,k} = u_k^g$ with (the inverse of) the third for $u_{3,k} = u_k^g + i/2$ and (the inverse of) the third for $u_{3,k} = u_k^g - i/2$. We then get the following equations for the center of the gluonic string u_k^g:

$$
1 = \prod_{\substack{j=1 \\ j \neq k}}^{N_g} \frac{u_k^g - u_j^g + i}{u_k^g - u_j^g - i} \prod_{j=1}^{K_4} \frac{(x_k^{g+} - x_{4,j}^+)(x_k^{g-} - x_{4,j}^+)}{(x_k^{g+} - x_{4,j}^-)(x_k^{g-} - x_{4,j}^-)} \prod_{j=1}^{N_F} \frac{u_k^g - u_{F,j} + i/2}{u_k^g - u_{F,j} - i/2}
$$
$$
\cdot \prod_{j=1}^{N_f} \frac{u_k^g - u_{f,j} + i/2}{u_k^g - u_{f,j} - i/2} \prod_{j=1}^{K_b} \frac{u_k^g - u_{b,j} + i/2}{u_k^g - u_{b,j} - i/2}. \tag{2.90}
$$

Making explicit the type-4 roots, we arrive at

$$
1 = \prod_{\substack{j=1 \\ j \neq k}}^{N_g} \frac{u_k^g - u_j^g + i}{u_k^g - u_j^g - i} \prod_{j=1}^{N_F} \frac{u_k^g - u_{F,j} + i/2}{u_k^g - u_{F,j} - i/2} \prod_{j=1}^{N_f} \frac{u_k^g - u_{f,j} + i/2}{u_k^g - u_{f,j} - i/2} \prod_{j=1}^{K_b} \frac{u_k^g - u_{b,j} + i/2}{u_k^g - u_{b,j} - i/2} \cdot
$$
$$
\cdot \prod_{h=1}^{H} \frac{(x_k^{g-} - x_h^-)(x_k^{g+} - x_h^-)}{(x_k^{g+} - x_h^+)(x_h^+ - x_k^{g-})} \prod_{j=1}^{K_b} (-1) \frac{(x_k^{g+} - x_{b,j}^{++})(x_{b,j}^{++} - x_k^{g-})}{(x_k^{g-} - x_{b,j}^{--})(x_k^{g+} - x_{b,j}^{--})}
$$
$$
\cdot \exp\left[i \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \chi(v, u_k^g | 1) \frac{d}{dv} \left(Z_4(v) - 2L_4(v) \right) \right]. \tag{2.91}
$$

Following what we did for (large) fermions, we multiply such expression by $1 = e^{iP}$. Then we use expression (2.69) for $Z_4(v)$. We observe the exact cancellation of terms involving the isotopic root u_b and eventually for the field $F_{+\perp}$ we obtain the equations

$$
1 = e^{iRP^{(g)}(u_k^g) + 2iD^{(g)}(u_k^g)} \prod_{j=1, j \neq k}^{N_g} S^{(gg)}(u_k^g, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(g\bar{g})}(u_k^g, u_j^{\bar{g}}) \prod_{h=1}^{H} S^{(gs)}(u_k^g, u_h)
$$
$$
\cdot \prod_{j=1}^{N_F} S^{(gF)}(u_k^g, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(g\bar{F})}(u_k^g, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(gf)}(u_k^g, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(g\bar{f})}(u_k^g, u_{\bar{f},j}), \tag{2.92}
$$

where

$$
P^{(g)}(u) = - \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\chi(v, u|1) + \chi(-v, u|1) \right] \left[1 - \frac{\sigma_{BES}(v)}{4} \right], \tag{2.93}
$$

$$
2D^{(g)}(u) = - \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\chi(v, u|1) + \chi(-v, u|1) \right] \frac{d}{dv} \tilde{P}(v). \tag{2.94}
$$

The procedure for the field $\bar{F}_{+\perp}$ is completely analogous, hence we give only the final equations

$$1 = e^{i R P^{(g)}(u_k^{\bar{g}}) + 2i D^{(g)}(u_k^{\bar{g}})} \prod_{j=1}^{N_g} S^{(\bar{g}g)}(u_k^{\bar{g}}, u_j^g) \prod_{j=1, j \neq k}^{N_{\bar{g}}} S^{(\bar{g}\bar{g})}(u_k^{\bar{g}}, u_j^{\bar{g}}) \prod_{h=1}^{H} S^{(\bar{g}s)}(u_k^{\bar{g}}, u_h)$$

$$\cdot \prod_{j=1}^{N_F} S^{(\bar{g}F)}(u_k^{\bar{g}}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{g}\bar{F})}(u_k^{\bar{g}}, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(\bar{g}f)}(u_k^{\bar{g}}, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{g}\bar{f})}(u_k^{\bar{g}}, u_{\bar{f},j}) .$$

(2.95)

Isotopic roots The equations for the isotopic roots u_a and u_c come directly from the second and the sixth of the Beisert–Staudacher equations and their derivation is completely analogous to the one loop case: the only difference is that in the general all loops case also small fermions are present.

$$1 = \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{N_F} \frac{u_{a,k} - u_{F,j} - i/2}{u_{a,k} - u_{F,j} + i/2} \prod_{j=1}^{N_f} \frac{u_{a,k} - u_{f,j} - i/2}{u_{a,k} - u_{f,j} + i/2}$$

$$\cdot \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - i/2}{u_{a,k} - u_{b,j} + i/2}$$

(2.96)

$$1 = \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{N_{\bar{F}}} \frac{u_{c,k} - u_{\bar{F},j} - i/2}{u_{c,k} - u_{\bar{F},j} + i/2} \prod_{j=1}^{n_{\bar{f}}} \frac{u_{c,k} - u_{\bar{f},j} - i/2}{u_{c,k} - u_{\bar{f},j} + i/2}$$

$$\cdot \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - i/2}{u_{c,k} - u_{b,j} + i/2}$$

(2.97)

Then, we consider the product of the third equation for $u_{3,k} = u_{b,k}$ with the fifth for $u_{5,k} = u_{b,k}$, the fourth for $u_{4,k} = u_{b,k} + i/2$ and the fourth for $u_{4,k} = u_{b,k} - i/2$. We arrive at the following equation

$$1 = \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - i/2}{u_{b,k} - u_{a,j} + i/2} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - i/2}{u_{b,k} - u_{c,j} + i/2} \prod_{j=1}^{K_4} \frac{u_{b,k} - u_{4,j} + i/2}{u_{b,k} - u_{4,j} - i/2}$$

$$\cdot \prod_{j=1}^{N_g} \frac{u_{b,k} - u_j^g - i/2}{u_{b,k} - u_j^g + i/2} \prod_{j=1}^{N_{\bar{g}}} \frac{u_{b,k} - u_j^{\bar{g}} - i/2}{u_{b,k} - u_j^{\bar{g}} + i/2} \left(\frac{x_{b,k}^{--}}{x_{b,k}^{++}} \right)^L$$

$$\cdot \prod_{j \neq k}^{K_4} \frac{x_{b,k}^{--} - x_{4,j}^+}{x_{b,k}^{++} - x_{4,j}^-} \frac{1 - \frac{g^2}{2 x_{b,k}^{++} x_{4,j}^-}}{1 - \frac{g^2}{2 x_{b,k}^{--} x_{4,j}^+}} \sigma^2(u_{b,k} + i/2, u_{4,j}) \sigma^2(u_{b,k} - i/2, u_{4,j})$$

$$\cdot \prod_{j=1}^{K_b} \left(\frac{x_{b,k}^{++} - x_{b,j}}{x_{b,k}^{--} - x_{b,j}} \right)^2 \prod_{j=1}^{N_F} \frac{x_{b,k}^{++} - x_{F,j}}{x_{b,k}^{--} - x_{F,j}} \prod_{j=1}^{N_{\bar{F}}} \frac{x_{b,k}^{++} - x_{\bar{F},j}}{x_{b,k}^{--} - x_{\bar{F},j}} \prod_{j=1}^{N_f} \frac{1 - \frac{x_{f,j}}{x_{b,k}^{++}}}{1 - \frac{x_{f,j}}{x_{b,k}^{--}}} \prod_{j=1}^{N_{\bar{f}}} \frac{1 - \frac{x_{\bar{f},j}}{x_{b,k}^{++}}}{1 - \frac{x_{\bar{f},j}}{x_{b,k}^{--}}}$$

$$\cdot \prod_{j=1}^{N_g} \frac{x_{b,k}^{++} - x_j^{g+}}{x_{b,k}^{--} - x_j^{g+}} \frac{x_{b,k}^{++} - x_j^{g-}}{x_{b,k}^{--} - x_j^{g-}} \prod_{j=1}^{N_{\bar{g}}} \frac{x_{b,k}^{++} - x_j^{\bar{g}+}}{x_{b,k}^{--} - x_j^{\bar{g}+}} \frac{x_{b,k}^{++} - x_j^{\bar{g}-}}{x_{b,k}^{--} - x_j^{\bar{g}-}},$$

(2.98)

where

$$L = H + 2 + N_F + N_{\bar{F}} + N_g + N_{\bar{g}}. \tag{2.99}$$

We have

$$\prod_{j=1}^{K_4} \frac{u_{b,k} - u_{4,j} + i/2}{u_{b,k} - u_{4,j} - i/2} = \prod_{\substack{j=1 \\ j \neq k}}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{h=1}^{H} \frac{u_{b,k} - u_h - i/2}{u_{b,k} - u_h + i/2} \left(1 + O(1/s^2)\right)$$

$$\cdot \exp\left[- \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{u_{b,k} - v + i/2}{u_{b,k} - v - i/2} \frac{d}{dv} (Z_4(v) - 2L_4(v)) \right], \tag{2.100}$$

where for $Z_4(v)$ it is convenient to use form (2.54). It is remarkable that, plugging (2.54) into the integral in the last term of (2.100), we find that

$$\exp\left[- \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{u_{b,k} - v + i/2}{u_{b,k} - v - i/2} \frac{d}{dv} Z_4(v) \right] \tag{2.101}$$

produces[10] massive cancellations in (2.98). On the other hand, the nonlinear term containing $L_4'(v)$ gives a negligible $O(1/s^2)$ contribution. Eventually, for the third isotopic root u_b we obtain the same equation as in the one loop:

$$1 = \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - i/2}{u_{b,k} - u_{a,j} + i/2} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - i/2}{u_{b,k} - u_{c,j} + i/2} \prod_{\substack{j=1 \\ j \neq k}}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i}$$

$$\cdot \prod_{h=1}^{H} \frac{u_{b,k} - u_h - i/2}{u_{b,k} - u_h + i/2}. \tag{2.102}$$

3. Conserved observables

Momentum was already obtained in previous sections: therefore, we concentrate on higher charges Q_r and in particular on anomalous dimensions $\gamma = Q_2$. Let us introduce the function

$$q_r(u) = \frac{ig^2}{r-1} \left[\left(\frac{1}{x^+(u)} \right)^{r-1} - \left(\frac{1}{x^-(u)} \right)^{r-1} \right], \quad r \geq 2, \tag{3.1}$$

whose Fourier transform reads

$$\hat{q}_r(k) = 2\pi i g^2 \left(\frac{\sqrt{2}}{ig} \right)^{r-1} e^{-\frac{|k|}{2}} \frac{J_{r-1}(\sqrt{2}gk)}{k}. \tag{3.2}$$

The r-th charge of an excited state over the GKP vacuum enjoys the expression

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[10] This cancellation was already noticed and proven by Basso in Appendix C.2 of [20].

$$Q_r = \sum_{j=1}^{K_4} q_r(u_{4,j}) = \sum_{j=1}^{K_b} [q_r(u_{b,j} + i/2) + q_r(u_{b,j} - i/2)] - \sum_{h=1}^{H} q_r(u_h)$$

$$- \int_{-\infty}^{+\infty} \frac{dv}{2\pi} q_r(v) \frac{d}{dv} [Z_4(v) - 2L_4(v)] + O\left(\frac{1}{s^2}\right), \tag{3.3}$$

where for $Z_4(v)$ we use expression (2.69). Doing this, we observe the exact cancellation of the dependence on the isotopic root u_b and we are left with the formula

$$Q_r = -\sum_{h=1}^{H} q_r(u_h) + \ln s \int \frac{dv}{2\pi} q_r(v) \frac{d}{dv} [4v - Z_{BES}(v)] + \int \frac{dv}{2\pi} q_r(v) \frac{d}{dv} 2\tilde{P}(v)$$

$$- \int \frac{dv}{2\pi} q_r(v) \frac{d}{dv} \Big[\sum_{h=1}^{H} \Theta'(v, u_h) + \sum_{j=1}^{N_g} F_G(v, u_j^g) + \sum_{j=1}^{N_{\bar{g}}} F_G(v, u_j^{\bar{g}})$$

$$+ \sum_{j=1}^{N_F} F_F(v, u_{F,j}) + \sum_{j=1}^{N_{\bar{F}}} F_F(v, u_{\bar{F},j}) + \sum_{j=1}^{N_f} F_f(v, u_{f,j}) + \sum_{j=1}^{N_{\bar{f}}} F_f(v, u_{\bar{f},j})$$

$$+ 2 \int dw\, G(v, w) L_4(w) - 2L_4(v) \Big]$$

$$= \ln s \int \frac{dv}{2\pi} q_r(v) \frac{d}{dv} [4v - Z_{BES}(v)] + \int \frac{dv}{2\pi} q_r(v) \frac{d}{dv} 2\tilde{P}(v)$$

$$+ \sum_{h=1}^{H} Q_r^{(s)}(u_h) + \sum_{j=1}^{N_g} Q_r^{(g)}(u_j^g) + \sum_{j=1}^{N_{\bar{g}}} Q_r^{(g)}(u_j^{\bar{g}}) + \sum_{j=1}^{N_F} Q_r^{(F)}(u_{F,j})$$

$$+ \sum_{j=1}^{N_{\bar{F}}} Q_r^{(F)}(u_{\bar{F},j}) + \sum_{j=1}^{N_f} Q_r^{(f)}(u_{f,j}) + \sum_{j=1}^{N_{\bar{f}}} Q_r^{(f)}(u_{\bar{f},j}) + O(1/s^2). \tag{3.4}$$

The first two terms in the right hand side of (3.4) are contributions from the GKP background. The remaining terms in (3.4) are the contributions that any single particle brings to the overall value of the r-th charge.

• For scalars we have

$$Q_r^{(s)}(u) = -q_r(u) - \int \frac{dv}{2\pi} q_r(v) \frac{d}{dv} \Theta'(v, u). \tag{3.5}$$

Restricting to r even, we use relation (2.21) of [23] to write

$$Q_r^{(s)}(u) = -q_r(u) - \int_{-\infty}^{+\infty} \frac{dk}{4\pi^2} \hat{q}_r(k) [\hat{\sigma}^{(1)}(k) + \hat{\sigma}(k; u)], \tag{3.6}$$

where the functions $\hat{\sigma}^{(1)}(k)$, $\hat{\sigma}(k; u)$ satisfy equations (2.19), (2.20) of [23], respectively. It is convenient to introduce the functions, defined for $k > 0$,

$$S^{(1)}(k) = \frac{\sinh \frac{k}{2}}{\pi k} \left[\hat{\sigma}^{(1)}(k) + \frac{\pi}{\sinh \frac{k}{2}} \left(1 - e^{-\frac{k}{2}} \right) \right], \tag{3.7}$$

$$S(k; u) = \frac{\sinh \frac{k}{2}}{\pi k} \left[\hat{\sigma}(k; u) - \frac{2\pi e^{-k}}{1 - e^{-k}} (\cos ku - 1) \right], \tag{3.8}$$

and to expand them in Neumann series

$$S^{(1)}(k) = \sum_{p=1}^{+\infty} S_p^{(1)} \frac{J_p(\sqrt{2}gk)}{k}, \quad S(k; u) = \sum_{p=1}^{+\infty} S_p'(u) \frac{J_p(\sqrt{2}gk)}{k}. \tag{3.9}$$

We use defining equations (3.10) of [47] for $S_p^{(1)}$ and (3.15) of [21] for $S_p'(u)$ to simplify (3.6) as follows

$$Q_r^{(s)}(u) = \frac{ig^2}{r-1} \left(\frac{\sqrt{2}}{ig} \right)^{r-1} \left[S_{r-1}^{(1)} + S_{r-1}'(u) \right]. \tag{3.10}$$

We could not find a formula analogous to (3.10) in the case r odd.

When $r = 2$, this simple expression can be connected with the first of (4.6) of [20]. We indeed remember formula (4.36) of [48] and that

$$S_1'(u) = -2\pi \sum_{n=1}^{+\infty} \frac{(-1)^n u^{2n}}{(2n)!} \tilde{S}_1^{(n)}, \tag{3.11}$$

where for $\tilde{S}_1^{(n)}$ we use formula (4.35) of [48] to connect with the solution of the BES equation. Operating in this way we get, after some algebra,

$$Q_2^{(s)}(u) = \gamma^{(s)}(u) = \int_0^{+\infty} \frac{dt}{t} \left[\frac{e^{-\frac{t}{2}} - \cos tu}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \gamma_-^{\emptyset}(\sqrt{2}gt) + \frac{\cos tu - e^{\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \gamma_+^{\emptyset}(\sqrt{2}gt) \right], \tag{3.12}$$

i.e. the first of (4.6) of [20]. Functions $\gamma_\pm^{\emptyset}$ are defined in (B.19), (B.20).

It can be of interest to express $\gamma^{(s)}(u)$ in the $O(6)$ limit [49]. We introduce

$$m(g) = \frac{2^{\frac{5}{8}} \pi^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)} g^{\frac{1}{4}} e^{-\frac{\pi g}{\sqrt{2}}} \left[1 + O\left(\frac{1}{g}\right) \right]. \tag{3.13}$$

In the $O(6)$ limit $\sqrt{2}g S_1^{(1)} = m(g) - 1$ and $\sqrt{2}g S_1'(u) = m(g) \left(\cosh \frac{\pi}{2} u - 1 \right)$. Substituting in (3.10) we get

$$\gamma^{(s)}(u) = m(g) \cosh \frac{\pi}{2} u - 1, \tag{3.14}$$

and for the complete anomalous dimension in presence only of scalar excitations

$$\gamma = \ln s f(g) + f_{sl}(g) + \sum_{h=1}^{H} \left(m(g) \cosh \frac{\pi}{2} u_h - 1 \right). \tag{3.15}$$

• For gluons we have

$$Q_r^{(g)}(u) = - \int \frac{dv}{2\pi} q_r(v) \frac{d}{dv} F^G(v, u). \tag{3.16}$$

Using equations (2.62), (2.63), we arrive at the formula

$$Q_r^{(g)}(u) = - \int \frac{dv}{2\pi} \left[\chi(v, u|1) + \Phi(v) \right] \frac{d}{dv} Q_r^{(s)}(v). \tag{3.17}$$

When $r = 2$ we have

$$Q_2^{(g)}(u) = \gamma^{(g)}(u) = \int\limits_{-\infty}^{\infty} \frac{dk}{4\pi^2} \frac{i\pi}{e^{\frac{k}{2}} - e^{-\frac{k}{2}}} \left(\gamma_+^\phi(\sqrt{2}gk) - \text{sgn}(k)\gamma_-^\phi(\sqrt{2}gk) \right)$$

$$\cdot \left[\frac{2\pi}{ik} e^{-|k|} e^{-iku} - \frac{2\pi}{ik} e^{-\frac{|k|}{2}} \sum_{n=1}^{\infty} \left(\left(\frac{g}{\sqrt{2}ix(u + \frac{i}{2})} \right)^n + \left(\frac{g}{\sqrt{2}ix(u - \frac{i}{2})} \right)^n \right) \right.$$

$$\left. \cdot J_n(\sqrt{2}gk) - \frac{2\pi}{ik} J_0(\sqrt{2}gk) e^{-\frac{|k|}{2}} \right]. \tag{3.18}$$

By means of the relation (3.40) in [20], and making use of the identity

$$\int\limits_0^{\infty} \frac{dk}{k} e^{-k\left(\frac{1}{2} \pm iu\right)} J_n(\sqrt{2}gk) = \frac{(\pm 1)^n}{n} \left(\frac{g}{\sqrt{2}ix(u \mp \frac{i}{2})} \right)^n, \tag{3.19}$$

the expression above becomes

$$\gamma^{(g)}(u) = \int\limits_0^{\infty} \frac{dk}{k} \frac{\gamma_+^\phi(\sqrt{2}gk)}{1 - e^{-k}} [\cos ku \, e^{-\frac{k}{2}} - 1] - \int\limits_0^{\infty} \frac{dk}{k} \frac{\gamma_-^\phi(\sqrt{2}gk)}{e^k - 1} [\cos ku \, e^{-\frac{k}{2}} - 1]. \tag{3.20}$$

• Large fermions and antifermions with rapidity u carry an amount of r-charge equal to

$$Q_r^{(F)}(u) = -\int \frac{dv}{2\pi} q_r(v) \frac{d}{dv} F_F(v, u). \tag{3.21}$$

Using equations (2.62), (2.63), we arrive at the formula

$$Q_r^{(F)}(u) = -\int \frac{dv}{2\pi} \left[\chi_F(v, u) + \Phi(v) \right] \frac{d}{dv} Q_r^{(s)}(v). \tag{3.22}$$

For small fermions we have

$$Q_r^{(f)}(u) = \int \frac{dv}{2\pi} \chi_H(v, u) \frac{d}{dv} Q_r^{(s)}(v). \tag{3.23}$$

When $r = 2$, the very same reasonings outlined above apply to large fermions, so that $(Q_2^{(F)}(u) = \gamma^{(F)}(u))$

$$\gamma^{(F)}(u) = \int\limits_0^{\infty} \frac{dk}{k} \frac{\gamma_+^\phi(\sqrt{2}gk) - \gamma_-^\phi(\sqrt{2}gk)}{e^k - 1} [\cos ku - 1]$$

$$+ \int\limits_0^{\infty} \frac{dk}{k} \gamma_+^\phi(\sqrt{2}gk) [\frac{1}{2} \cos ku - 1]. \tag{3.24}$$

Analogously for $r = 2$ and small fermions

$$Q_2^{(f)}(u) = \gamma^{(f)}(u) = -\frac{1}{2} \int\limits_0^{\infty} \frac{dk}{k} \gamma_+^\phi(\sqrt{2}gk) \cos ku. \tag{3.25}$$

3.1. One loop

All the previous expression are explicitly computed at one loop, upon introducing the following notation for the derivatives of the digamma function

$$\psi^{(n)}(z) \equiv \left(\frac{d}{dz}\right)^n \psi(z), \tag{3.26}$$

$$\psi^{(0)}(z) \equiv \psi(z). \tag{3.27}$$

- Scalars

$$Q_r^{(s)}(u) = -\frac{i^r g^2}{(r-1)!} \Big[\psi^{(r-2)}(1/2 - iu)$$
$$+ (-1)^r \psi^{(r-2)}(1/2 + iu) - \psi^{(r-2)}(1)(1 + (-1)^r) \Big]; \tag{3.28}$$

- Fermions and antifermions

$$Q_r^{(F)}(u) = -\frac{i^r g^2}{(r-1)!} \Big[\psi^{(r-2)}(1 - iu)$$
$$+ (-1)^r \psi^{(r-2)}(1 + iu) - \psi^{(r-2)}(1)(1 + (-1)^r) \Big]; \tag{3.29}$$

- Gluons

$$Q_r^{(g)}(u) = -\frac{i^r g^2}{(r-1)!} \Big[\psi^{(r-2)}(3/2 - iu)$$
$$+ (-1)^r \psi^{(r-2)}(3/2 + iu) - \psi^{(r-2)}(1)(1 + (-1)^r) \Big]. \tag{3.30}$$

4. Strong coupling regimes of 2D scattering factors

In this section we want to give a detailed analysis of the different strong coupling limits of the 2D scattering factors $S^{**}(u, v)$ of Section 2.3. In fact there are different ways of performing the $g \rightarrow +\infty$ limit as these give rise to different results or regimes, so parallelling what happens to the energy/momentum dispersion relations [20].

First, we shall discuss the regime, relevant only for scalars (as the other excitations decouples towards very high energy), where we keep their rapidities fixed, namely the so-called non-perturbative regime. In this case integrations inside the expressions for the various scattering factors receive the leading contributions from the region where the integration variables are fixed (while sending $g \rightarrow +\infty$). This regime is dominated by scalars which are the only ones to have a non-trivial (finite) S-factor, whilst the other S-factors involving other excitations reduce to one. Here we find out the (usual) $O(6)$ non-linear sigma model scattering theory as low energy string theory [49,47,50].[11] Alternatively, we can first rescale the external rapidities $u = \sqrt{2}g\bar{u}$, $v = \sqrt{2}g\bar{v}$ and then send $g \rightarrow +\infty$. If the rescaled variables, $\bar{u}$ and $\bar{v}$, have modulus smaller than one we are (with the exception of scalars, see discussion below) in the perturbative string regime (where the irrelevant and relevant perturbations of the $O(6)$ non-linear sigma prevail on

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[11] At next approximation it would be perturbed by irrelevant fields as suggested by the expansion of the energy in inverse powers of the size (R) [21] (*cf.* also the dispersion relation in [20] and the effective field theory of [51]).

it putting at zero its mass); while if their modulus is greater than one we are in the so-called giant hole (semiclassical soliton) regime. In both cases, in order to have the maximum contribution to the integrals, after rescaling external rapidities, we have to perform the same rescaling of the integration variables $u_i = \sqrt{2}g\bar{u}_i$ and eventually take the limit $g \to +\infty$.

4.1. Scalars

Scalars in the non-perturbative regime   We report the strong coupling limit of the scalar–scalar scattering factor in the non-perturbative regime, *i.e.* $g \to +\infty$, with u, v fixed (details on the calculation can be found in [23], see also [21] and [22]):

$$g \to +\infty \quad \Rightarrow \quad \Theta(u, v) \to \Theta_{np}(u - v)$$

$$= -i \ln \frac{\Gamma\left(1 - i\frac{u-v}{4}\right)\Gamma\left(\frac{1}{2} + i\frac{u-v}{4}\right)}{\Gamma\left(1 + i\frac{u-v}{4}\right)\Gamma\left(\frac{1}{2} - i\frac{u-v}{4}\right)} - \mathrm{gd}\left(\frac{\pi(u-v)}{2}\right), \tag{4.1}$$

which depends only on the difference of the rapidities and coincide with the pre-factor of the S-matrix, as derived in [52], of the $O(6)$ non-linear sigma model upon the identification (of the hyperbolic rapidities) $\theta = \pi u/2$ and $\theta' = \pi v/2$. This definitely supports the proposal of the latter model by [49] as that describing the string at low energy (see also subsequent studies [47,50]).

Scalars in the perturbative regime   As in [20] the perturbative regime for scalars is recovered by introducing a new rapidity z as

$$u = \frac{2}{\pi} \ln \frac{z}{m(g)}, \quad \text{for } u > 0 \text{ (right mover)};$$

$$u = \frac{2}{\pi} \ln \frac{m(g)}{z}, \quad \text{for } u < 0 \text{ (left mover)}, \tag{4.2}$$

$m(g)$ being the non-perturbative mass (3.13). The rapidity z is kept fixed in the region $m(g) < z < 1$ as $g \to +\infty$. If $m(g) < z < 1$ formula (4.1) is valid, therefore the function Θ in the perturbative regime is obtained by plugging (4.2) into (4.1).

Scalars in the scaling regimes   We rescale the rapidities $u = \sqrt{2}g\bar{u}$, $v = \sqrt{2}g\bar{v}$ and then send $g \to +\infty$, with $\bar{u}, \bar{v}$ fixed and $|\bar{u}| > 1$, $|\bar{v}| > 1$. This is the so-called giant hole regime. Details on the calculation can be found in [23]. We give the final result for the double derivative, which will be useful for next computations

$$\frac{d}{d\bar{u}}\frac{d}{d\bar{v}}\Theta(\sqrt{2}g\bar{u}, \sqrt{2}g\bar{v}) = \sqrt{2}g\frac{\left(\frac{\bar{u}+1}{\bar{u}-1}\right)^{1/4}\left(\frac{\bar{v}-1}{\bar{v}+1}\right)^{1/4} + \left(\frac{\bar{u}-1}{\bar{u}+1}\right)^{1/4}\left(\frac{\bar{v}+1}{\bar{v}-1}\right)^{1/4}}{\bar{u}-\bar{v}} + O(g^0). \tag{4.3}$$

This result (4.3) agrees with corresponding formula coming from using the scattering phase (2.34) of [53].

Another possibility is to define rescaled rapidities (with a bar) $u = \sqrt{2}g\bar{u}$, $v = \sqrt{2}g\bar{v}$ and then send $g \to +\infty$, with $\bar{u}, \bar{v}$ fixed and $|\bar{u}| < 1$, $|\bar{v}| < 1$. Although for the other particles this second possibility gives rise to the perturbative string regime (giving for understood an obvious modification $u \to x(u)$ for the (small) fermion, *cf.* below), it does not in the case of scalars as

given in [20], because of their non-perturbative, dynamically generated mass. Yet, we need to consider the scalar Θ in this regime at least to access the other S-matrix elements (depending on it). In fact, we may write the limiting value

$$\frac{d^2}{d\bar{u}d\bar{v}}\Theta(u, v) = 2\pi \frac{d}{d\bar{u}}\delta(\bar{u} - \bar{v}) + \frac{1}{\sqrt{2}g}\frac{d^2}{d\bar{v}^2}P\frac{1}{\bar{v} - \bar{u}} + O(1/g^2). \tag{4.4}$$

Importantly, formula (4.4) is valid also in the domains $|\bar{u}| < 1$, $|\bar{v}| > 1$ and $|\bar{u}| > 1$, $|\bar{v}| < 1$. We will make frequent use of (4.4) in this section.

From (4.4) we can infer

$$\Theta(u, v) = -\pi \operatorname{sgn}(u - v) - \frac{1}{\sqrt{2}g}\frac{1}{\bar{v} - \bar{u}} + O(1/g^2) \Rightarrow S^{(ss)}(u, v) = e^{\frac{i}{\sqrt{2}g}\frac{1}{\bar{v}-\bar{u}}+O(1/g^2)} \tag{4.5}$$

4.2. Gluons

Gluons in the perturbative regime   We want to study the gluon-gluon scattering factor (C.13) in the limit $g \to +\infty$, with $u = \bar{u}\sqrt{2}g$, $v = \bar{v}\sqrt{2}g$, $\bar{u}$, $\bar{v}$ fixed and $\bar{u}^2 < 1$, $\bar{v}^2 < 1$ (perturbative regime).

We have

$$i \ln\left(-S^{(gg)}(u, v)\right) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{4.6}$$

where

$$\mathcal{I}_1 = \tilde{\chi}(u, v|1, 1) = -2 \arctan \sqrt{2}g(\bar{v} - \bar{u})$$

$$= -\pi \operatorname{sgn}(\bar{v} - \bar{u}) + \frac{\sqrt{2}}{g(\bar{v} - \bar{u})} + O(1/g^3). \tag{4.7}$$

Passing to study $\mathcal{I}_2$ and $\mathcal{I}_3$, we first remark that in the perturbative regime

$$x^{\mp}(u) = \left[\frac{g}{\sqrt{2}} + \frac{1}{4\sqrt{1 - \bar{u}^2}}\right](\bar{u} \mp i\sqrt{1 - \bar{u}^2}) + O(1/g). \tag{4.8}$$

Since we have to work out $\chi(w, u|1) + \Phi(w)$, in addition to (4.8) we need also to know the behaviour of $x^{\pm}(w)$ when $w = \bar{w}\sqrt{2}g$ and $g \to +\infty$. When $|\bar{w}| < 1$ we can use (4.8). On the other hand, for $|\bar{w}| > 1$ we have

$$x^{\pm}(w) = \sqrt{2}g\bar{x}(\bar{w}) \pm \frac{i}{4}\frac{1 + \sqrt{1 - \frac{1}{\bar{w}^2}}}{\sqrt{1 - \frac{1}{\bar{w}^2}}} + O(1/g), \quad \bar{x}(\bar{w}) = \frac{\bar{w}}{2}\left[1 + \sqrt{1 - \frac{1}{\bar{w}^2}}\right]. \tag{4.9}$$

Using results (4.8), (4.9), we arrive at the relations, valid for $w = \bar{w}\sqrt{2}g$ and $g \to +\infty$:

$$\chi(w, u|1) + \Phi(w) = \frac{1}{\sqrt{2}g(\bar{u} - \bar{w})} + O(1/g^2), \quad \text{when } |\bar{w}| > 1, \tag{4.10}$$

$$\chi(w, u|1) + \Phi(w) = O(1/g), \quad \text{when } |\bar{w}| < 1. \tag{4.11}$$

Therefore, we have

$$\mathcal{I}_2 = -\int\limits_{-\infty}^{+\infty} \frac{dw}{2\pi} [\chi(w, u|1) + \Phi(w)] \frac{d}{dw}[\chi(w, v|1) + \Phi(w)] = O(1/g^2). \tag{4.12}$$

For what concerns the last term $\mathcal{I}_3$ in the rhs of (C.13), we find convenient to perform the change of variables $w = \sqrt{2}g\bar{w}$, $z = \sqrt{2}g\bar{z}$:

$$\mathcal{I}_3 = \int\limits_{-\infty}^{+\infty} \frac{d\bar{w}}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{d\bar{z}}{2\pi} \left[\chi(w, u|1) + \Phi(w)\right] \left[\frac{d}{d\bar{w}} \frac{d}{d\bar{z}} \Theta(\sqrt{2}g\bar{w}, \sqrt{2}g\bar{z})\right]$$
$$\cdot \left[\chi(z, v|1) + \Phi(z)\right]. \tag{4.13}$$

Now, from formulæ (4.3), (4.4), we deduce that the leading behaviour of the double derivative of the scalar–scalar phase is realised in the giant hole regime $|\bar{w}| > 1$, $|\bar{z}| > 1$. Therefore, we can write

$$\mathcal{I}_3 \cong \int\limits_{|\bar{w}|>1} \frac{d\bar{w}}{2\pi} \int\limits_{|\bar{z}|>1} \frac{d\bar{z}}{2\pi} \frac{1}{\sqrt{2}g\bar{w} - \sqrt{2}g\bar{u}} \frac{1}{\sqrt{2}g\bar{z} - \sqrt{2}g\bar{v}} \frac{d}{d\bar{w}} \frac{d}{d\bar{z}} \Theta(\sqrt{2}g\bar{w}, \sqrt{2}g\bar{z}). \tag{4.14}$$

Plugging (4.3) into (4.14) and performing the integrations we arrive at

$$\mathcal{I}_3 = \frac{1}{2\sqrt{2}g(\bar{u} - \bar{v})} \left[2 - \left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{1/4} \left(\frac{1-\bar{v}}{1+\bar{v}}\right)^{1/4} - \left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{1/4} \left(\frac{1+\bar{v}}{1-\bar{v}}\right)^{1/4}\right]. \tag{4.15}$$

Now, summing up (4.7), (4.12), (4.15) we obtain the final result for the gluon–gluon scattering phase at the order $O(1/g)$:

$$S^{(gg)}(u, v) = \exp\left[\frac{i}{\sqrt{2}g(\bar{u} - \bar{v})}\right.$$
$$\left.\left(1 + \frac{1}{2}\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{1/4}\left(\frac{1-\bar{v}}{1+\bar{v}}\right)^{1/4} + \frac{1}{2}\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{1/4}\left(\frac{1+\bar{v}}{1-\bar{v}}\right)^{1/4}\right)\right]. \tag{4.16}$$

The expression above agrees with the correspondent result of Basso, Sever, Vieira [34].

Gluons in the giant hole regime   We now want to compute the gluon–gluon scattering factor (C.13) in the limit $g \to +\infty$, with $u = \bar{u}\sqrt{2}g$, $v = \bar{v}\sqrt{2}g$, $\bar{u}$, $\bar{v}$ fixed and $\bar{u}^2 > 1$, $\bar{v}^2 > 1$ (giant hole regime).

As a preliminary calculation we consider the quantity $\chi(v, u|1) + \Phi(v)$. For its expression we refer to (A.7). Computing the scaling limit $u = \bar{u}\sqrt{2}g$, $v = \bar{v}\sqrt{2}g$, $g \to +\infty$, $\bar{u}$, $\bar{v}$ fixed and $\bar{u}^2 > 1$, $\bar{v}^2 > 1$, we find that

$$\chi(v, u|1) + \Phi(v) = \pi \, \mathrm{sgn}(v - u) - \pi \, \mathrm{sgn}\, v + O(1/g). \tag{4.17}$$

Since $\chi(v, u|1) + \Phi(v)$ is at most $O(g^0)$, properties (4.3), (4.4) imply that the following part of the integral $\mathcal{I}_3$,

$$\mathcal{I}_3^> = \int\limits_{|\bar{w}|>1} \frac{d\bar{w}}{2\pi} \int\limits_{|\bar{z}|>1} \frac{d\bar{z}}{2\pi} \left[\chi(w, u|1) + \Phi(w)\right] \left[\frac{d}{d\bar{w}} \frac{d}{d\bar{z}} \Theta(\sqrt{2}g\bar{w}, \sqrt{2}g\bar{z})\right]$$
$$\cdot \left[\chi(z, v|1) + \Phi(z)\right], \tag{4.18}$$

gives actually the dominant contribution (proportional to g) to $i \ln\left(-S^{(gg)}(u, v)\right)$. The integrations in (4.18) are easily performed and the final result[12] is

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[12] In order to get (4.19) we use the properties $\Theta(u, \pm\sqrt{2}g) = \Theta(\pm\sqrt{2}g, v) = 0$ which are proven using expressions given in [53].

$$i \ln \left(-S^{(gg)}(u, v) \right) = i \ln \left(-S^{(g\bar{g})}(u, v) \right) = \Theta(u, v) + O(g^0)$$
$$= i \ln \left(-S^{(ss)}(u, v) \right) + O(g^0). \tag{4.19}$$

Gluons in the non-scaling regime In this regime we send $g \to +\infty$ keeping the excitations rapidities fixed. For what concerns gluons, if we send $g \to +\infty$, with gluons and scalar rapidities, u, v respectively, fixed, we get that

$$\chi(v, u|1) + \Phi(v) = O(1/g^2). \tag{4.20}$$

In order to get (4.20), relation

$$g \to +\infty, \quad u \text{ fixed} \quad \Rightarrow x^{\pm}(u) = \pm \frac{ig}{\sqrt{2}} + \frac{u \pm \frac{i}{2}}{2} \mp i \frac{\left(u \pm \frac{i}{2} \right)^2}{4\sqrt{2}g} + O(1/g^3), \tag{4.21}$$

is useful. Result (4.20) means that in this regime the gluon–gluon scattering phase $S^{(gg)}(u, v)$ reduces to $\frac{u-v+i}{u-v-i}$.

4.3. Fermions

Fermions in the perturbative regime We want to find the strong coupling limit of the fermion–fermion scattering factor in the perturbative regime. As we will show in a moment, this regime fits in the small fermion case. We start from

$$S^{(ff)}(u, v) = \exp\left\{ i \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \chi_H(w, u) \frac{d}{dw} \chi_H(w, v) \right.$$
$$\left. - i \int \frac{dw}{2\pi} \frac{dz}{2\pi} \chi_H(w, u) \frac{d^2}{dwdz} \Theta(w, z) \chi_H(z, v) \right\}, \tag{4.22}$$

where

$$\chi_H(w, u) = -i \ln \frac{1 - \frac{x_f(u)}{x^+(w)}}{1 - \frac{x_f(u)}{x^-(w)}}, \quad x_f(u) = \frac{u}{2} \left[1 - \sqrt{1 - \frac{2g^2}{u^2}} \right]. \tag{4.23}$$

In the perturbative regime the fermion rapidity scales as $x_f(u) = \sqrt{2}g\bar{x}_f(\bar{u})$, $u = \sqrt{2}g\bar{u}$, with

$$\bar{x}_f(\bar{u}) = \frac{\bar{u}}{2} \left[1 - \sqrt{1 - \frac{1}{\bar{u}^2}} \right], \quad |\bar{u}| \geq 1, \quad |\bar{x}_f(\bar{u})| \leq \frac{1}{2}. \tag{4.24}$$

It is then clear that we are in the small fermion case.

For what concerns scalar rapidity, we make the rescaling $w = \sqrt{2}g\bar{w}$ and we develop at strong coupling. We have to distinguish two cases.

• If $|\bar{w}| > 1$, then

$$x^{\pm}(w) = \sqrt{2}g\bar{x}(\bar{w}) \pm \frac{i}{4} \frac{1 + \sqrt{1 - \frac{1}{\bar{w}^2}}}{\sqrt{1 - \frac{1}{\bar{w}^2}}} + O(1/g),$$
$$\bar{x}(\bar{w}) = \frac{\bar{w}}{2} \left[1 + \sqrt{1 - \frac{1}{\bar{w}^2}} \right]. \tag{4.25}$$

In this case we have

$$\chi_H(w, u) = -i \ln \frac{1 - \frac{x_f(u)}{x^+(w)}}{1 - \frac{x_f(u)}{x^-(w)}} \cong -\frac{\bar{x}_f(\bar{u})}{\sqrt{2}g} \frac{1}{\bar{w}\sqrt{1 - \frac{1}{\bar{w}^2}}} \frac{1}{\bar{x}_f(\bar{u}) - \bar{x}(\bar{w})} + O(1/g^2). \quad (4.26)$$

• If $|\bar{w}| < 1$, then

$$x^{\pm}(w) = \frac{g}{\sqrt{2}}[\bar{w} \pm i\sqrt{1 - \bar{w}^2}] + O(g^0). \quad (4.27)$$

In this second case

$$\chi_H(w, u) = -i \ln \frac{1 - 2\bar{x}_f(\bar{w} - i\sqrt{1 - \bar{w}^2})}{1 - 2\bar{x}_f(\bar{w} + i\sqrt{1 - \bar{w}^2})} + O(1/g). \quad (4.28)$$

In general we write $-i \ln S^{(ff)} = I_1 + I_2$, where

$$I_1 = \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \chi_H(w, u) \frac{d}{dw} \chi_H(w, v) = I_1^> + I_1^<, \quad (4.29)$$

where

$$I_1^> = \int_{|\bar{w}|>1} \frac{dw}{2\pi} \chi_H(w, u) \frac{d}{dw} \chi_H(w, v) \sim O(1/g^2), \quad (4.30)$$

$$I_1^< = \int_{|\bar{w}|<1} \frac{dw}{2\pi} \chi_H(w, u) \frac{d}{dw} \chi_H(w, v), \quad (4.31)$$

and

$$I_2 = -\int \frac{dw}{2\pi} \frac{dz}{2\pi} \chi_H(w, u) \frac{d^2}{dwdz} \Theta(w, z) \chi_H(z, v) = I_2^> + I_2^{rest}, \quad (4.32)$$

where

$$I_2^> = -\int_{|\bar{w}|,|\bar{z}|\geq 1} \frac{d\bar{w}}{2\pi} \frac{d\bar{z}}{2\pi} \chi_H(w, u) \frac{d^2}{d\bar{w}d\bar{z}} \Theta(w, z) \chi_H(z, v), \quad (4.33)$$

$$I_2^{rest} = -\int_{|\bar{w}| \, or \, |\bar{z}| \leq 1} \frac{d\bar{w}}{2\pi} \frac{d\bar{z}}{2\pi} \chi_H(w, u) \frac{d^2}{d\bar{w}d\bar{z}} \Theta(w, z) \chi_H(z, v). \quad (4.34)$$

In order to evaluate I_2 we need first to compute (the second derivative of) $\Theta(w, z)$. This happens to depend on the domain of w, z. When $|\bar{w}|$ and $|\bar{z}|$ are both greater than one, we can use formula (2.33) of the letter [23]: in particular, in this domain $\frac{d^2}{d\bar{w}d\bar{z}} \Theta(w, z) = O(g)$. In the remaining domains (*i.e.* $|\bar{w}|$ and $|\bar{z}|$ are not both greater than one), we have formula (4.4):

$$\frac{d^2}{d\bar{w}d\bar{z}} \Theta(w, z) = 2\pi \frac{d}{d\bar{w}} \delta(\bar{w} - \bar{z}) + \frac{1}{\sqrt{2}g} \frac{d^2}{d\bar{z}^2} P \frac{1}{\bar{z} - \bar{w}} + O(1/g^2). \quad (4.35)$$

Using this formula we can estimate I_2^{rest}. We have

$$
I_2^{rest} = - \int\limits_{|\bar{w}|<1} \frac{dw}{2\pi} \chi_H(w,u) \frac{d}{dw} \chi_H(w,v)
$$

$$
- \frac{1}{\sqrt{2}g} \int\limits_{|\bar{w}|,|\bar{z}|\leq 1} \frac{d\bar{w}}{2\pi} \frac{d\bar{z}}{2\pi} \chi_H(w,u) \frac{d^2}{d\bar{w}^2} P \frac{1}{\bar{z}-\bar{w}} \chi_H(z,v) + O(1/g^2), \tag{4.36}
$$

where we used the fact that $\chi_H(w,u)$ is $O(1/g)$ when $|\bar{w}|>1$. The first term in (4.36) cancels $I_1^<$. The second term equals

$$
- \frac{1}{\sqrt{2}g} \int\limits_{|\bar{w}|,|\bar{z}|\leq 1} \frac{d\bar{w}}{2\pi} \frac{d\bar{z}}{2\pi} \chi_H(w,u) \frac{d^2}{d\bar{w}^2} P \frac{1}{\bar{z}-\bar{w}} \chi_H(z,v)
$$

$$
= - \frac{2\sqrt{2}}{g} (\bar{x}_f(\bar{u}) - \bar{x}_f(\bar{v})) \frac{1 + 4\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})}{1 - 4\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})} \frac{\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})}{(1-4\bar{x}_f(\bar{u})^2)(1-4\bar{x}_f(\bar{v})^2)}. \tag{4.37}
$$

In order to get this result, we made use of the approximation

$$
\frac{d}{d\bar{w}} \chi_H(w,u) = \frac{\bar{w} - 2\bar{x}_f(\bar{u})}{\sqrt{1-\bar{w}^2}(\bar{w}-\bar{u})} - \frac{1}{2\sqrt{2}g(\bar{w}-\bar{u})^2} + O\left(\frac{1}{g^2}\right) \tag{4.38}
$$

and of the integral (B.31). For what concerns $I_2^>$, we have

$$
I_2^> = - \frac{1}{\sqrt{2}g} \int\limits_{|\bar{w}|\geq 1} \frac{d\bar{w}}{2\pi} \int\limits_{|\bar{z}|\geq 1} \frac{d\bar{z}}{2\pi} \frac{1}{\bar{w}\sqrt{1-\frac{1}{\bar{w}^2}}} \frac{\bar{x}_f(\bar{u})}{\bar{x}_f(\bar{u}) - \bar{x}(\bar{w})}
$$

$$
\cdot \frac{\left(\frac{\bar{w}-1}{\bar{w}+1}\right)^{\frac{1}{4}} \left(\frac{\bar{z}+1}{\bar{z}-1}\right)^{\frac{1}{4}} + \left(\frac{\bar{w}+1}{\bar{w}-1}\right)^{\frac{1}{4}} \left(\frac{\bar{z}-1}{\bar{z}+1}\right)^{\frac{1}{4}}}{\bar{w}-\bar{z}} \frac{1}{\bar{z}\sqrt{1-\frac{1}{\bar{z}^2}}} \frac{\bar{x}_f(\bar{v})}{\bar{x}_f(\bar{v}) - \bar{x}(\bar{z})}. \tag{4.39}
$$

Now, we use the identity

$$
\frac{\left(\frac{\bar{w}-1}{\bar{w}+1}\right)^{\frac{1}{4}} \left(\frac{\bar{z}+1}{\bar{z}-1}\right)^{\frac{1}{4}} + \left(\frac{\bar{w}+1}{\bar{w}-1}\right)^{\frac{1}{4}} \left(\frac{\bar{z}-1}{\bar{z}+1}\right)^{\frac{1}{4}}}{\bar{w}-\bar{z}}
$$

$$
= \frac{1}{\bar{x}(\bar{w}) - \bar{x}(\bar{z})} \frac{\sqrt{1+\sqrt{1-\frac{1}{\bar{w}^2}}}\sqrt{1+\sqrt{1-\frac{1}{\bar{z}^2}}}}{\left(1-\frac{1}{\bar{w}^2}\right)^{\frac{1}{4}}\left(1-\frac{1}{\bar{z}^2}\right)^{\frac{1}{4}}} \tag{4.40}
$$

and arrive at

$$
I_2^> = - \frac{1}{\sqrt{2}g} \int\limits_{|\bar{w}|\geq 1} \frac{d\bar{w}}{2\pi} \int\limits_{|\bar{z}|\geq 1} \frac{d\bar{z}}{2\pi} \frac{1}{\bar{w}\sqrt{1-\frac{1}{\bar{w}^2}}} \frac{\bar{x}_f(\bar{u})}{\bar{x}_f(\bar{u}) - \bar{x}(\bar{w})}
$$

$$
\cdot \frac{1}{\bar{x}(\bar{w}) - \bar{x}(\bar{z})} \frac{\sqrt{1+\sqrt{1-\frac{1}{\bar{w}^2}}}\sqrt{1+\sqrt{1-\frac{1}{\bar{z}^2}}}}{\left(1-\frac{1}{\bar{w}^2}\right)^{\frac{1}{4}}\left(1-\frac{1}{\bar{z}^2}\right)^{\frac{1}{4}}} \frac{1}{\bar{z}\sqrt{1-\frac{1}{\bar{z}^2}}} \frac{\bar{x}_f(\bar{v})}{\bar{x}_f(\bar{v}) - \bar{x}(\bar{z})}. \tag{4.41}
$$

We now use the symmetry properties of the integrand under the exchange $\bar{w}$ with $\bar{z}$ and factorise the integral as

$$I_2^> = -\frac{1}{2\sqrt{2}g}\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})[\bar{x}_f(\bar{v}) - \bar{x}_f(\bar{u})]\beth(\bar{u},\bar{v})^2, \tag{4.42}$$

where

$$\beth(\bar{u},\bar{v}) = \int\limits_{|\bar{w}|\geq 1} \frac{d\bar{w}}{2\pi} \frac{\sqrt{1 + \sqrt{1 - \frac{1}{\bar{w}^2}}}}{\bar{w}\left(1 - \frac{1}{\bar{w}^2}\right)^{\frac{3}{4}}} \frac{1}{(\bar{x}_f(\bar{u}) - \bar{x}(\bar{w}))(\bar{x}_f(\bar{v}) - \bar{x}(\bar{w}))}. \tag{4.43}$$

We change variable of integration from $\bar{w}$ to $\bar{x}(\bar{w}) = y$. We get

$$\beth(\bar{u},\bar{v}) = \int\limits_{|y|\geq 1/2} \frac{dy}{2\pi y} \sqrt{\frac{2}{1 - \frac{1}{4y^2}}} \frac{1}{\bar{x}_f(\bar{u}) - y} \frac{1}{\bar{x}_f(\bar{v}) - y}, \tag{4.44}$$

which can be exactly computed by means of (B.31):

$$\beth(\bar{u},\bar{v}) = \frac{\sqrt{2}}{\bar{x}_f(\bar{u}) - \bar{x}_f(\bar{v})} \left[\frac{1}{\sqrt{1 - 4\bar{x}_f(\bar{u})^2}} - \frac{1}{\sqrt{1 - 4\bar{x}_f(\bar{v})^2}} \right]. \tag{4.45}$$

Therefore, we obtain

$$I_2^> = -\frac{1}{\sqrt{2}g} \frac{\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})}{\bar{x}_f(\bar{v}) - \bar{x}_f(\bar{u})} \left[\frac{1}{\sqrt{1 - 4\bar{x}_f(\bar{v})^2}} - \frac{1}{\sqrt{1 - 4\bar{x}_f(\bar{u})^2}} \right]^2. \tag{4.46}$$

Adding (4.37), (4.46) we arrive at the final formula

$$
\begin{aligned}
S^{(ff)}(u,v) = \exp\Bigg\{ &-\frac{2i\sqrt{2}}{g}(\bar{x}_f(\bar{u}) - \bar{x}_f(\bar{v}))\frac{1 + 4\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})}{1 - 4\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})} \\
&\cdot \frac{\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})}{(1 - 4\bar{x}_f(\bar{u})^2)(1 - 4\bar{x}_f(\bar{v})^2)} - \frac{i}{\sqrt{2}g}\frac{\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})}{\bar{x}_f(\bar{v}) - \bar{x}_f(\bar{u})} \\
&\cdot \left[\frac{1}{\sqrt{1 - 4\bar{x}_f(\bar{v})^2}} - \frac{1}{\sqrt{1 - 4\bar{x}_f(\bar{u})^2}} \right]^2 + O(1/g^2) \Bigg\}.
\end{aligned}
\tag{4.47}
$$

Fermions in the giant hole regime   In the giant hole regime which fits into the large fermion case the fermion rapidity scales as $x_F(u) = \sqrt{2}g\bar{x}(\bar{u})$, $u = \sqrt{2}g\bar{u}$, with

$$\bar{x}(\bar{u}) = \frac{\bar{u}}{2}\left[1 + \sqrt{1 - \frac{1}{\bar{u}^2}} \right], \quad |\bar{u}| \geq 1, \quad |\bar{x}(\bar{u})| \geq \frac{1}{2}, \tag{4.48}$$

where $\bar{x}(\bar{u})$ was already defined in the second of (4.9). Referring then to formula (C.11) for large fermions, we first show that if $w = \bar{w}\sqrt{2}g$, $u = \bar{u}\sqrt{2}g$ and $g \to +\infty$, with $|\bar{w}| > 1$, then

$$\chi_F(w,u) + \Phi(w) = \pi\,\text{sgn}(w - u) - \pi\,\text{sgn}(w) + O(1/g). \tag{4.49}$$

In order to prove (4.49), it is convenient to start from (A.9) and then use (4.9). Therefore, the situation is completely analogous to the gluon case: relation (4.49) implies that in (C.11) the dominant contribution comes from integrations in the second term in the region $|\bar{w}| > 1$, $|\bar{x}| > 1$, where the scalar–scalar factor Θ is proportional to g. The final result is

$$i \ln S^{(FF)}(u, v) = \Theta(u, v) + O(g^0) = i \ln(-S^{(ss)}(u, v)) + O(g^0)$$
$$= i \ln(-S^{(gg)}(u, v)) + O(g^0). \tag{4.50}$$

Fermions in the non-scaling regime   We send $g \to +\infty$ by keeping fixed all the rapidities. For fermions, rapidities are the variables x: therefore if we keep x fixed, we are necessarily in the small fermion case, *i.e.* $|x_f| < g/\sqrt{2}$. We can then show that

$$\lim_{g \to +\infty} \chi_H(u, v) = x_f(v) \left[-\frac{2\sqrt{2}}{g} + \frac{1}{g^2} + O(1/g^3) \right]. \tag{4.51}$$

This means that in this regime $i \ln S^{(ff)}(u, v) = O(1/g^2)$.

4.4. Mixed factors

Scalar–gluon
• Perturbative regime
   We start from the exact expression (C.18)

$$i \ln[S^{(sg)}(u, v)] = \chi(u, v|1) + \Phi(u) - \int \frac{dw}{2\pi} \frac{d\Theta}{dw}(u, w)[\chi(w, v|1) + \Phi(w)], \tag{4.52}$$

where both the scalar and the gluon are in the perturbative regime. This means that the u rapidity is parametrised as (4.2) and the v rapidity is scaled as $v = \sqrt{2}g\bar{v}$. In these hypothesis we have that

$$\chi(u, v|1) + \Phi(u) = \frac{1}{\sqrt{2}g} \frac{1}{\bar{v} - \mathrm{sgn}(u)}. \tag{4.53}$$

In addition, in first approximation, we can integrate in the region $|w| < \frac{2}{\pi} \ln m$ and use for Θ the expression (4.1) in which rapidities are parametrised as (4.2). The final result is

$$i \ln[S^{(sg)}(u, v)] = \frac{1}{2\pi\sqrt{2}g} \frac{1}{\bar{v} - \mathrm{sgn}(u)} \left[\pi - \Theta_{np} \left(\frac{2}{\pi} \ln z \right) \right], \tag{4.54}$$

where $|u| = \frac{2}{\pi} \ln \frac{z}{m(g)}$, $m(g) < z < 1$ being the scalar rapidity in the perturbative regime.
• Giant hole regime
   In the giant hole regime we use formula (4.17) for the limiting expression of $\chi(v, u|1) + \Phi(v)$ when both $|\bar{v}|$ and $|\bar{u}|$ are greater than one. Then the leading (*i.e.* $O(g)$) contribution to $i \ln S^{(sg)}(u, v)$ comes from integration in the second term of (C.18) in the region $|\bar{w}| > 1$. This integration is easily done and the result is

$$i \ln S^{(sg)}(u, v) = i \ln S^{(s\bar{g})}(u, v) = \Theta(u, v) + O(g^0). \tag{4.55}$$

• Non-scaling regime
   In order to compute the scattering phase $S^{(sg)}(u, v)$ in the non-scaling regime, we have to plug the expressions (4.4) and (4.20) into (C.18). Since $\chi(u, v|1) + \Phi(u)$ is of order $O(1/g^2)$,

we claim that

$$i \ln S^{(sg)}(u, v) = [\chi(u, v|1) + \Phi(u)] - \int \frac{dw}{2\pi} \frac{d}{dw} \Theta(u, w) [\chi(w, v|1) + \Phi(w)]$$
$$= O\left(\frac{1}{g^2}\right). \tag{4.56}$$

Gluons–fermions

• Perturbative regime

We study the scattering factor between gluons with rapidity u and (small) fermions with rapidity $x_f(v)$ in perturbative regime of the strong coupling limit, *i.e.* $u = \sqrt{2}g\bar{u}$, with $|\bar{u}| \leq 1$ and $x_f(v) = \sqrt{2}g\bar{x}_f(\bar{v})$, $v = \sqrt{2}g\bar{v}$, with $|\bar{v}| \geq 1$, $|\bar{x}_f(\bar{v})| \leq 1/2$.

We start from

$$i \ln\left(-S^{(gf)}(u, v)\right) = I_1 + I_2 + I_3, \tag{4.57}$$

where

$$I_1 = 2 \arctan 2(u - v), \tag{4.58}$$

$$I_2 = \int\limits_{-\infty}^{+\infty} \frac{dw}{2\pi} [\chi(w, u|1) + \Phi(w)] \frac{d}{dw} \chi_H(w, v), \tag{4.59}$$

$$I_3 = -\int \frac{dw}{2\pi} \frac{dz}{2\pi} [\chi(w, u|1) + \Phi(w)] \frac{d^2}{dw\,dz} \Theta(w, z) \chi_H(z, v). \tag{4.60}$$

In the perturbative regime

$$I_1 = \pi \operatorname{sgn}(\bar{u} - \bar{v}) - \frac{1}{\sqrt{2}g(\bar{u} - \bar{v})} + O(1/g^2), \tag{4.61}$$

$$I_2 = \int\limits_{|\bar{w}|<1} \frac{dw}{2\pi} [\chi(w, u|1) + \Phi(w)] \frac{d}{dw} \chi_H(w, v) + O(1/g^2), \tag{4.62}$$

$$I_3 = \int\limits_{|\bar{w}|,|\bar{z}|\geq 1} \frac{d\bar{w}}{2\pi} \frac{d\bar{z}}{2\pi} \frac{1}{\bar{u} - \bar{w}} \frac{\left(\frac{\bar{w}+1}{\bar{w}-1}\right)^{\frac{1}{4}} \left(\frac{\bar{z}-1}{\bar{z}+1}\right)^{\frac{1}{4}} + \left(\frac{\bar{w}-1}{\bar{w}+1}\right)^{\frac{1}{4}} \left(\frac{\bar{z}+1}{\bar{z}-1}\right)^{\frac{1}{4}}}{\bar{w} - \bar{z}} \frac{\bar{x}_f(\bar{v})}{\sqrt{2}g\bar{z}\sqrt{1 - \frac{1}{\bar{z}^2}}}$$
$$\cdot \frac{1}{\bar{x}_f(\bar{v}) - \bar{x}(\bar{z})} - \int\limits_{|\bar{w}|<1} \frac{dw}{2\pi} [\chi(w, u|1) + \Phi(w)] \frac{d}{dw} \chi_H(w, v) + O(1/g^2). \tag{4.63}$$

We evaluate the sum $I_2 + I_3$ by first performing integration in $\bar{w}$ with the help of (B.33), getting

$$I_2 + I_3 = -\frac{\bar{x}_f(\bar{v})}{2g} \int\limits_{|\bar{z}|\geq 1} \frac{d\bar{z}}{2\pi} \frac{1}{\bar{z}\sqrt{1 - \frac{1}{\bar{z}^2}}} \frac{1}{\bar{x}_f(\bar{v}) - \bar{x}(\bar{z})} \frac{\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}} \left(\frac{\bar{z}-1}{\bar{z}+1}\right)^{\frac{1}{4}} - \left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}} \left(\frac{\bar{z}+1}{\bar{z}-1}\right)^{\frac{1}{4}}}{\bar{u} - \bar{z}}$$
$$+ O(1/g^2). \tag{4.64}$$

Then, we integrate in $\bar{z}$, using (B.34). We obtain

$$I_2 + I_3 = \frac{1}{4g} \frac{\sqrt{\frac{1-2\bar{x}_f(\bar{v})}{1+2\bar{x}_f(\bar{v})}}\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}} + \sqrt{\frac{1+2\bar{x}_f(\bar{v})}{1-2\bar{x}_f(\bar{v})}}\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}} - \sqrt{2}}{\bar{v} - \bar{u}} + O(1/g^2). \tag{4.65}$$

Summing up $I_1 + I_2 + I_3$ we get the final result

$$S^{(gf)}(u,v) = \exp\left[\frac{i}{4g}\frac{\sqrt{2} + \sqrt{\frac{1-2\bar{x}_f}{1+2\bar{x}_f}}\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}} + \sqrt{\frac{1+2\bar{x}_f}{1-2\bar{x}_f}}\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}}}{\bar{u} - \bar{v}} + O(1/g^2)\right]. \tag{4.66}$$

For what concerns $i \ln S^{(\bar{g}f)}(u,v) = I_2 + I_3$, we have

$$S^{(\bar{g}f)}(u,v) = \exp\left[\frac{i}{4g}\frac{\sqrt{\frac{1-2\bar{x}_f}{1+2\bar{x}_f}}\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}} + \sqrt{\frac{1+2\bar{x}_f}{1-2\bar{x}_f}}\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}} - \sqrt{2}}{\bar{u} - \bar{v}} + O(1/g^2)\right]. \tag{4.67}$$

- Giant hole regime

Since both $\chi(v,u|1) + \Phi(v)$ and $\chi_F(v,u) + \Phi(v)$ have the same limiting nonzero expression (4.17) when $|\bar{v}| > 1$, $|\bar{u}| > 1$, the leading expressions for $i \ln(-S^{(gF)}(u,v))$ and $i \ln(S^{(\bar{g}F)}(u,v))$ coincide with the one for $i \ln S^{(FF)}(u,v)$. Therefore,

$$i \ln(-S^{(gF)}(u,v)) = i \ln(S^{(\bar{g}F)}(u,v)) + O(g^0) = \Theta(u,v) + O(g^0). \tag{4.68}$$

- Non-scaling regime

As written before, in this regime fermions are necessarily small. Then, since $\chi(w,u|1) + \Phi(w)$ is $O(1/g^2)$ and $\chi_H(w,v)$ is $O(1/g)$, integrals I_2, I_3 are both $O(1/g^3)$. For what concerns I_1, since fermionic rapidities $u_{f,k}$ are bounded by the inequality $u_{f,k}^2 > 2g^2$, we can safely approximate $-e^{iI_1} = 1 + O(1/g)$. Therefore, in the non-scaling regime $S^{(gf)}(u,v) = S^{(\bar{g}\bar{f})}(u,v) = 1 + O(1/g)$ and $S^{(\bar{g}f)}(u,v) = S^{(g\bar{f})}(u,v) = 1 + O(1/g^3)$.

Scalars–fermions
- Perturbative regime

We start from the exact expression (C.16),

$$-i \ln S^{(sf)}(u,v) = \chi_H(u,v) - \int_{-\infty}^{+\infty} \frac{d\bar{w}}{2\pi} \frac{d\Theta}{d\bar{w}}(u,w)\chi_H(w,v), \tag{4.69}$$

and make the parameterisations (4.2) for u and $v = \sqrt{2}g\bar{v}$. At leading order we have

$$\chi_H(u,v) = 2^{\frac{3}{4}}\sqrt{\frac{\ln\frac{g}{\sqrt{2}}}{\pi g}} \frac{1}{\bar{v} - 1 + \sqrt{\bar{v}^2 - 1}}. \tag{4.70}$$

Then, as for the scalar–gluon case, in first approximation, we can integrate in the region $|w| < \frac{2}{\pi} \ln m(g)$ and use for Θ the expression (4.1) in which rapidities are parametrised as (4.2). The final result is

$$-i \ln S^{(sf)}(u,v) = i \ln S^{(fs)}(v,u)$$

$$= 2^{-\frac{1}{4}} \sqrt{\frac{\ln \frac{g}{\sqrt{2}}}{\pi g}} \frac{1}{\bar{v} - 1 + \sqrt{\bar{v}^2 - 1}} \left[1 - \frac{1}{\pi} \Theta_{np} \left(\frac{2}{\pi} \ln z \right) \right], \tag{4.71}$$

with $|u| = \frac{2}{\pi} \ln \frac{z}{m(g)}$.

• Giant hole regime

Since $\chi_F(v,u) + \Phi(v)$ and $\chi(v,u|1) + \Phi(v)$ go to the same limit, *i.e.* $\pi \, \text{sgn}(v-u) - \pi \, \text{sgn}(v)$, by comparing (C.16) and (C.18) we get the equality $i \ln S^{(sF)}(u,v) = -i \ln S^{(gs)}(v,u) + O(g^0)$, which, together with (4.55), gives

$$i \ln S^{(sF)}(u,v) = \Theta(u,v) + O(g^0). \tag{4.72}$$

• Non-scaling regime

We perform the non-perturbative limit of the scalar–fermion scattering phase

$$i \ln S^{(sf)}(u,v) = -\chi_H(u,v) + \int_{-\infty}^{\infty} \frac{dw}{2\pi} \frac{d}{dw} \Theta(u,w) \chi_H(w,v), \tag{4.73}$$

by taking $g \longrightarrow \infty$ while keeping the scalar rapidities finite, whereas the modulus of the fermionic rapidities x_f must be $|x_f| < g/\sqrt{2}$. Under these assumptions, we can make use of the approximations (4.51) and (4.1) for $\chi_H(u,v)$ and $\Theta(u,v)$; eventually, we find:

$$i \ln S^{(sf)}(u,v) = O\left(\frac{1}{g^3} \right). \tag{4.74}$$

4.5. Remark on the non-scaling regime

We showed that in the non-scaling regime all the factors $S^{**'}(u,v)$ go as $1 + O(1/g^2)$, with the exception of the scalar–scalar one which goes as

$$S^{(ss)}(u,v) = -\frac{\Gamma\left(1 + i\frac{u-v}{4}\right) \Gamma\left(\frac{1}{2} - i\frac{u-v}{4}\right)}{\Gamma\left(1 - i\frac{u-v}{4}\right) \Gamma\left(\frac{1}{2} + i\frac{u-v}{4}\right)} \exp\left[i \, \text{gd} \frac{\pi}{2}(u-v)\right] [1 + O(1/g)]. \tag{4.75}$$

In addition to that, we recall that the fermionic rapidities $u_{f,k}$ satisfy the inequalities $u_{f,k}^2 > 2g^2$. Therefore in this regime all the rational factors involving fermionic rapidities (which appear in the quantisation conditions for fermions and in the equations for isotopic roots u_a and u_c) go to one. In addition all the exponentials of momenta and defect ($= e^i(P + D)$) go to 1, with the exception of those for scalars. Summarising, in the limit $g \to +\infty$ with rapidities fixed (and finite), the non-trivial equations are

$$1 = e^{i R P^{(s)}(u_h) + i D^{(s)}(u_h)} \prod_{j=1}^{K_b} \frac{u_h - u_{b,j} + \frac{i}{2}}{u_h - u_{b,j} - \frac{i}{2}} \prod_{\substack{h'=1 \\ h' \neq h}}^{H} S^{(ss)}(u_h, u_{h'}),$$

$$1 = \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - i/2}{u_{a,k} - u_{b,j} + i/2},$$

$$\prod_{h=1}^{H} \frac{u_{b,k} - u_h + i/2}{u_{b,k} - u_h - i/2} = \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - i/2}{u_{b,k} - u_{a,j} + i/2} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - i/2}{u_{b,k} - u_{c,j} + i/2} \prod_{\substack{j=1 \\ j \neq k}}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i},$$

$$1 = \prod_{\substack{j \neq k}}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - i/2}{u_{c,k} - u_{b,j} + i/2},$$

$$1 = \prod_{\substack{j \neq k}}^{N_g} \frac{u_k^g - u_j^g + i}{u_k^g - u_j^g - i},$$

$$1 = \prod_{\substack{j \neq k}}^{N_{\bar{g}}} \frac{u_k^{\bar{g}} - u_j^{\bar{g}} + i}{u_k^{\bar{g}} - u_j^{\bar{g}} - i}, \tag{4.76}$$

with $S^{(ss)}$ given by (4.75). Since equations for gluons have no solutions for finite rapidities (*i.e.* $N_g = N_{\bar{g}} = 0$), equations (4.76) show that in the non-perturbative regime the only active excitations are the six scalars. The other excitations are obliged to assume infinite rapidities and thus decouple to very high energy from the scalars. The latter satisfy the above ABA (4.76) which is the same we can derive from the $O(6)$ non-linear sigma model S-matrix of [52]. Therefore, also the exact TBA would be that of the $O(6)$ model (if we can neglect the exchange of the $g \to +\infty$ limit with the thermodynamics).

5. Particle momentum in different forms

Momentum was already thoroughly discussed by Basso in [20]. The aim of this section is to show that the expressions for momenta of the various excitations we found (in our notations) in previous sections agree with corresponding formulæ of [20].

• Scalars

We found (2.74) that the momentum of a scalar excitation is

$$P^{(s)}(u) = 2u - \frac{1}{2} Z_{BES}(u). \tag{5.1}$$

Now, using the mapping (B.19), valid for $k > 0$,

$$\frac{i \sinh \frac{k}{2}}{\pi} \hat{Z}_{BES}(k) = \frac{\gamma_+^\emptyset(\sqrt{2}gk) + \gamma_-^\emptyset(\sqrt{2}gk)}{k}, \tag{5.2}$$

between our quantities and quantities used in [20], we immediately write (5.1) in the form reported in [20] (second of the (4.6)).

• Gluons

For a gluon with rapidity u we found for the momentum the expression (2.93), which we can write in Fourier space as

$$P^{(g)}(u) = \int_{-\infty}^{+\infty} \frac{dk}{4\pi^2} \left[-2\pi \frac{\sin ku}{k} e^{-|k|} + i \sum_{n=1}^{\infty} \left(\frac{g}{i\sqrt{2}x^-(u)} \right)^n \frac{2\pi}{k} e^{-\frac{|k|}{2}} J_n(\sqrt{2}gk) \right.$$

$$\left. + i \sum_{n=1}^{\infty} \left(\frac{g}{i\sqrt{2}x^+(u)} \right)^n \frac{2\pi}{k} e^{-\frac{|k|}{2}} J_n(\sqrt{2}gk) \right] (-4\pi \delta(k) + \frac{1}{2}\hat{\sigma}_{BES}(k))$$

$$= 2u - \left(\frac{g^2}{x^-} + \frac{g^2}{x^+} \right) + \int\limits_{-\infty}^{+\infty} \frac{dk}{8\pi^2} \left[-2\pi \frac{\sin ku}{k} e^{-|k|} \right.$$

$$+ i \sum_{\substack{n=1 \\ n \text{ odd}}}^{+\infty} \left(\frac{g}{i\sqrt{2}x^-(u)} \right)^n \frac{2\pi}{k} e^{-\frac{|k|}{2}} J_n(\sqrt{2}gk)$$

$$+ i \sum_{\substack{n=1 \\ n \text{ odd}}}^{+\infty} \left(\frac{g}{i\sqrt{2}x^+(u)} \right)^n \frac{2\pi}{k} e^{-\frac{|k|}{2}} J_n(\sqrt{2}gk) \bigg] \hat{\sigma}_{BES}(k) \,.$$

Now, we use the equalities

$$\int\limits_{-\infty}^{+\infty} \frac{dk}{k} e^{-\frac{|k|}{2}} J_n(\sqrt{2}gk) \hat{\sigma}_{BES}(k) = 4\pi [\sqrt{2}g \delta_{n,1} - \gamma_n^{\emptyset}], \quad n \text{ odd}, \tag{5.3}$$

and

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left[\left(\frac{g}{i\sqrt{2}x^-} \right)^n + \left(\frac{g}{i\sqrt{2}x^+} \right)^n \right] \gamma_n^{\emptyset} = -i \int\limits_0^{+\infty} \frac{dk}{k} \sin ku \, e^{-\frac{k}{2}} \gamma_-^{\emptyset}(\sqrt{2}gk), \tag{5.4}$$

to eventually obtain

$$P^{(g)}(u) = 2u - \int\limits_0^{+\infty} \frac{dk}{k} \sin ku \, e^{-\frac{k}{2}} \left[\frac{\gamma_-^{\emptyset}(\sqrt{2}gk)}{1 - e^{-k}} + \frac{\gamma_+^{\emptyset}(\sqrt{2}gk)}{e^k - 1} \right], \tag{5.5}$$

which agrees with the second of (4.9) of [20].

• Large fermions

The momentum associated to a large fermion with rapidity u enjoys the expression (2.82). In Fourier space it reads

$$P^{(F)}(u) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \left[\frac{\sin ku}{k} e^{-\frac{|k|}{2}} + \sum_{n=1}^{\infty} \left(\frac{g}{i\sqrt{2}x(u)} \right)^n \frac{e^{-\frac{|k|}{2}}}{ik} J_n(\sqrt{2}gk) \right]$$

$$\times \left(4\pi \delta(k) - \frac{\hat{\sigma}_{BES}(k)}{2} \right)$$

$$= 2u - \int\limits_0^{+\infty} \frac{dk}{k} \sin ku \, \frac{\gamma_+^{\emptyset}(\sqrt{2}gk) + \gamma_-^{\emptyset}(\sqrt{2}gk)}{e^k - 1} - \frac{1}{2} \int\limits_0^{+\infty} \frac{dk}{k} \sin ku \, \gamma_-^{\emptyset}(\sqrt{2}gk),$$

which recalls the second of (4.10) of [20]. In order to obtain the equation in the last line, we made use of the relation

$$\int\limits_0^{\infty} \frac{dk}{2k} \sin ku \, \gamma_-^{\emptyset}(\sqrt{2}gk) = i \sum_{n=1}^{\infty} \left(\frac{g}{\sqrt{2}ix(u)} \right)^{2n-1} \gamma_{2n-1}^{\emptyset}, \tag{5.6}$$

which holds for $u^2 > 2g^2$.

• Small fermions

The reasonings for the momentum of a small fermion with rapidity u mimic very closely the large fermion case. We start from our expression (2.87) and in Fourier space we eventually get the result ($u^2 > 2g^2$)

$$P^{(f)}(u) = \frac{1}{2} \int\limits_{0}^{+\infty} \frac{dk}{k} \, \sin(ku) \, \gamma_-^{\emptyset}(\sqrt{2}gk), \tag{5.7}$$

therefore matching the second of (4.12) of [20].

6. Strong coupling analysis of the defect term

We now perform a quantitative analysis of the strong coupling limit of the defect which appears in the Bethe equations on the GKP vacuum.

6.1. Scalars

It is convenient to concentrate on the function Z_4 (2.69) in absence of excitations, which equals

$$Z_4(u)|_{NE} = -2 \ln s \, P^{(s)}(u) - 2D^{(s)}(u). \tag{6.1}$$

The study of this function, which relies also on previous results, provides information on both the momentum and the defect of the scalar.

Non-perturbative regime In this regime we send $g \to +\infty$, keeping the rapidity u fixed. We can use results from [21] where the non-perturbative regime for the pure $sl(2)$ sector is studied. We found that the function (6.1) has the form

$$Z_4(u)|_{NE} = -2m(g) \ln \frac{2\sqrt{2}s}{g} \, \sinh \frac{\pi}{2}u + O(m(g)^3), \tag{6.2}$$

where $m(g)$ is given by (3.13). Therefore, the contribution of the two defect is proportional to the momentum

$$D^{(s)}(u) = m(g) \ln \frac{2\sqrt{2}}{g} \, \sinh \frac{\pi}{2}u + O(m(g)^3), \tag{6.3}$$

so allowing us to fully re-absorb them into a re-definition of the size $R(g)$ as in [54,21]

Perturbative regime We obtain interesting formulæ in this regime by plugging (4.2) in (6.3), namely:

$$D^{(s)}(u) = \frac{z}{2} \ln \frac{2\sqrt{2}}{g} + \dots, \tag{6.4}$$

for right movers ($u > 0$), while

$$D^{(s)}(u) = -\frac{z}{2} \ln \frac{2\sqrt{2}}{g} + \dots, \tag{6.5}$$

for left movers ($u < 0$), where evidently the dots … mean sub-leading corrections.

Scaling regimes   We introduce the density $\sigma_4(u)|_{NE} = \frac{d}{du} Z_4(u)|_{NE}$ and rescale the rapidity $u = \sqrt{2}g\bar{u}$. By using techniques developed in [15], we eventually find

$$
\sigma_4(\sqrt{2}g\bar{u})|_{NE}
$$

$$
= \int\limits_{0}^{+\infty} \frac{d\bar{h}}{\sqrt{2}g} \cos\bar{h}\bar{u} \left[-\frac{e^{-\frac{\bar{h}}{2\sqrt{2}g}}}{\cosh\frac{\bar{h}}{\sqrt{2}g}} \Gamma_+(\bar{h}) + \frac{e^{\frac{\bar{h}}{2\sqrt{2}g}}}{\cosh\frac{\bar{h}}{\sqrt{2}g}} \Gamma_-(\bar{h}) \right] - 4\ln s
$$

$$
+ \frac{2\pi}{\cosh\pi\sqrt{2}g\bar{u}} + 2\psi\left(\frac{1}{2} + i\bar{u}\sqrt{2}g\right) + 2\psi\left(\frac{1}{2} - i\bar{u}\sqrt{2}g\right) + \psi\left(\frac{5}{8} + \frac{i\sqrt{2}g\bar{u}}{4}\right)
$$

$$
+ \psi\left(\frac{5}{8} - \frac{i\sqrt{2}g\bar{u}}{4}\right) - \psi\left(\frac{7}{8} + \frac{i\sqrt{2}g\bar{u}}{4}\right) - \psi\left(\frac{7}{8} - \frac{i\sqrt{2}g\bar{u}}{4}\right), \tag{6.6}
$$

where the functions $\Gamma_\pm$ satisfy the relations, valid when $|\bar{u}| < 1$:

$$
\int\limits_{0}^{+\infty} \frac{d\bar{h}}{\sqrt{2}g} \sin\bar{h}\bar{u}[\Gamma_-(\bar{h}) + \Gamma_+(\bar{h})] = -2i[\psi(1 - i\bar{u}\sqrt{2}g) - \psi(1 + i\bar{u}\sqrt{2}g)] \tag{6.7}
$$

$$
\int\limits_{0}^{+\infty} \frac{d\bar{h}}{\sqrt{2}g} \cos\bar{h}\bar{u}[\Gamma_-(\bar{h}) - \Gamma_+(\bar{h})] = 4\ln s - 2[\psi(1 - i\bar{u}\sqrt{2}g) + \psi(1 + i\bar{u}\sqrt{2}g)]. \tag{6.8}
$$

Going to the strong coupling limit $g \to +\infty$, with $\bar{u}$ fixed, we find

$$
\sigma_4(\sqrt{2}g\bar{u})|_{NE}
$$

$$
= \int\limits_{0}^{+\infty} \frac{d\bar{h}}{\sqrt{2}g} \cos\bar{h}\bar{u} \left[-\frac{e^{-\frac{\bar{h}}{2\sqrt{2}g}}}{\cosh\frac{\bar{h}}{\sqrt{2}g}} \Gamma_+(\bar{h}) + \frac{e^{\frac{\bar{h}}{2\sqrt{2}g}}}{\cosh\frac{\bar{h}}{\sqrt{2}g}} \Gamma_-(\bar{h}) \right] - 4\ln\frac{s}{g} + O(g^0), \tag{6.9}
$$

where

$$
\int\limits_{0}^{+\infty} \frac{d\bar{h}}{\sqrt{2}g} \sin\bar{h}\bar{u}[\Gamma_-(\bar{h}) + \Gamma_+(\bar{h})] = O(g^0), \quad |\bar{u}| < 1, \tag{6.10}
$$

$$
\int\limits_{0}^{+\infty} \frac{d\bar{h}}{\sqrt{2}g} \cos\bar{h}\bar{u}[\Gamma_-(\bar{h}) - \Gamma_+(\bar{h})] = 4\ln\frac{s}{g} + O(g^0), \quad |\bar{u}| < 1. \tag{6.11}
$$

Solutions to these equations go differently according to the value of $|\bar{u}|$. If $|\bar{u}| < 1$ we have

$$
\sigma_4(u)|_{NE} = -\frac{\sqrt{2}\pi}{g} \delta(\bar{u}) + O(1/g^2), \tag{6.12}
$$

which means that $P^{(s)}(u)$ is exponentially small and that $D^{(s)}(u) = -\frac{\sqrt{2}\pi}{g}\delta(\bar{u}) + O(1/g^2)$.
   On the other hand, if $|\bar{u}| > 1$, we have

$$
\sigma_4(u)|_{NE} = -2\ln\frac{s}{g}\frac{d}{du}P^{(s)}(u) + O(g^0), \tag{6.13}
$$

with $P^{(s)}(u) = O(g)$ or, alternatively, the proportionality to the momentum

$$D^{(s)}(u) = -P^{(s)}(u) \ln g + O(g), \tag{6.14}$$

which would allow us to re-absorb, at this order (only), fully the defect into a simple redefinition of the size R. Using now (5.27) of [20] we can then express $D^{(s)}(u)$ in terms of the rapidity $\bar{x}(\bar{u}) = \frac{1}{2}\bar{u} + \frac{1}{2}\bar{u}\sqrt{1 - \frac{1}{\bar{u}^2}}$ (defined in the second of (4.9)) as

$$D^{(s)}(u) = -\sqrt{2}g \ln g \left[2\bar{x}(\bar{u})\sqrt{1 - \frac{1}{4\bar{x}^2(\bar{u})}} - \arctan\left(2\bar{x}(\bar{u})\sqrt{1 - \frac{1}{4\bar{x}^2(\bar{u})}} \right) \right] + O(g). \tag{6.15}$$

6.2. Gluons

Perturbative and giant hole regimes   We start from the formula

$$2 \ln s \, P^{(g)}(u) + 2D^{(g)}(u) = \int \frac{dv}{2\pi} [\chi(v, u|1) + \chi(-v, u|1)] \frac{1}{2}\sigma_4(v)|_{NE}. \tag{6.16}$$

Scaling $u = \sqrt{2}g\bar{u}$ and $v = \sqrt{2}g\bar{v}$, we have that $\chi(v, u|1) + \Phi(v)$ is $O(1/g)$ if $|\bar{u}| < 1$ and (at most) $O(g^0)$ if $|\bar{u}| > 1$. Referring to (6.12), (6.13), we remark that the integration receives leading contribution from the region $|\bar{v}| > 1$. We conclude that

$$|\bar{u}| < 1 \quad \Rightarrow \quad 2 \ln s \, P^{(g)}(u) + 2D^{(g)}(u) = 2 \ln \frac{s}{g} \, P^{(g)}(u) + O(g^0), \tag{6.17}$$

$$|\bar{u}| > 1 \quad \Rightarrow \quad 2 \ln s \, P^{(g)}(u) + 2D^{(g)}(u) = 2 \ln \frac{s}{g} \, P^{(g)}(u) + O(g). \tag{6.18}$$

We can now refer to formulæ of [20] for the momentum of the gauge field and arrive at the final expressions

$$|\bar{u}| < 1 \quad \Rightarrow \quad D^{(g)}(u) = -\ln g \, P^{(g)}(u) + O(g^0)$$
$$= -\frac{\ln g}{\sqrt{2}} \left[\left(\frac{1 + \bar{u}}{1 - \bar{u}} \right)^{\frac{1}{4}} - \left(\frac{1 - \bar{u}}{1 + \bar{u}} \right)^{\frac{1}{4}} \right] + O(g^0), \tag{6.19}$$

$$|\bar{u}| > 1 \quad \Rightarrow \quad D^{(g)}(u) = -\ln g \, P^{(g)}(u) + O(g)$$
$$= -\sqrt{2}g \ln g \left[2\bar{x}(\bar{u})\sqrt{1 - \frac{1}{4\bar{x}^2(\bar{u})}} - \arctan\left(2\bar{x}(\bar{u})\sqrt{1 - \frac{1}{4\bar{x}^2(\bar{u})}} \right) \right] + O(g), \tag{6.20}$$

$\bar{x}(\bar{u})$ being defined in the second of (4.9).

6.3. Fermions

Perturbative regime   In this regime the scaled rapidity needs to satisfy $|\bar{x}_f(\bar{u})| < 1/2$, so that it belongs to a small fermion. Therefore, we start from the formula

$$2 \ln s \, P^{(f)}(u) + 2D^{(f)}(u) = -\int \frac{dv}{2\pi} [\chi_H(v, u) + \chi_H(-v, u)] \frac{1}{2} \frac{d}{dv} Z_4(v)|_{NE}. \tag{6.21}$$

Scaling $v = \sqrt{2}g\bar{v}$, we have that $\chi_H(v, u)$ is $O(1/g)$ if $|\bar{v}| > 1$ and $O(g^0)$ if $|\bar{v}| < 1$. Referring to (6.12), (6.13), we remark that the integration receives leading contribution from the region $|\bar{v}| > 1$. We conclude that

$$2\ln s\, P^{(f)}(u) + 2D^{(f)}(u) = 2\ln\frac{s}{g}\, P^{(f)}(u) + O(g^0) \tag{6.22}$$

and

$$D^{(f)}(u) = -\ln g\, P^{(f)}(u) + O(g^0) = -2\ln g\,\frac{\bar{x}_f(\bar{u})}{\sqrt{1 - 4\bar{x}_f(\bar{u})^2}} + O(g^0)\,. \tag{6.23}$$

Giant hole regime On the contrary, here the scaled rapidity is $|\bar{x}_F(\bar{u})| > 1/2$, so to characterise a large fermion. Therefore, we start from the formula

$$2\ln s\ P^{(F)}(u) + 2D^{(F)}(u) = \int \frac{dv}{2\pi}[\chi_F(v, u) + \chi_F(-v, u)]\frac{1}{2}\sigma_4(v)|_{NE}\,. \tag{6.24}$$

Scaling $v = \sqrt{2}g\bar{v}$, we now have that $\chi_H(v, u)$ is $O(g^0)$ for all $|\bar{v}|$. Referring to (6.12), (6.13), we remark that the integration receives leading contribution from the region $|\bar{v}| > 1$. We conclude that

$$2\ln s\, P^{(F)}(u) + 2D^{(F)}(u) = 2\ln\frac{s}{g}\ P^{(F)}(u) + O(g) \tag{6.25}$$

and, consequently, that

$$
\begin{aligned}
D^{(F)}(u) &= -\ln g\, P^{(F)}(u) \\
&= -\sqrt{2}g\ln g\left[2\bar{x}(\bar{u})\sqrt{1 - \frac{1}{4\bar{x}^2(\bar{u})}} - \arctan\left(2\bar{x}(\bar{u})\sqrt{1 - \frac{1}{4\bar{x}^2(\bar{u})}}\right)\right] \\
&\quad + O(g)\,.
\end{aligned}
\tag{6.26}
$$

7. The SU(4) symmetry

The particles we are addressing to (scalars, gluons, fermions and anti-fermions) belong to a specific multiplet under the $SU(4)$ symmetry (**6**, **1**, **4** and $\bar{\mathbf{4}}$, respectively). In fact, the scattering matrix possess this symmetry. Starting from the scattering matrices derived in the previous section, the Bethe equations may be assembled for every sort of excitation; anyway, they are actually able to catch only a single state in each multiplet, precisely the one corresponding to the highest weight state of the representation. In this section the focus moves to a few sectors of the complete theory, which include just one type (or two at most) of excitations along with the set of isotopic roots, aiming at elucidating the behaviour of the different kinds of particle under $SU(4)$.

Following [44], a set of Bethe equations can be formulated for any spin chain associated to a simple Lie algebra. Therefore, given the set of simple roots of a simple Lie algebra $\{\alpha_q\}$, and chosen a representation ρ by fixing its highest weight $\vec{w}_\rho$ (or equivalently a tern of positive integer Dynkin labels), the relative Bethe equations arise, with a further generalisation stemming from the introduction of a set of inhomogeneities (labelled by their rapidities u_m) along the spin chain:

$$\prod_m \frac{u_{q,k} - u_m + i\vec{\alpha}_q \cdot \vec{w}_\rho}{u_{q,k} - u_m - i\vec{\alpha}_q \cdot \vec{w}_\rho}$$
$$= \prod_{j \neq k}^{K_q} \frac{u_{q,k} - u_{q,j} + i\vec{\alpha}_q \cdot \vec{\alpha}_q}{u_{q,k} - u_{q,j} - i\vec{\alpha}_q \cdot \vec{\alpha}_q} \prod_{q' \neq q} \prod_{j=1}^{K_{q'}} \frac{u_{q,k} - u_{q',j} + i\vec{\alpha}_q \cdot \vec{\alpha}_{q'}}{u_{q,k} - u_{q',j} - i\vec{\alpha}_q \cdot \vec{\alpha}_{q'}}. \tag{7.1}$$

Turning to the $su(4)$ algebra, we perform a choice of three simple roots $\vec{\alpha}_k$, along with the three simple roots $\vec{\varphi}_k$, resulting from the defining condition $\frac{2\vec{\alpha}_j \cdot \vec{\varphi}_k}{(\vec{\alpha}_j)^2} = \delta_{kj}$:

$$\vec{\alpha}_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right) \qquad\qquad \vec{\varphi}_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}} \right)$$

$$\vec{\alpha}_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0 \right) \qquad\qquad \vec{\varphi}_2 = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{6}} \right)$$

$$\vec{\alpha}_3 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{2}{\sqrt{6}} \right) \qquad\qquad \vec{\varphi}_3 = \left(0, 0, \frac{3}{2\sqrt{6}} \right) \qquad . \tag{7.2}$$

To sum up, the Bethe equations in (7.1) specialise to the $su(4)$ algebra:

$$\prod_m \frac{u_{a,k} - u_m + i\vec{\alpha}_1 \cdot \vec{w}_\rho}{u_{a,k} - u_m - i\vec{\alpha}_1 \cdot \vec{w}_\rho} = \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - i/2}{u_{a,k} - u_{b,j} + i/2}$$

$$\prod_m \frac{u_{b,k} - u_m + i\vec{\alpha}_2 \cdot \vec{w}_\rho}{u_{b,k} - u_m - i\vec{\alpha}_2 \cdot \vec{w}_\rho}$$
$$= \prod_{j \neq k}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - i/2}{u_{b,k} - u_{a,j} + i/2} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - i/2}{u_{b,k} - u_{c,j} + i/2}$$

$$\prod_m \frac{u_{c,k} - u_m + i\vec{\alpha}_3 \cdot \vec{w}_\rho}{u_{c,k} - u_m - i\vec{\alpha}_3 \cdot \vec{w}_\rho} = \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - i/2}{u_{c,k} - u_{b,j} + i/2} \tag{7.3}$$

- **Scalar sector:**

When considering a system composed only of H scalar excitations with rapidities $\{u_h\}$, together with K_a roots u_a, K_b roots u_b and K_c roots u_c, the equations for the isotopic roots (2.49), (2.50), (2.53) take the form:

$$1 = \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}}$$

$$\prod_{h=1}^{H} \left(\frac{u_{b,k} - u_h + \frac{i}{2}}{u_{b,k} - u_h - \frac{i}{2}} \right) = \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}}$$

$$1 = \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}} \tag{7.4}$$

A comparison with (7.3) promptly reveals that equations (7.4) coincide with those for a spin chain associated to the antisymmetric (**6**) representation of $su(4)$, whose highest weight is, ac-

cording to our convention, $\vec{w}_6 = \vec{\varphi}_2$. The H hole rapidities (u_h, $h = 1, \ldots, H$) can be read as inhomogeneities along the spin chain, and their dynamics are regulated by equations (2.73), suitably adapted to the case at hand.

• **(Large) fermionic sector**

Let us stick now to a system composed of N_F large fermions $u_{F,j}$, $j = 1, \ldots, N_F$, together with K_a roots u_a, K_b roots u_b and K_c roots u_c. While the fermions satisfy Bethe equations (2.81), the auxiliary roots obey relations:

$$\prod_{j=1}^{N_F} \left(\frac{u_{a,k} - u_{F,j} + \frac{i}{2}}{u_{a,k} - u_{F,j} - \frac{i}{2}} \right) = \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}}$$

$$1 = \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}} \tag{7.5}$$

A look at (7.3) suggests equations (7.5) should be associated to a spin chain related to the fundamental representation (**4**) of $su(4)$ (with highest weight $\vec{w}_4 = \vec{\varphi}_1$), where the large fermions behave as inhomogeneities, with rapidities $u_{F,j}$, $j = 1, \ldots, N_F$.

Otherwise, when only large antifermions (in number of $N_{\bar{F}}$) appear in the vacuum, again accompanied by K_a isotopic roots u_a, K_b roots u_b and K_c roots u_c, the system is described by the set of Bethe equations (2.84) together with the isotopic roots equations:

$$1 = \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}}$$

$$\prod_{j=1}^{N_{\bar{F}}} \left(\frac{u_{c,k} - u_{\bar{F},j} + \frac{i}{2}}{u_{c,k} - u_{\bar{F},j} - \frac{i}{2}} \right) = \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}}. \tag{7.6}$$

Where (7.6) are in fact the equations for $\bar{\mathbf{4}}$ spin chain (highest weight $\vec{w}_{\bar{4}} = \vec{\varphi}_3$), as may be read from (7.3).

Some interest should be paid to a system including both N_F (large) fermions and $N_{\bar{F}}$ (large) antifermions; in this case, the isotopic roots satisfy the relations:

$$\prod_{j=1}^{N_F} \left(\frac{u_{a,k} - u_{F,j} + \frac{i}{2}}{u_{a,k} - u_{F,j} - \frac{i}{2}} \right) = \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}}$$

$$\prod_{j=1}^{N_{\bar{F}}} \left(\frac{u_{c,k} - u_{\bar{F},j} + \frac{i}{2}}{u_{c,k} - u_{\bar{F},j} - \frac{i}{2}} \right) = \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}}. \tag{7.7}$$

On the basis of $su(4)$ simple roots and fundamental weights (7.2), we can claim that equations (7.7) are associated to a spin chain, related to the representation of $su(4)$ whose Dynkin labels are $(1, 0, 1)$; in other terms, we found as its highest weight $\vec{w}_{15} = \vec{\varphi}_1 + \vec{\varphi}_3$, and that leads to the **15**. The reason lies in the way how fermions (in the **4**) and antifermions (in the $\bar{\mathbf{4}}$) scatter, since the process can be decomposed into two channels, according to the rule

$$\mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{1} \oplus \mathbf{15}; \tag{7.8}$$

the singlet **1** channel is not explicitly appearing in (7.7), but it can be revealed upon imposing some constraints on the isotopic roots (see next section).

We eventually remark that analogous considerations and expressions hold if we replace large fermions/antifermions with small fermions/antifermions.

- **Gauge field sector**

When only N_g gluons (with rapidities u_j^g) are excited over the vacuum, the isotopic roots decouple from them, since

$$1 = \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}}$$

$$1 = \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}}. \tag{7.9}$$

Therefore, gluon excitations behave like singlets (**1**) under $SU(4)$. The very same reasoning applies to barred-gluons.

8. Eigenvalues

While commenting on equations (7.7), we hinted the role the $SU(4)$ symmetry takes in the scattering between fermions and antifermions. Now we are going to examine in some more detail several scattering processes involving different kinds of particles. In general, given two types of particles α and β, transforming under the representations of $su(4)$ ρ_α and ρ_β, which act respectively on the spaces V_α and V_β, their scattering decomposes according to the Clebsch–Gordan rule

$$\rho_\alpha \otimes \rho_\beta = \bigoplus_\Omega \rho_\Omega. \tag{8.1}$$

Recalling [44], the scattering matrix $\hat{S}^{(\alpha\beta)}$ (defined on $V_\alpha \otimes V_\beta$) between excitations α and β enjoys the spectral decomposition

$$\hat{S}^{(\alpha\beta)} = \sum_\Omega S_\Omega^{(\alpha\beta)} P_\Omega, \tag{8.2}$$

where $S_\Omega^{(\alpha\beta)}$ are the eigenvalues of the matrix $\hat{S}^{(\alpha\beta)}$, relatives to the (normalised) eigenvectors P_Ω, which act as projectors onto the space V_Ω, *i.e.* $P_\Omega(V_\alpha \otimes V_\beta) = V_\Omega$. In this section we list the eigenvalues corresponding to the scattering between excitations on the top of the GKP string.

Scalar–scalar The scalar–scalar scattering was completely clarified in [22], here we list eigenvalues and corresponding isotopic roots; since scalars belong to the **6**, the decomposition follows:

$$6 \otimes 6 = 1 \oplus 15 \oplus 20. \tag{8.3}$$

The singlet **1** channel involves two type-b isotopic roots, which shall be related to the hole rapidities u_h, $u_{h'}$ according to

$$u_{b,1} = \frac{1}{2} \left(u_h + u_{h'} - \sqrt{\frac{1+(u_h-u_{h'})^2}{3}} \right) \quad \text{and} \quad u_{b,2} = \frac{1}{2} \left(u_h + u_{h'} + \sqrt{\frac{1+(u_h-u_{h'})^2}{3}} \right),$$

together with a and c roots $u_a = u_c = \frac{u_h+u_{h'}}{2}$. These constraints on the isotopic roots lead us to the eigenvalue

$$S_1^{(ss)}(u_h, u_{h'}) = \frac{u_h - u_{h'} + 2i}{u_h - u_{h'} - 2i} \frac{u_h - u_{h'} + i}{u_h - u_{h'} - i} S^{(ss)}(u_h, u_{h'}), \tag{8.4}$$

where the scalar factor $S^{(ss)}(u_h, u_{h'})$ can be read from (C.10). The adjoint channel $\Omega = \mathbf{15}$ requires one b-type root, satisfying $u_b = \frac{u_h+u_{h'}}{2}$ and no a nor c roots $K_a = K_c = 0$. Eventually, the resulting eigenvalue follows

$$S_{15}^{(ss)}(u_h, u_{h'}) = \frac{u_h - u_{h'} + i}{u_h - u_{h'} - i} S^{(ss)}(u_h, u_{h'}). \tag{8.5}$$

Finally, the $\Omega = \mathbf{20}$ channel request no isotopic roots ($K_a = K_b = K_c = 0$), so that the eigenvalue simply coincides with (C.10)

$$S_{20}^{(ss)}(u_h, u_{h'}) = S^{(ss)}(u_h, u_{h'}). \tag{8.6}$$

Fermion–fermion In the fermion–fermion scattering, we have two eigenvalues corresponding to the decomposition $4 \otimes 4 = 10 \oplus 6$. The first one, for $\Omega = \mathbf{10}$, corresponds to no isotopic roots and therefore it holds

$$S_{10}^{(FF)}(u_{F,1}, u_{F,2}) = S^{(FF)}(u_{F,1}, u_{F,2}),$$

where the scalar factor corresponds to (C.11). The second one, for the $\Omega = \mathbf{6}$ channel, is obtained from the solution with $K_a = 1$, $K_b = 2$, $K_c = 0$, such that $u_a = \frac{u_{F,1}+u_{F,2}}{2}$, while $u_{b,1} = u_{F,1}$ and $u_{b,2} = u_{F,2}$; consequently, we find that:

$$S_6^{(FF)}(u_{F,1}, u_{F,2}) = \frac{u_{F,1} - u_{F,2} + i}{u_{F,1} - u_{F,2} - i} S^{(FF)}(u_{F,1}, u_{F,2}).$$

Fermion–antifermion As previously mentioned, the fermion–antifermion scattering is associated to the decomposition $4 \otimes \bar{4} = 15 \oplus 1$. Turning to the $\Omega = \mathbf{15}$ channel, no isotopic roots are involved, therefore the eigenvalue equals the scalar factor:

$$S_{15}^{(F\bar{F})}(u_{F,1}, u_{\bar{F},1}) = S^{(FF)}(u_{F,1}, u_{\bar{F},1}). \tag{8.7}$$

The singlet channel instead is obtained from the solution with $K_a = K_b = K_c = 1$, where the isotopic roots satisfy the constraints $u_a = \frac{3}{4}u_{F,1} + \frac{1}{4}u_{\bar{F},1}$, $u_{b,1} = \frac{1}{2}u_{F,1} + \frac{1}{2}u_{\bar{F},1}$ and $u_{c,1} =$

$\frac{1}{4}u_{F,1} + \frac{3}{4}u_{\bar{F},1}$. As a consequence, we obtain as the eigenvalue for the $\Omega = \mathbf{1}$ channel:

$$S_{\mathbf{1}}^{(F\bar{F})}(u_{F,1}, u_{\bar{F},1}) = \frac{u_{F,1} - u_{\bar{F},1} + 2i}{u_{F,1} - u_{\bar{F},1} - 2i} \, S^{(FF)}(u_{F,1}, u_{\bar{F},1}) \,. \tag{8.8}$$

The same result, *i.e.*

$$S_{\mathbf{1}}^{(f\bar{f})}(u_{f,1}, u_{\bar{f},1}) = \frac{u_{f,1} - u_{\bar{f},1} + 2i}{u_{f,1} - u_{\bar{f},1} - 2i} \, S^{(ff)}(u_{f,1}, u_{\bar{f},1}) \,, \tag{8.9}$$

holds for small fermions. We highlighted (8.9), since this factor is responsible of the appearance of a mass two particle at $g = +\infty$. This issue will be discussed in the next section.

9. Classification of possible bound states

9.1. String solutions at large size

In the large size limit, $R \to +\infty$, solutions to Bethe Ansatz equations show many (numerical and analytic) evidences that they organise into strings or stacks (generalised strings with different isospin or nested degrees of freedom). Their derivation follows as customary [55]. Let a complex rapidity u_k^* exists, whose imaginary part be different from zero, but with sign so that the factor $e^{iRP(u_k^*)}$ goes to zero (infinity) in the large R limit: then another rapidity $u_j^{*'}$ must exist with the same real part, but imaginary part lowered if $\text{Im}\,u_k^* > 0$ or, otherwise, raised if $\text{Im}\,u_k^* < 0$, by an appropriate quantity, in order to drive rational factors in $S^{**'}(u_k^*, u_j^{*'})$ to infinity (zero), thus balancing the ABA equations. The process can continue by involving a finite number of extra rapidities displaced at regular distances until a string of m roots disposed around a real 'center' is closed. Since the 'wave function' of a string of m roots is by construction rapidly decreasing at $\pm\infty$, we naturally associate this configuration with a bound state of m 'elementary' excitations.

In this section we discuss some possible bound states, with the important caveat that the list below is not meant to provide a complete classification of the particles living in the theory. This is indeed an interesting problem in itself and will be possibly dealt with in a future publication.

We also remark that, strictly speaking, the complexes of solutions we provide below are meant to be valid for finite values of the coupling constant, *i.e.* $g \neq 0$ and $g < \infty$. At $g = +\infty$ the situation ought to be different, as it can be inferred from considerations on the classical (quadratic) string theory action. Indeed, its small fluctuations in the bosonic sector consists of two mass $= \sqrt{2}$ (real) bosons and one mass 2 (real) boson, besides the five massless bosons (of the $O(6)$ non-linear sigma model) [6,49]. Seemingly, this mass 2 boson degree of freedom is missing in the gauge theory,[13] but, in the following, we find evidence that, with this mass, there is indeed a composite state made up of a small fermion and a small antifermion in their singlet channel. By means of the string mechanism discussed before, this bound state cannot exist at *finite g*, since divergences (zeroes) of the phase factor $e^{iRP^{(f)}(u_f)}$ of small fermions for complex rapidities go together with divergences (zeroes) of the S matrices, instead of compensating each other: thus this is a *resonance*. Indeed, after specialising the isotopic roots to the values defining the singlet channel of formula (8.9) and after switching off for simplicity's sake all the excitations (dots in the equation below) but a fermion and an antifermion, we have to cope with the coupled

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[13] Before us, many authors shared this concern, as, for instance, [49,20,51,36].

equations

$$1 = e^{i R P^{(f)}(u_{f,1}) + 2i D^{(f)}(u_{f,1})} \frac{u_{f,1} - u_{\bar{f},1} + 2i}{u_{f,1} - u_{\bar{f},1} - 2i} \, S^{(ff)}(u_{f,1}, u_{\bar{f},1}) \ldots,$$

$$1 = e^{i R P^{(f)}(u_{\bar{f},1}) + 2i D^{(f)}(u_{\bar{f},1})} \frac{u_{\bar{f},1} - u_{f,1} + 2i}{u_{\bar{f},1} - u_{f,1} - 2i} \, S^{(ff)}(u_{\bar{f},1}, u_{f,1}) \ldots . \qquad (9.1)$$

Thus, we can verify that the candidate string-like solutions $u_{f,1} = u_{M,1} \pm i$, $u_{\bar{f},1} = u_{M,1} \mp i$, and related isotopic rapidities of the singlet (8.9), with $u_{M,1}$ real, cannot satisfy (at large R) equations (9.1), since Im $P^{(f)}(u_f) > 0$ if Im $u_f < 0$ and viceversa.[14] However, this remains true as long as g is finite, while this bound state can appear as a new 'particle' when the value of g is strictly $+\infty$. In fact, the point $g = +\infty$ is rather peculiar and singular, as complex scaled rapidities ($\bar{u} = u/\sqrt{2}g$) all collapse into the real axis, thus making possible a solution of ABA equations with a stack with two (small) fermion–antifermion rapidities (besides the isotopic rapidities): the two constituents possess no binding energy and no 'breathing', but they are just 'one on the other'. This is indeed a new (real) scalar, named here 'meson', coming to life only in the classical string regime $g = +\infty$.[15] In summary, our following analysis of the ABA scattering on the GKP vacuum shows evidence for the existence of this bosonic particle with mass 2 as long as $g \to +\infty$. And not only: the same mechanism at $g \to +\infty$ sustains the existence of a bound state of $k = 1, 2, 3, \ldots$ mesons with mass $m_k = 2k$ (zero static binding energy, as well as for gluon bound states). This is a new bosonic sector with respect to the classical (quadratic) string spectrum (and, *a fortiori*, to previous gauge theory analyses), but yet indispensable to be considered in the BSV series for 4D amplitudes, – as we shall see –, for making checks with and reproducing the string minimal area solution (in other words the Thermodynamic Bubble Ansatz (TBA)). On the other end if this is an important way to check the validity of the series, it also confirms the pentagonal amplitude values and the 2D scattering factors entailing them. Eventually, the formation of mesons and bound states thereof shows a sort of *confinement* phenomenon at strong coupling as for the 4D amplitudes/Wilson loops, in that the contribution of the constituents, the fermions, to them is subtlety sub-dominant (as $g \to +\infty$, *cf.* [36] and below). In fact, this negligibility is not true for the 2D scattering amplitudes in themselves, but in their contribution to the 4D ones.

A more mathematical understanding of the small fermion–antifermion state ought to arrive [58] from the collision of the poles into the integration (real) axis [59–62,36]: this will give us the opportunity to explain the meson bound states and hence the *confinement* under a different light.[16]

• **Gluonic strings:**

A first example is provided by strings made up of gluons or, alternatively, barred gluons, as equations (E.6) and (E.7) suggest. In this case one remarks the emergence of complex of solutions characterised by length m and real centre $u_k^{g,m}$:

$$u_{k'}^{g,m} = u_k^{g,m} + \frac{i}{2}(m - 1 - 2k'), \quad k' = 0, \ldots, m-1 ; \qquad (9.2)$$

the very same structure may be built by assembling barred-gluon rapidities, too. We will study more extensively bound states of gluons in next subsection, where we will show that they can

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[14] We take up the occasion to highlight that we have found in the previous part of the paper that this opposition of signs happens only for the small fermions

[15] Only at this value its rapidity, otherwise virtual [56], enters the physical domain [57,56].

[16] We are particularly grateful to I. Kostov and J.-E. Bourgine for explanatory discussions on this point.

be also obtained starting from the BMN vacuum by considering stacks of roots of types 1, 2 and 3.

● **More bound states:**

Along with gluonic strings, whose structure is quite ordinary, the ABA equations also admit the existence of more peculiar kinds of complexes whose composition and length result completely determined by the SU(4) symmetry of the vacuum. In fact, the structure of the Bethe equations for the SU(4) spin chain (7.1), reflected in the equations for auxiliary roots (E.8), (E.9), (E.10), prevents these strings from including more than two massive roots (or exceptionally three, as in (9.7) below), intertwined with isotopic roots which are spaced by a constant distance fully fixed by (7.1). The presence of isotopic roots is necessary for these strings to effectively represent solutions of the ABA equations and live in some definite scattering channel (see Section 8), thus behaving in a broad sense like bound states which belong to some SU(4) 'isospin' multiplet. Below, such peculiar strings are listed according to their composition and SU(4) behaviour.

● **Bound states of large fermions in the 6 channel:**

$$u_{F,k,\pm} = u_k \pm \frac{i}{2} \qquad u_{a,k} = u_k \qquad u_{b,k,\pm} = u_k \pm \frac{i}{2};\qquad(9.3)$$

the same structure occurs with antifermions too, upon substituting fermions with antifermions and the central a-root with a c-root.

● **Bound states of large fermions in the singlet channel[17]:**

$$u_{F,k} = u_{M,k} + i \qquad u_{a,k} = u_{M,k} + \frac{i}{2} \qquad u_{b,k} = u_{M,k}$$

$$u_{\bar{F},k} = u_{M,k} - i \qquad u_{c,k} = u_{M,k} - \frac{i}{2} \qquad\qquad (9.4)$$

(the complex conjugate of (9.4) is a solution, too).

● **Bound states of scalars in the 15 channel:**

$$u_{h,k,\pm} = u_k \pm \frac{i}{2} \qquad u_{b,k} = u_k\ .\qquad(9.5)$$

● **Bound states of scalars in the singlet channel:**

$$u_{h,k,\pm} = u_k \pm i \qquad u_{b,k,\pm} = u_k \pm \frac{i}{2} \qquad u_{a,k} = u_{c,k} = u_k\ ;\qquad(9.6)$$

it is important to point out that these strings made of holes do not survive to the strong coupling limit in the non-perturbative regime, as they are destroyed by poles of (4.1). Indeed, it is a well known result that in this regime the scalar dynamics is regulated by the O(6) non-linear σ-model, which lacks bound states.

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[17] Anyway we remark that bound states of this sort do not play any role in the strong coupling perturbative regime and, noticeably, their centres need to lie on the real axis in the region $|u_{M,k}| < \sqrt{2}g$, hence inside a square root branch cut in the large fermionic u-rapidity plane, so that perhaps they should not even be considered physical. It is thus far from being obvious that any relation exist with what in the following we will refer to as 'meson' bound states, which exclusively subsist at $g = \infty$ and are made of small fermions, instead.

• **Mixed bound states of large fermions and gluons in the 10 channel:**

$$u_{F,k,\pm} = u_k \pm \frac{i}{2} \qquad u_k^g = u_k .$$
(9.7)

(the same also holds for barred-gluons and large antifermions).[18]

• **(Purely) magnonic strings:**

Also three distinct kinds of massless strings made of isotopic roots only, one for each type, can be found:

$$u_{a,k,j}^A = u_{a,k}^A + \frac{i}{2}(A - 1 - 2j), \quad j = 0, \ldots, A - 1 ;$$

$$u_{b,k,j}^B = u_{b,k}^B + \frac{i}{2}(B - 1 - 2j), \quad j = 0, \ldots, B - 1 ;$$
(9.8)

$$u_{c,k,j}^C = u_{c,k}^C + \frac{i}{2}(C - 1 - 2j), \quad j = 0, \ldots, C - 1 .$$

9.2. Bound states of gluons

On the BMN vacuum with a sea of u_4 roots bound states of excitations $F_{+\perp}$ with rapidity $u_k^{g,m}$ can be constructed [20] as stacks involving type 1, type 2 and type 3 roots:

$$u_{1,k} = u_k^{g,m} + \frac{i}{2}(m - 2 - 2k'), \quad k' = 0, \ldots, m - 2$$

$$u_{2,k} = u_k^{g,m} + \frac{i}{2}(m - 1 - 2k'), \quad k' = 0, \ldots, m - 1$$
(9.9)

$$u_{3,k} = u_k^{g,m} + \frac{i}{2}(m - 2k'), \quad k' = 0, \ldots, m .$$

Analogously, bound states of gauge fields $\bar{F}_{+\perp}$ with rapidity $u_k^{\bar{g},m}$ are obtained from (9.9), with $g \to \bar{g}$ and $u_1, u_2, u_3 \to u_5, u_6, u_7$. In presence of bound states of gluons Bethe equations should be modified as follows.

Bethe equations for bound states of $F_{+\perp}$ (N_g^l ($N_{\bar{g}}^l$) is the number of bound states of $F_{+\perp}$ ($\bar{F}_{+\perp}$) with length l: their centers are indicated with $u_j^{g,l}$ ($u_j^{\bar{g},l}$)) are

$$1 = e^{i R P_m^{(g)}(u_k^{g,m}) + 2i D_m^{(g)}(u_k^{g,m})} \prod_{l=1}^{+\infty} \prod_{j=1}^{N_g^l} S_{ml}^{(gg)}(u_k^{g,m}, u_j^{g,l}) \prod_{l=1}^{+\infty} \prod_{j=1}^{N_{\bar{g}}^l} S_{ml}^{(g\bar{g})}(u_k^{g,m}, u_j^{\bar{g},l})$$

$$\cdot \prod_{h=1}^{H} S_m^{(gs)}(u_k^{g,m}, u_h) \prod_{j=1}^{N_F} S_m^{(gF)}(u_k^{g,m}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S_m^{(g\bar{F})}(u_k^{g,m}, u_{\bar{F},j})$$

$$\cdot \prod_{j=1}^{N_f} S_m^{(gf)}(u_k^{g,m}, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S_m^{(g\bar{f})}(u_k^{g,m}, u_{\bar{f},j}),$$
(9.10)

---

[18] In addition to the string configurations listed above, several further complexes of solutions could be found, although strictly speaking they should not be considered actual bound states, since they are not endowed with real valued momenta; an example is offered by strings made of one single scalar and one fermion (or antifermion) bound together, whose distance gets fixed by the SU(4) symmetry to $\frac{3i}{2}$, and which could be probably related to a similar state described in [36].

where momentum and defect are given by

$$P_m^{(g)}(u) = -\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\chi(v,u|m) + \chi(-v,u|m)\right]\left[1 - \frac{\sigma_{BES}(v)}{4}\right], \tag{9.11}$$

$$2D_m^{(g)}(u) = -\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\chi(v,u|m) + \chi(-v,u|m)\right]\frac{d}{dv}\tilde{P}(v) \tag{9.12}$$

and the various scattering factors are listed in Appendix C.

Exchanging g with $\bar{g}$ we get Bethe equations for bound states of $\bar{F}_{+\perp}$. The other equations can be obtained from the equations written when simple gluons are present (and collected in Appendix E) by means of the replacements:

$$\prod_{j=1}^{N_g} S^{*g}(u_*, u_j^g) \rightarrow \prod_{l=1}^{+\infty}\prod_{j=1}^{N_g^{(l)}} S_l^{(*g)}(u_*, u_j^{g,l}),$$

$$\prod_{j=1}^{N_{\bar{g}}} S^{*\bar{g}}(u_*, u_j^{\bar{g}}) \rightarrow \prod_{l=1}^{+\infty}\prod_{j=1}^{N_{\bar{g}}^{(l)}} S_l^{(*\bar{g})}(u_*, u_j^{\bar{g},l}). \tag{9.13}$$

We now show that equations (9.10) and others, which constrain centers of the string (9.9), are not independent of equations describing excitations on the GKP vacuum, but actually can be obtained from these by considering strings involving gluons.

In order to obtain (9.10), we shall first consider equations (2.92) for gluons,

$$1 = e^{iRP^{(g)}(u_{k'}^g)+2iD^{(g)}(u_{k'}^g)} \prod_{j=1,\neq k'}^{N_g} \frac{u_{k'}^g - u_j^g + i}{u_{k'}^g - u_j^g - i} S_{red}^{(gg)}(u_{k'}^g, u_j^g)$$

$$\cdot \prod_{j=1}^{N_F} \frac{u_{k'}^g - u_{F,j} + \frac{i}{2}}{u_{k'}^g - u_{F,j} - \frac{i}{2}} S_{red}^{(gF)}(u_{k'}^g, u_{F,j}) \prod_{j=1}^{N_f} \frac{u_{k'}^g - u_{f,j} + \frac{i}{2}}{u_{k'}^g - u_{f,j} - \frac{i}{2}} S_{red}^{(gf)}(u_{k'}^g, u_{f,j})$$

$$\cdot \prod_{h=1}^{H} S^{(gs)}(u_{k'}^g, u_h) \prod_{j=1}^{N_{\bar{g}}} S^{(g\bar{g})}(u_{k'}^g, u_j^{\bar{g}}) \prod_{j=1}^{N_{\bar{F}}} S^{(g\bar{F})}(u_{k'}^g, u_{\bar{F},j}) \prod_{j=1}^{N_{\bar{f}}} S^{(g\bar{f})}(u_{k'}^g, u_{\bar{f},j}), \tag{9.14}$$

where S_{red} (C.13), (C.19) stand for the S factors deprived of the rational factors $-e^{\pm i\chi_0}$. The appearance of the rational factor

$$\frac{u_{k'}^g - u_j^g + i}{u_{k'}^g - u_j^g - i} \tag{9.15}$$

suggests that strings of gluons rapidities with the same real part and imaginary parts separated of i form. In specific,

$$u_{k'}^g = u_k^{g,m} + \frac{i}{2}(m - 1 - 2k'), \quad k' = 0, \ldots, m-1, \tag{9.16}$$

$$u_{k'}^{\bar{g}} = u_k^{\bar{g},m} + \frac{i}{2}(m - 1 - 2k'), \quad k' = 0, \ldots, m-1, \tag{9.17}$$

where the real centers of the strings are in the region $|u_k^{g,m}| < \sqrt{2}g$, $|u_k^{\bar{g},m}| < \sqrt{2}g$.

Performing the products over k' we arrive at the equation

$$
1 = e^{iRP_m^{(g)}(u_k^{g,m})+2iD_m^{(g)}(u_k^{g,m})} \prod_{j=1}^{N_g} \frac{u_k^{g,m} - u_j^g + \frac{i}{2}(m+1)}{u_k^{g,m} - u_j^g - \frac{i}{2}(m+1)} \frac{u_k^{g,m} - u_j^g + \frac{i}{2}(m-1)}{u_k^{g,m} - u_j^g - \frac{i}{2}(m-1)}
$$

$$
\cdot S_{red,m}^{(gg)}(u_k^{g,m}, u_j^g) \prod_{j=1}^{N_F} \frac{u_k^{g,m} - u_{F,j} + \frac{im}{2}}{u_k^{g,m} - u_{F,j} - \frac{im}{2}} S_{red,m}^{(gF)}(u_k^{g,m}, u_{F,j})
$$

$$
\cdot \prod_{j=1}^{N_f} \frac{u_k^{g,m} - u_{f,j} + \frac{im}{2}}{u_k^{g,m} - u_{f,j} - \frac{im}{2}} S_{red,m}^{(gf)}(u_k^{g,m}, u_{f,j}) \prod_{h=1}^{H} S_m^{(gs)}(u_k^{g,m}, u_h)
$$

$$
\cdot \prod_{j=1}^{N_{\bar{g}}} S_m^{(g\bar{g})}(u_k^{g,m}, u_j^{\bar{g}}) \prod_{j=1}^{N_{\bar{F}}} S_m^{(g\bar{F})}(u_k^{g,m}, u_{\bar{F},j}) \prod_{j=1}^{N_{\bar{f}}} S_m^{(g\bar{f})}(u_k^{g,m}, u_{\bar{f},j}), \tag{9.18}
$$

where

$$
S_{red,m}^{(g*)}(u,v) = \prod_{l=-\frac{m-1}{2}}^{\frac{m-1}{2}} S_{red}^{(g*)}(u+il,v), \quad S_m^{(g*)}(u,v) = \prod_{l=-\frac{m-1}{2}}^{\frac{m-1}{2}} S_m^{(g*)}(u+il,v). \tag{9.19}
$$

It is now immediate to recognise the scattering factors between a bound state of $F_{+\perp}$ with center u and 'length' m and a fermion or a scalar:

$$
S_m^{(gF)}(u,v) = \frac{u-v+\frac{im}{2}}{u-v-\frac{im}{2}} S_{red,m}^{(gF)}(u,v), \quad S_m^{(gf)}(u,v) = \frac{u-v+\frac{im}{2}}{u-v-\frac{im}{2}} S_{red,m}^{(gf)}(u,v),
$$

$$
S_m^{(g\bar{F})}(u,v) = S_{red,m}^{(g\bar{F})}(u,v), \quad S_m^{(g\bar{f})}(u,v) = S_{red,m}^{(g\bar{f})}(u,v), \quad S_m^{(gs)}(u,v) = S_{red,m}^{(gs)}(u,v). \tag{9.20}
$$

By means of (C.35) one shows that these factors equal the ones appearing in (9.10). Equations (9.18) are then completed by taking into account that rapidities u_j^g, $u_j^{\bar{g}}$ appear into strings (9.16), (9.17). Because of the properties

$$
\prod_{k'=0}^{l-1} \frac{u_k^{g,m} - u_j^{g,l} - \frac{i}{2}(l-1-2k') + \frac{i}{2}(m+1)}{u_k^{g,m} - u_j^{g,l} - \frac{i}{2}(l-1-2k') - \frac{i}{2}(m+1)} \frac{u_k^{g,m} - u_j^{g,l} - \frac{i}{2}(l-1-2k') + \frac{i}{2}(m-1)}{u_k^{g,m} - u_j^{g,l} - \frac{i}{2}(l-1-2k') - \frac{i}{2}(m-1)}
$$

$$
\cdot S_{red,m}^{(gg)}(u_k^{g,m}, u_j^{g,l} + \frac{i}{2}(l-1-2k')) = S_{ml}^{(gg)}(u_k^{g,m}, u_j^{g,l}), \tag{9.21}
$$

$$
\prod_{k'=0}^{l-1} S_m^{(g\bar{g})}(u_k^{g,m}, u_j^{\bar{g},l} + \frac{i}{2}(l-1-2k')) = S_{ml}^{(g\bar{g})}(u_k^{g,m}, u_j^{\bar{g},l}) \tag{9.22}
$$

which follow from (C.35), one finally finds that equations (9.10) are reproduced. In analogous fashion, other expressions, derived from the BMN vacuum, can be reproduced, as well, starting from equations on the GKP vacuum.

The energy   We end this part by spending some words on the energy of the gluon and of its bound states (the same will apply to its barred companion). Immediately, we can put it in the general form at any g

$$E_m^{(g)}(u) = 2m - \int \frac{dv}{2\pi} \left[\chi(v, u|m) + \Phi(v) \right] \frac{d}{dv} \gamma^{(s)}(v), \tag{9.23}$$

which we rewrite as

$$E_m^{(g)}(u) = 2m + \int\limits_{-\infty}^{\infty} \frac{dk}{4\pi^2} \frac{i\pi}{e^{\frac{k}{2}} - e^{-\frac{k}{2}}} (\gamma_+^{\emptyset}(\sqrt{2}gk)$$

$$- \operatorname{sgn}(k)\gamma_-^{\emptyset}(\sqrt{2}gk)) \left[\frac{2\pi}{ik} e^{-|k|\frac{m+1}{2}} e^{-iku} \right.$$

$$- \frac{2\pi}{ik} e^{-\frac{|k|}{2}} \sum_{n=1}^{\infty} \left(\left(\frac{g}{\sqrt{2}ix(u + \frac{im}{2})} \right)^n + \left(\frac{g}{\sqrt{2}ix(u - \frac{im}{2})} \right)^n \right) J_n(\sqrt{2}gk)$$

$$\left. - \frac{2\pi}{ik} J_0(\sqrt{2}gk) e^{-\frac{|k|}{2}} \right]. \tag{9.24}$$

By means of the relation (3.40) in [20] and of the Fourier transform

$$\int\limits_{0}^{\infty} \frac{dk}{k} e^{-k(\frac{m}{2} \pm iu)} J_n(\sqrt{2}gk) = \frac{(\pm 1)^n}{n} \left(\frac{g}{\sqrt{2}ix(u \mp \frac{im}{2})} \right)^n, \tag{9.25}$$

the expression above becomes

$$E_m^{(g)}(u) = 2m + \int\limits_{0}^{\infty} \frac{dk}{k} \frac{\gamma_+^{\emptyset}(\sqrt{2}gk)}{1 - e^{-k}} [\cos ku \, e^{-k\frac{m}{2}} - 1]$$

$$- \int\limits_{0}^{\infty} \frac{dk}{k} \frac{\gamma_-^{\emptyset}(\sqrt{2}gk)}{e^k - 1} [\cos ku \, e^{-k\frac{m}{2}} - 1], \tag{9.26}$$

in agreement with [20].

Perturbative strong coupling regime   In this limit we can use the following results

$$\exp\left[-i\tilde{\chi}(u, v|m, l)\right] = \exp\left[\frac{\sqrt{2}ml}{g(\bar{v} - \bar{u})} + O(1/g^3) \right] \tag{9.27}$$

and

$$\chi(w, u|m) + \Phi(w) = \frac{m}{\sqrt{2}g(\bar{u} - \bar{w})} + O(1/g^2), \quad \text{when } |\bar{w}| > 1, \tag{9.28}$$

$$\chi(w, u|m) + \Phi(w) = O(1/g), \quad \text{when } |\bar{w}| < 1. \tag{9.29}$$

Repeating all the steps we did for one single gluon, we write the scattering factor between two bound states of m and l gluons, respectively, in the perturbative regime (up to terms $O(1/g^2)$) as

$$S_{ml}^{(gg)}(u, v) = \exp\left[\frac{iml}{\sqrt{2}g(\bar{u} - \bar{v})}\left(1 + \frac{1}{2}\left(\frac{1 + \bar{u}}{1 - \bar{u}}\right)^{1/4}\left(\frac{1 - \bar{v}}{1 + \bar{v}}\right)^{1/4}\right.\right.$$

$$\left.\left. + \frac{1}{2}\left(\frac{1 - \bar{u}}{1 + \bar{u}}\right)^{1/4}\left(\frac{1 + \bar{v}}{1 - \bar{v}}\right)^{1/4}\right)\right]. \tag{9.30}$$

For what concerns energy (9.26) and momentum (9.11) of a bound state of m gluons, we have the simple relativistic result

$$E_m^{(g)}(\theta) = 2m\cosh\theta, \qquad P_m^{(g)}(\theta) = 2m\sinh\theta, \tag{9.31}$$

in terms of the hyperbolic variable θ defined by $u = \sqrt{2}g\tanh 2\theta$. As already written, the results above apply to the barred gluon as well.

9.3. The meson and its bound states

As hinted before, at infinite coupling in the perturbative regime a bound state of a small fermion and a small anti-fermion arises. As anticipated, we will call this state 'meson'.

Now, we elaborate a proposal for the complete set of ABA equations satisfied by mesons. One meson is described by the stack (9.4) in which, of course, the index F is replaced by f, *i.e.*

$$u_{f,k} = u_{M,k} + i \qquad u_{a,k} = u_{M,k} + \frac{i}{2} \qquad u_{b,k} = u_{M,k}$$

$$u_{\bar{f},k} = u_{M,k} - i \qquad u_{c,k} = u_{M,k} - \frac{i}{2}. \tag{9.32}$$

We already emphasised that this stack does not give rise to a genuine bound state for finite g, but describes a virtual state: in this perspective, we can formally write the ABA equations describing the scattering between virtual states (9.32) with themselves and with other excitations at generic g. Of course, we should start by deducing – in the usual manner – the scattering between two virtual mesons:

$$S^{(MM)}(u, v) = \frac{u - v + i}{u - v - i}S^{(ff)}(u + i, v + i)S^{(ff)}(u - i, v + i)$$

$$\cdot S^{(ff)}(u + i, v - i)S^{(ff)}(u - i, v - i)$$

$$= \frac{u - v + i}{u - v - i}\exp\left\{i\int\frac{dw}{2\pi}\chi_M(w, u|1)\frac{d}{dw}\chi_M(w, v|1)\right.$$

$$\left. - i\int\frac{dw}{2\pi}\frac{dz}{2\pi}\chi_M(w, u|1)\frac{d^2}{dwdz}\Theta(w, z)\chi_M(z, v|1)\right\}, \tag{9.33}$$

with the definitions

$$\chi_M(v, u|m) = \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}}\chi_M(v, u + ik|1) \qquad \text{with } m \geq 1$$

$$\chi_M(v, u|1) = -\chi_H(v, u + i) - \chi_H(v, u - i). \tag{9.34}$$

Then, we can introduce also the other excitations and derive, at least formally, the ABA equations involving N_M mesons

$$1 = e^{i R P^{(M)}(u_{M,k})} \, e^{2i D^{(M)}(u_{M,k})} \prod_{h=1}^{H} S^{(Ms)}(u_{M,k}, u_h) \prod_{j=1}^{N_g} S^{(Mg)}(u_{M,k}, u_j^g)$$

$$\cdot \prod_{j=1}^{N_{\bar{g}}} S^{(M\bar{g})}(u_{M,k}, u_j^{\bar{g}}) \prod_{j=1}^{N_f} S^{(Mf)}(u_{M,k}, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(M\bar{f})}(u_{M,k}, u_{\bar{f},j})$$

$$\cdot \prod_{j=1, j \neq k}^{N_M} S^{(MM)}(u_{M,k}, u_{M,j}), \tag{9.35}$$

where $P^{(M)}(u) = P^{(f)}(u+i) + P^{(f)}(u-i)$, $D^{(M)}(u) = D^{(f)}(u+i) + D^{(f)}(u-i)$ and the scattering factors read

$$S^{(Ms)}(u,v) = \frac{u-v+\frac{i}{2}}{u-v-\frac{i}{2}} \exp\left\{ i\chi_M(v,u|1) - i\int \frac{dw}{2\pi}\frac{d\Theta}{dw}(v,w)\,\chi_M(w,u|1) \right\}, \tag{9.36}$$

$$S^{(Mg)}(u,v) = \frac{u-v+\frac{3i}{2}}{u-v+\frac{i}{2}} \exp\left\{ i\int \frac{dw}{2\pi}\chi_M(w,u|1)\frac{d}{dw}\left[\chi(w,v|1)+\Phi(x)\right] \right.$$
$$\left. -i\int \frac{dw}{2\pi}\frac{dz}{2\pi}\chi_M(w,u|1)\frac{d^2}{dwdz}\Theta(w,z)\left[\chi(z,v|1)+\Phi(z)\right] \right\}, \tag{9.37}$$

$$S^{(M\bar{g})}(u,v) = \frac{u-v-\frac{i}{2}}{u-v-\frac{3i}{2}} \exp\left\{ i\int \frac{dw}{2\pi}\chi_M(w,u|1)\frac{d}{dw}\left[\chi(w,v|1)+\Phi(x)\right] \right.$$
$$\left. -i\int \frac{dw}{2\pi}\frac{dz}{2\pi}\chi_M(w,u|1)\frac{d^2}{dwdz}\Theta(w,z)\left[\chi(z,v|1)+\Phi(z)\right] \right\}, \tag{9.38}$$

$$S^{(Mf)}(u,v) = \frac{u-v+i}{u-v} \exp\left\{ -i\int \frac{dw}{2\pi}\chi_M(w,u|1)\frac{d}{dw}\chi_H(w,v) \right.$$
$$\left. +i\int \frac{dw}{2\pi}\frac{dz}{2\pi}\chi_M(w,u|1)\frac{d^2}{dwdz}\Theta(w,z)\chi_H(z,v) \right\}, \tag{9.39}$$

$$S^{(M\bar{f})}(u,v) = \frac{u-v}{u-v-i} \exp\left\{ -i\int \frac{dw}{2\pi}\chi_M(w,u|1)\frac{d}{dw}\chi_H(w,v) \right.$$
$$\left. +i\int \frac{dw}{2\pi}\frac{dz}{2\pi}\chi_M(w,u|1)\frac{d^2}{dwdz}\Theta(w,z)\chi_H(z,v) \right\}. \tag{9.40}$$

Eventually, these ABA equations (9.35)[19] will acquire the aforementioned physical meaning in the infinite coupling limit where it is possible to parametrise $u_M = \sqrt{2}g \coth 2\theta_M$ in terms of the hyperbolic rapidity, and then $P^{(M)}(\theta_M) = 2\sinh\theta_M$, $D^{(M)}(\theta_M) = -P^{(M)}(\theta_M)\ln g$ and

$$S^{(Ms)}(\theta_M, u_h) = \exp\left[\frac{i}{\sqrt{2}g}\frac{1}{\coth 2\theta_M - 1} \right] [S^{(fs)}(\theta_M, u_h)]^2, \tag{9.41}$$

---

[19] When writing them, we have deliberately neglected the multiplication by $S^{(MF)}(u,v)$ and $S^{(M\bar{F})}(u,v)$ (which could be immediately obtained respectively from (9.39) and (9.40) via the usual substitution $-\chi_H(w,u) \longrightarrow \chi_F(w,u) + \Phi(w)$), since we know (*cf.* (4.50) and (4.47)) that this contribution from the large fermions, forced to live in the giant hole regime at strong coupling, would be exponentially small.

$$S^{(Mg)}(\theta_M, \theta^g) = S^{(fg)}(\theta_M, \theta^g) \, S^{(\bar{f}g)}(\theta_M, \theta^g), \tag{9.42}$$

$$S^{(M\bar{g})}(\theta_M, \theta^{\bar{g}}) = S^{(f\bar{g})}(\theta_M, \theta^{\bar{g}}) \, S^{(\bar{f}\bar{g})}(\theta_M, \theta^{\bar{g}}), \tag{9.43}$$

$$S^{(Mf)}(\theta_M, \theta_f) = \exp\left[\frac{i}{\sqrt{2}g} \frac{1}{(\coth\theta_M - \coth\theta_f)} + O\left(\frac{1}{g^2}\right)\right] [S^{(ff)}(\theta_M, \theta_f)]^2, \tag{9.44}$$

$$S^{(M\bar{f})}(\theta_M, \theta_{\bar{f}}) = \exp\left[\frac{i}{\sqrt{2}g} \frac{1}{(\coth\theta_M - \coth\theta_{\bar{f}})} + O\left(\frac{1}{g^2}\right)\right] [S^{(ff)}(\theta_M, \theta_{\bar{f}})]^2, \tag{9.45}$$

$$S^{(MM)}(\theta_M, \theta'_M) = \exp\left[\frac{i\sqrt{2}}{g} \frac{1}{(\coth\theta_M - \coth\theta'_M)} + O\left(\frac{1}{g^2}\right)\right] [S^{(ff)}(\theta_M, \theta'_M)]^4. \tag{9.46}$$

Explicit expressions for the scattering $S^{(fs)}(\theta_M, u_h)$ is given by (4.71), with $\bar{v} = \coth 2\theta_M$ and for all the other S factors (entering the previous r.h.s.) in Appendix C.3.

Now, if we come back to (9.33), we can notice that, in analogy to equations (9.1) for small fermion/antifermion (and to (9.14) for gluons), a rational pre-factor in front of the scattering factor is present. This suggests that the same mechanism, which produces (virtual) mesons and bound states of gluons should also produce (virtual) bound states of mesons: in fact, these will be represented by strings with the customary form

$$u_{M,k} = u_M^m + \frac{i}{2}(m - 1 - 2k), \qquad k = 0, \ldots, m-1, \tag{9.47}$$

where all roots, once scaled, collapse towards the real axis at infinite coupling. At generic coupling, these virtual states enjoy scattering phases which, formally, do not distance themselves from the ones previously found for actual particles; indeed, when considering a process involving a bound state of l mesons with center u and a bound state of m mesons with center v, the resulting scattering matrix emerges from the fusion of (9.33):

$$S_{lm}^{(MM)}(u, v) = \prod_{j=0}^{l-1} \prod_{k=0}^{m-1} S^{(MM)}\left(u + \frac{i}{2}(l-1-2j), v + \frac{i}{2}(m-1-2k)\right)$$

$$= \exp\left\{-i\tilde{\chi}(u, v|l, m) + i \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \chi_M(w, u|l) \frac{d}{dw} \chi_M(w, v|m)\right.$$

$$\left. - i \int \frac{dw}{2\pi} \frac{dz}{2\pi} \chi_M(w, u|l) \frac{d^2}{dw\,dz} \Theta(w, z) \chi_M(z, v|m)\right\}, \tag{9.48}$$

with the definitions (9.34). Similarly, from a generalisation of (9.36)–(9.38) we can deduce the scattering phases concerning bound states of m mesons against scalars or bound states of l gluons:

$$S_m^{(Ms)}(u, v) = \frac{u - v + \frac{im}{2}}{u - v - \frac{im}{2}} \exp\left\{i\chi_M(v, u|m) - i \int \frac{dw}{2\pi} \frac{d\Theta}{dw}(v, w) \chi_M(w, u|m)\right\} \tag{9.49}$$

$$S_{ml}^{(Mg)}(u,v) = \prod_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{u-v+\frac{i}{2}(l+2k+2)}{u-v-\frac{i}{2}(l+2k-2)}$$

$$\cdot \exp\left\{ i \int \frac{dw}{2\pi} \chi_M(w,u|m) \frac{d}{dw}\left[\chi(w,v|l) + \Phi(x) \right] \right.$$

$$\left. - i \int \frac{dw}{2\pi} \frac{dz}{2\pi} \chi_M(w,u|m) \frac{d^2}{dwdz} \Theta(w,z)\left[\chi(z,v|l) + \Phi(z) \right] \right\}, \qquad (9.50)$$

$$S_{ml}^{(M\bar{g})}(u,v) = \prod_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{u-v+\frac{i}{2}(l+2k-2)}{u-v-\frac{i}{2}(l+2k+2)}$$

$$\cdot \exp\left\{ i \int \frac{dw}{2\pi} \chi_M(w,u|m) \frac{d}{dw}\left[\chi(w,v|l) + \Phi(x) \right] \right.$$

$$\left. - i \int \frac{dw}{2\pi} \frac{dz}{2\pi} \chi_M(w,u|m) \frac{d^2}{dwdz} \Theta(w,z)\left[\chi(z,v|l) + \Phi(z) \right] \right\}. \qquad (9.51)$$

It is crucial for the following (TBA) noticing the drastic simplification of the fusion into a multiplication when $g = \infty$ (zero shifts)

$$S_{lm}^{(MM)}(\theta,\theta') = \left[S^{(MM)}(\theta,\theta') \right]^{lm}, \qquad (9.52)$$

along with the scattering phases between a bound state of l mesons and a bound state of m gluons which become

$$S_{lm}^{(Mg)}(\theta,\theta') = [S_{ml}^{(gM)}(\theta',\theta)]^{-1} = [S^{(Mg)}(\theta,\theta')]^{lm},$$

$$S_{lm}^{(M\bar{g})}(\theta,\theta') = [S_{ml}^{(\bar{g}M)}(\theta',\theta)]^{-1} = [S^{(M\bar{g})}(\theta,\theta')]^{lm}. \qquad (9.53)$$

Explicit expressions for (9.52), (9.53) in terms of hyperbolic variables are given by formulæ (C.45), (C.46) in Appendix C.

We need to conclude this section with the dispersion relations. At least in principle, we can write the energy and momentum of the virtual meson (of mass $= 2$) and of its m bound state at finite g, albeit these are not stable particle: the energy as

$$E_m^M(u) = 2m + \gamma_m^{(M)}(u) , \qquad (9.54)$$

with

$$\gamma_m^{(M)}(u) = \int_0^{+\infty} \frac{dk}{k} \frac{\cos ku}{\sinh \frac{k}{2}} \left(\cosh k \, e^{-\frac{m}{2}k} - \cosh \frac{k}{2} \right) \gamma_+^{\phi}(\sqrt{2}gk) \quad \text{for } m \text{ odd},$$

$$\gamma_m^{(M)}(u) = \int_0^{+\infty} \frac{dk}{k} \frac{\cos ku}{\sinh \frac{k}{2}} \left(\cosh k \, e^{-\frac{m}{2}k} - 1 \right) \gamma_+^{\phi}(\sqrt{2}gk) \quad \text{for } m \text{ even}; \qquad (9.55)$$

and the momentum as

$$P_m^{(M)}(u) = \int\limits_0^{+\infty} \frac{dk}{k} \frac{\sin ku}{\sinh \frac{k}{2}} \left(\cosh \frac{k}{2} - \cosh k \, e^{-\frac{m}{2}k} \right) \gamma_-^{\emptyset}(\sqrt{2}gk) \quad \text{for } m \text{ odd,}$$

$$P_m^{(M)}(u) = \int\limits_0^{+\infty} \frac{dk}{k} \frac{\sin ku}{\sinh \frac{k}{2}} \left(1 - \cosh k \, e^{-\frac{m}{2}k} \right) \gamma_-^{\emptyset}(\sqrt{2}gk) \quad \text{for } m \text{ even,} \tag{9.56}$$

both in terms of the functions $\gamma_\pm^{\emptyset}$ defined in (B.19), (B.20). The (rest) masses of these resonances are protected at all values of the coupling to be the purely additive values $E_m^{(M)}(u = \pm\infty) = 2m$ (zero binding energy), which are indeed stable as long as the momentum is zero. Yet, these acquire full (any momentum) stability in the strong coupling limit where boost invariance is recovered. In fact, if we enter the perturbative regime $g \to +\infty$, $u = \sqrt{2}g\bar{u}$, with $|\bar{u}| > 1$, we obtain the relativistic dispersion relations

$$E_m^{(M)}(\theta) = 2m\cosh\theta, \quad P_m^{(M)}(\theta) = 2m\sinh\theta, \tag{9.57}$$

in terms of the hyperbolic rapidity θ, defined via $\bar{u} = \coth 2\theta$.

10. Pentagonal amplitudes at strong coupling (perturbative regime) and confinement

An important application of the above scattering data, which implies a non-trivial check of them, is the construction of the so-called pentagonal amplitudes, P [34–38]. The latter, in their turn, are the building blocks of an infinite expansion – the BSV series – of the gluonic (MHV) scattering amplitudes. In this section, we want to compute the pentagonal factors, P, relevant at large g, so to prepare the analysis of the BSV series (at strong coupling) in next section.

The BSV series is a sum over the (intermediate) multi-particle states, where the particles may be, – at generic finite coupling –, scalars, fermions, gluons and bound states thereof, as analysed above. The simplest example is provided by the six-particle amplitude (or, in other terms, the equivalent hexagonal Wilson loop)

$$W_{hex} = \sum_{N=0}^{+\infty} \frac{1}{N!} \sum_{a_1} \cdots \sum_{a_N} \int \prod_{k=1}^{N} \left[\frac{du_k}{2\pi} \mu_{a_k}(u_k) e^{-\tau E_{a_k}(u_k) + i\sigma p_{a_k}(u_k) + im_{a_k}\phi} \right]$$
$$\times P_{a_1 \dots a_N}(0|u_1 \dots u_N) P_{a_1 \dots a_N}(-u_N \dots - u_1|0), \tag{10.1}$$

which is expressed by means of the measures $\mu_{a_i}(u_i)$ (corresponding to quadrangular amplitudes) and the multi-particle pentagonal amplitudes $P_{a_1 \dots a_N}(0|u_1 \dots u_N)$, representing the transition from the vacuum to an intermediate state with N particles of the kinds listed above, each one associated to a label a_i. When we go to the strong coupling limit, we have to disentangle the integrations over internal rapidities by performing the limit $g \to +\infty$ in the integrand. This procedure means that we have to add different contributions.

The first one comes from performing the limit $g \to +\infty$ with integration variables fixed. This part depends on excitations in the non-perturbative regime and is dominated by scalars, and may reserve very interesting surprises as anticipated in [37]. In fact, this contribution would come from a (genuinely) quantised string in S^5 and would elude the minimal area argument of the AdS_5 string. However, this regime misses contributions from regions in which rapidities are large: these are recovered by adding the integrals in which integration rapidities are scaled before taking the limit $g \to +\infty$.

More precisely, if we scale the integration rapidity $u = \sqrt{2}g\bar{u}$, with $\bar{u}$ fixed, we have the two following regimes. If $|\bar{u}| > 1$ we are in the giant hole regime. In this regime all the excitations behave in the same way. In particular, as we showed in Section 4, scattering phases $-i \ln S$ between any pair of excitations are all the same and are all proportional to the coupling g (4.3). The same happens to energies and momenta [20]. This property is crucial, since it implies that contributions to scattering amplitudes coming from integrations in these regions (all scaled rapidities $|\bar{u}| > 1$) are exponentially suppressed.

Instead, things are different if $|\bar{u}| < 1$, *i.e.* in the (string) perturbative regime (for all particles except scalars: the rapidity of the latter in this regime does not scale, but instead $\bar{u} = u + \frac{2}{\pi} \ln m(g)$, where $m(g) \sim g^{1/4} e^{-\frac{\pi}{\sqrt{2}}g}$, as seen above). In this regime, energy, momentum and scattering factors are expanded in inverse powers of the coupling constant g. Additional structure is added when expressing the pentagonal transition between an M particle state to an N particle state in terms of the one-particle-to-one-particle transitions because of the matrix representation carried by the single particle (thus the singlets makes an exception to this). In this operation polynomials in the rapidity appear as denominators, taking into account the different representations to which the S matrices can belong. For instance, in the case of the hexagon (10.1), extensively discussed below, as we start from the GKP vacuum, we need consider only pentagonal amplitudes to the other possible singlet states. In particular, this polynomial is a monomial in the case of the transition (from the vacuum) to a two particle state of a fermion and an antifermion (which, though, belong to the 4 and $\bar{4}$, respectively). This monomial 'squares' in the integrand of the amplitude contribution to (10.1)

$$P^{(f\bar{f})}(0|u,v)\,P^{(\bar{f}f)}(-v,-u|0) = \frac{1}{(u-v)^2+4}\,\frac{1}{P^{(f\bar{f})}(u|v)\,P^{(\bar{f}f)}(v|u)}. \qquad (10.2)$$

Instead, the transition from the vacuum into the two scalar singlet is even more depressed, albeit the rapidity does not scale (for a scalar, but is added a g-depending constant). In fact, the P factor contains at the denominator a polynomial of degree 2 multiplied by g^2 (*cf.* [35,36] for details), and then the two scalar contribution to the hexagonal amplitude writes down:

$$W_{hex}^{(ss)} = 3 \int \frac{du\,dv}{(2\pi)^2}\,\frac{\mu^s(u)\mu^s(v)}{g^4[(u-v)^2+4][(u-v)^2+1]}\,\frac{e^{-\tau[E^s(u)+E^s(v)]+i\sigma[p^s(u)+p^s(v)]}}{P^{(ss)}(u|v)\,P^{(ss)}(v|u)}, \qquad (10.3)$$

where we ought to consider that $\mu^s(u) = O(g)$ and $P^{(ss)}(u|v) = O(1/g)$. Hence, this integral turns out to be of order $W_{hex}^{(ss)} = O(g^0)$, then subdominant with respects to semi-classical approximation (contributed by the gluons, for instance). Actually, while the scalar contributions are really subdominant in the perturbative regime, on the contrary fermion ones behave in a subtle manner: in fact, the lorentzian function in front of (10.2) would entail a contribution from the singularity $\bar{u} - \bar{v} = \pm\sqrt{2}i/g$ pinching the real axis when $g \to +\infty$ [36]. But in our picture this is the contribution given by their bound state, the meson indeed. Moreover, also the greater multi-fermion coalescence are taken into account by the multi-meson and meson-bound-state contributions, *cf.* below. In summary, we are in the presence of a phenomenon in which the fermions coalesces at least in a fermion and antifermion couple and disappear from the spectrum as free particles: as anticipated, this is a sort of *confinement* typical of MHV gluon scattering amplitudes/Wilson loops at strong coupling, not evident at first glance from the 2D scattering factors. To be fully precise, although the string theory minimal surface confirms this disappearance, and the appearance of the meson [49,6] and its bound states [39,41,40], nevertheless a detailed multi-fermion description is missing so far [58].

Concluding this preamble, these polynomials in the denominator produce in the general case negative powers of the coupling constant after scaling[20] the rapidities and thus 'depress' the amplitude. Of course, these polynomials are absent if the excitations belong to an $SU(4)$ singlet (see also [35,36] for a detailed analysis of the two particle case). Therefore, we can argue that the leading contributions in the perturbative regime are due to particles behaving as singlets under $SU(4)$, indeed. They are gluons and their bound states, as already proven by the detailed two particle analysis of [36]. But, at strong coupling, we have shown necessary to add mesons and their bound states to the spectrum, as well.

Now, the pentagonal amplitudes P enjoy at general coupling a series of axioms depending on the S-matrix entries. Therefore for the latter we need to use our previous (strong coupling) perturbative expansions at leading order and 'solve' the axioms. For exposition's sake, we give in the following the complete list of the P factors (gluon–gluon, gluon–meson, meson–meson and bound states, contributing at leading order), leaving the details of their derivation in Appendix C.[21]

We start from the gluon and then the bound states of ℓ of them. In this gluonic sector the rapidity enjoys (at perturbative strong coupling) the parametrisation $u = \sqrt{2}g \tanh 2\theta$. Thanks to this, the three axioms (6–8) in [34] for the gluon (g) and its barred companion ($\bar{g}$, the other component of the massless spin 1 field) simplify their arguments:

$$P^{(gg)}(-\theta|-\theta') = P^{(gg)}(\theta'|\theta), \quad P^{(g\bar{g})}(-\theta|-\theta') = P^{(g\bar{g})}(\theta'|\theta), \tag{10.4}$$

$$P^{(gg)}(\theta|\theta') = S^{(gg)}(\theta,\theta')P^{(gg)}(\theta'|\theta), \quad P^{(g\bar{g})}(\theta|\theta') = S^{(g\bar{g})}(\theta,\theta')P^{(g\bar{g})}(\theta'|\theta), \tag{10.5}$$

$$P^{(gg)}(\theta - i\pi/2|\theta') = P^{(g\bar{g})}(\theta'|\theta), \tag{10.6}$$

and we can solve them with input the leading order expansion of the gluon–gluon scattering matrix (C.40), (C.41). We obtain

$$\alpha P^{(gg)}(\theta,\theta') = 1 + \frac{i}{2\sqrt{2}g}\frac{\cosh 2\theta \cosh 2\theta'}{\sinh(2\theta - 2\theta')}\left[1 + \cosh(\theta - \theta') - i\sinh(\theta - \theta')\right]$$
$$+ O(1/g^2), \tag{10.7}$$

$$\alpha P^{(g\bar{g})}(\theta,\theta') = 1 + \frac{i}{2\sqrt{2}g}\frac{\cosh 2\theta \cosh 2\theta'}{\sinh(2\theta - 2\theta')}\left[-1 + \cosh(\theta - \theta') - i\sinh(\theta - \theta')\right]$$
$$+ O(1/g^2), \tag{10.8}$$

and the symmetric channels $P^{(\bar{g}\bar{g})}(\theta,\theta') = P^{(gg)}(\theta,\theta')$, $P^{(\bar{g}g)}(\theta,\theta') = P^{(g\bar{g})}(\theta,\theta')$. The constant α may equal ± 1: its precise value is not fixed by the axioms, but by the comparison with data derived from the Thermodynamic Bubble Ansatz (TBA) in [41]. As we wrote above, formulæ (10.7), (10.8) with $\alpha = 1$ have been already reported in [34].

For what concerns P factors of gluon bound states, we may conjecture, along the lines of the previous equations (10.6) for the single gluons, the following functional relations as axioms: $P_{ml}^{(gg)}(\theta,\theta') = P_{ml}^{(\bar{g}\bar{g})}(\theta,\theta')$, $P_{ml}^{(g\bar{g})}(\theta,\theta') = P_{lm}^{(\bar{g}g)}(\theta,\theta')$ and moreover

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[20] Rapidity of scalars do not need to scale.

[21] We have to say that the expansion of the gluon–gluon P factors – formulæ (10.7), (10.8) – previously appeared in [34].

$$P_{lm}^{(gg)}(-\theta, -\theta') = P_{ml}^{(gg)}(\theta', \theta), \quad P_{lm}^{(gg)}(\theta - i\pi/2, \theta') = P_{ml}^{(g\bar{g})}(\theta', \theta),$$

$$P_{lm}^{(gg)}(\theta, \theta') = S_{lm}^{(gg)}(\theta, \theta') P_{ml}^{(gg)}(\theta', \theta), \quad P_{lm}^{(g\bar{g})}(\theta, \theta') = S_{lm}^{(g\bar{g})}(\theta, \theta') P_{ml}^{(\bar{g}g)}(\theta', \theta). \quad (10.9)$$

Moreover, we recall that the S-matrix factors are simply multiplicative at perturbative strong coupling: $S_{ml}^{(gg)}(\theta, \theta') = [S^{(gg)}(\theta, \theta')]^{ml}$ and $S_{ml}^{(g\bar{g})}(\theta, \theta') = [S^{(g\bar{g})}(\theta, \theta')]^{ml}$. Therefore, solutions to (10.9) should enjoy the same property, which entails upon expansion for large g

$$\alpha_{ml} P_{ml}^{(gg)}(\theta, \theta') = 1 + \frac{iml}{2\sqrt{2}g} \frac{\cosh 2\theta \cosh 2\theta'}{\sinh(2\theta - 2\theta')} \left[1 + \cosh(\theta - \theta') - i \sinh(\theta - \theta') \right]$$
$$+ O(1/g^2), \quad (10.10)$$

$$\alpha_{ml} P_{ml}^{(g\bar{g})}(\theta, \theta') = 1 - \frac{iml}{2\sqrt{2}g} \frac{\cosh 2\theta \cosh 2\theta'}{\sinh(2\theta - 2\theta')} \left[1 - \cosh(\theta - \theta') + i \sinh(\theta - \theta') \right]$$
$$+ O(1/g^2), \quad (10.11)$$

or in barred rapidities $\bar{u} = \tanh 2\theta$

$$\alpha_{ml} P_{ml}^{(gg)}(\bar{u}, \bar{u}') = 1 + \frac{iml}{2\sqrt{2}g} \frac{1}{\bar{u} - \bar{u}'} \left[1 + \frac{1}{2}(1 - i) \left(\frac{1 + \bar{u}}{1 - \bar{u}} \right)^{1/4} \left(\frac{1 - \bar{u}'}{1 + \bar{u}'} \right)^{1/4} \right.$$
$$\left. + \frac{1}{2}(1 + i) \left(\frac{1 + \bar{u}'}{1 - \bar{u}'} \right)^{1/4} \left(\frac{1 - \bar{u}}{1 + \bar{u}} \right)^{1/4} \right] \quad (10.12)$$

$$\alpha_{ml} P_{ml}^{(g\bar{g})}(\bar{u}, \bar{u}') = 1 - \frac{iml}{2\sqrt{2}g} \frac{1}{\bar{u} - \bar{u}'} \left[1 - \frac{1}{2}(1 - i) \left(\frac{1 + \bar{u}}{1 - \bar{u}} \right)^{1/4} \left(\frac{1 - \bar{u}'}{1 + \bar{u}'} \right)^{1/4} \right.$$
$$\left. - \frac{1}{2}(1 + i) \left(\frac{1 + \bar{u}'}{1 - \bar{u}'} \right)^{1/4} \left(\frac{1 - \bar{u}}{1 + \bar{u}} \right)^{1/4} \right] \quad (10.13)$$

Overall constants $\alpha_{ml} = \alpha_{lm}$ can be equal to ± 1 and are constrained by the comparison with the TBA of [41].

Let us now consider the meson and its bound states, and in particular recall that for all of them the rapidity enjoys the perturbative parametrisation $u = \sqrt{2}g \coth 2\theta$. As we discussed before, these are self-conjugate particles and this property allows us to postulate the following set of functional relations (which now will be meaningful only in the perturbative strong coupling regime, where the particle does exist) for the single meson P factor:

$$P^{(MM)}(\theta, \theta') = P^{(MM)}(-\theta', -\theta),$$
$$P^{(MM)}(\theta, \theta') = S^{(MM)}(\theta, \theta') P^{(MM)}(\theta', \theta),$$
$$P^{(MM)}(\theta - i\pi/2, \theta') = P^{(MM)}(\theta', \theta), \quad (10.14)$$

where $S^{(MM)}$ is given by (C.45). We write the solution of (10.14) as

$$\beta P^{(MM)}(\theta, \theta') = 1 - \frac{1}{\sqrt{2}g} \frac{i \sinh 2\theta \sinh 2\theta'}{\sinh(2\theta - 2\theta')} \sqrt{2} \cosh \left(\theta - \theta' - i \frac{\pi}{4} \right) + O(1/g^2), \quad (10.15)$$

where $\beta = \pm 1$. For mesons bound states, we have anew the multiplicativity of the scattering factors in the perturbative regime, namely $S_{ml}^{(MM)}(\theta, \theta') = [S^{(MM)}(\theta, \theta')]^{ml}$. Which, in its turn,

imply the same property on P factors, *i.e.* upon expanding at large g

$$\beta_{ml} P_{ml}^{(MM)}(\theta, \theta') = 1 - \frac{ml}{\sqrt{2}g} \frac{i \sinh 2\theta \sinh 2\theta'}{\sinh(2\theta - 2\theta')} \sqrt{2} \cosh\left(\theta - \theta' - i\frac{\pi}{4}\right) + O(1/g^2), \tag{10.16}$$

or in barred variables $\bar{u} = \coth 2\theta$

$$\beta_{ml} P_{ml}^{(MM)}(\bar{u}, \bar{u}') = 1 + \frac{iml}{2\sqrt{2}g} \frac{1}{\bar{u} - \bar{u}'} \left[(1 - i) \left(\frac{\bar{u} + 1}{\bar{u} - 1}\right)^{1/4} \left(\frac{\bar{u}' - 1}{\bar{u}' + 1}\right)^{1/4} \right.$$
$$\left. + (1 + i) \left(\frac{\bar{u} - 1}{\bar{u} + 1}\right)^{1/4} \left(\frac{\bar{u}' + 1}{\bar{u}' - 1}\right)^{1/4} \right], \tag{10.17}$$

where $\beta_{ml} = \beta_{lm}$ can equal ± 1.

Eventually,[22] we consider the scattering between (bound states of) mesons and (bound states of) gluons. We are now looking for functions $P_{ml}^{(Mg)}$, $P_{ml}^{(M\bar{g})}$, $P_{ml}^{(gM)}$, $P_{ml}^{(\bar{g}M)}$ which may conjecturally satisfy the functional properties (meaningful only at perturbative strong coupling)

$$P_{ml}^{(ab)}(-\theta, -\theta') = P_{lm}^{(ba)}(\theta', \theta),$$
$$P_{ml}^{(Mg)}(\theta, \theta') = S_{ml}^{(Mg)}(\theta, \theta') P_{lm}^{(gM)}(\theta', \theta), \quad P_{ml}^{(M\bar{g})}(\theta, \theta') = S_{ml}^{(M\bar{g})}(\theta, \theta') P_{lm}^{(\bar{g}M)}(\theta', \theta),$$
$$P_{ml}^{(Mg)}\left(\theta - \frac{i\pi}{2}, \theta'\right) = P_{lm}^{(\bar{g}M)}(\theta', \theta), \quad P_{ml}^{(M\bar{g})}\left(\theta - \frac{i\pi}{2}, \theta'\right) = P_{lm}^{(gM)}(\theta', \theta). \tag{10.19}$$

We write solutions to these equations in the form $\gamma_{ml}^{(ab)} P_{ml}^{(ab)} = 1 + \frac{ml}{\sqrt{2}g} K^{(ab)} + O\left(\frac{1}{\lambda}\right)$, where $\gamma_{ml}^{(ab)}$ can equal ± 1, $\gamma_{ml}^{(Mg)} = \gamma_{lm}^{(gM)} = \gamma_{ml}^{(M\bar{g})} = \gamma_{lm}^{(\bar{g}M)}$ and

$$K^{(Mg)}(\theta, \theta') = K^{(M\bar{g})}(\theta, \theta') = \frac{\sinh 2\theta \cosh 2\theta'}{\sqrt{2} \cosh(2\theta - 2\theta')} [\sinh(\theta - \theta') + i \cosh(\theta - \theta')], \tag{10.20}$$

$$K^{(gM)}(\theta', \theta) = K^{(\bar{g}M)}(\theta', \theta) = \frac{\sinh 2\theta \cosh 2\theta'}{\sqrt{2} \cosh(2\theta - 2\theta')} [\sinh(\theta - \theta') - i \cosh(\theta - \theta')], \tag{10.21}$$

or alternatively in the barred variables

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[22] Even if contributions of small fermions to amplitudes is suppressed at strong coupling with respect to gluons and mesons, we give also the strong coupling limit of their P factors. We refer to formulæ (38) of [36] and use formulæ (C.42), (C.49), (C.50) for the (strong coupling) perturbative regime of the fermion-(anti)fermion scattering factor and its mirror, respectively. We eventually obtain

$$[P^{(ff)}(\theta, \theta')]^2$$
$$= \frac{\cosh\theta \cosh\theta' \sinh 2\theta \sinh 2\theta'}{g^2 \sinh(\theta - \theta') \sinh(2\theta - 2\theta')} \left[1 + \frac{i}{2\sqrt{2}g} \frac{\sinh 2\theta \sinh 2\theta'}{\sinh(2\theta - 2\theta')} \left(1 - \cosh(\theta - \theta') + i \sinh(\theta - \theta')\right)\right]$$
$$[P^{(f\bar{f})}(\theta, \theta')]^2 = \frac{\sinh\theta \sinh\theta'}{\cosh(\theta - \theta')} \left[1 + \frac{i}{2\sqrt{2}g} \frac{\sinh 2\theta \sinh 2\theta'}{\sinh(2\theta - 2\theta')} \left(1 - \cosh(\theta - \theta') + i \sinh(\theta - \theta')\right)\right]. \tag{10.18}$$

$$K^{(Mg)}(\bar{u}, \bar{u}') = \frac{1}{2\sqrt{2}} \frac{1}{\bar{u} - \bar{u}'} \left[(1+i) \left(\frac{\bar{u}+1}{\bar{u}-1} \right)^{1/4} \left(\frac{1-\bar{u}'}{1+\bar{u}'} \right)^{1/4} \right.$$
$$\left. - (1-i) \left(\frac{\bar{u}-1}{\bar{u}+1} \right)^{1/4} \left(\frac{1+\bar{u}'}{1-\bar{u}'} \right)^{1/4} \right] \tag{10.22}$$

$$K^{(gM)}(\bar{u}', \bar{u}) = \frac{1}{2\sqrt{2}} \frac{1}{\bar{u} - \bar{u}'} \left[(1-i) \left(\frac{\bar{u}+1}{\bar{u}-1} \right)^{1/4} \left(\frac{1-\bar{u}'}{1+\bar{u}'} \right)^{1/4} \right.$$
$$\left. - (1+i) \left(\frac{\bar{u}-1}{\bar{u}+1} \right)^{1/4} \left(\frac{1+\bar{u}'}{1-\bar{u}'} \right)^{1/4} \right]. \tag{10.23}$$

11. Hexagon at strong coupling

11.1. Aim and assumptions

Now, we want to compute the hexagonal Wilson loop as the series proposed by [34–36]

$$W_{hex} = \sum_{N=0}^{+\infty} \frac{1}{N!} \sum_{a_1} \cdots \sum_{a_N} \int \prod_{k=1}^{N} \left[\frac{du_k}{2\pi} \mu_{a_k}(u_k) e^{-\tau E_{a_k}(u_k) + i\sigma p_{a_k}(u_k) + im_{a_k}\phi} \right]$$
$$\times P_{a_1 \ldots a_N}(0|u_1 \ldots u_N) P_{a_1 \ldots a_N}(-u_N \ldots -u_1|0) \tag{11.1}$$

at strong coupling. As argued above, in this regime intermediate states which contribute are gluons and their bound states, together with mesons and their bound states. All of them are singlets and then for their pentagonal amplitudes a simple product and inversions hold when changing a rapidity from in to out:

$$P_{a_1 \ldots a_N}(0|u_1 \ldots u_N) P_{a_1 \ldots a_N}(-u_N \ldots -u_1|0) = \prod_{i<j}^{N} \frac{1}{P_{a_i,a_j}(u_i|u_j) P_{a_j,a_i}(u_j|u_i)}. \tag{11.2}$$

This formula entails an easy product to appear inside the hexagonal amplitude:

$$W_{hex} = \sum_{N=0}^{+\infty} \frac{1}{N!} \sum_{a_1} \cdots \sum_{a_N} \int \prod_{k=1}^{N} \left[\frac{du_k}{2\pi} \mu_{a_k}(u_k) e^{-\tau E_{a_k}(u_k) + i\sigma p_{a_k}(u_k) + im_{a_k}\phi} \right]$$
$$\cdot \prod_{i<j}^{N} \frac{1}{P_{a_i,a_j}(u_i|u_j) P_{a_j,a_i}(u_j|u_i)}, \tag{11.3}$$

where the indices a_k label the species of different particles (including bound states): this is the formula we want first to match with initially, and then to sum up.

For the gluon and the bound states of ℓ of them, rapidity may be parametrised as $u = \sqrt{2}g \tanh 2\theta$. Then, we can recall their relativistic energy and momentum (9.31) and notice how they are purely additive

$$E_\ell^g(u) = \sqrt{2}\ell \cosh\theta = \ell E_1^g(u), \qquad p_\ell^g(u) = \sqrt{2}\ell \sinh\theta = \ell p_1^g(u). \tag{11.4}$$

In the following, the expression will be written in terms of

$$\frac{\sqrt{\lambda}}{2\pi} = \sqrt{2}g$$

in order to facilitate the comparison with the existing literature on this subject. Gluonic measures appearing in (11.3) is given by ($\ell = 1$ is the gluon)

$$\frac{du}{2\pi}\mu_\ell^g(u) = \frac{d\theta}{2\pi}\mu_\ell^g(\theta) = \frac{i}{\lim_{\theta' \to \theta}(\theta' - \theta)P_{\ell\ell}^{(gg)}(\theta, \theta')}\frac{d\theta}{2\pi}, \tag{11.5}$$

with $P_{\ell\ell}^{(gg)}$ given by (10.10). In order to have agreement with TBA it is enough to choose $\alpha_{\ell\ell} = (-1)^{\ell-1}$. Then, in terms of the single gluon measure (at strong coupling)

$$\mu^g(u) = -1 + \dots \tag{11.6}$$

we can express the ℓ gluon bound-state ones

$$\frac{du}{2\pi}\mu_\ell^g(u) = \frac{\sqrt{\lambda}}{2\pi}(-1)^\ell\frac{d\theta}{\pi\ell^2\cosh^2 2\theta} = \frac{du}{2\pi}\frac{(-1)^{\ell-1}}{\ell^2}\mu^g(u) = \frac{d\theta}{2\pi}\frac{(-1)^{\ell-1}}{\ell^2}\mu^g(\theta). \tag{11.7}$$

On the other hand, let us remind that the meson and its bound-states enjoy the rapidity parametrisation $u = \sqrt{2}g\coth 2\theta$ with relativistic energy and momentum (9.57) which are purely additive (zero binding energy) as well

$$E_m^M(u) = 2m\cosh\theta = mE_1^M(u), \qquad p_m^M(u) = 2m\sinh\theta = mp_1^M(u). \tag{11.8}$$

The measure for bound states of m mesons is

$$\frac{du}{2\pi}\mu_m^M(u) = \frac{d\theta}{2\pi}\mu_m^M(\theta) = \frac{i}{\lim_{\theta' \to \theta}(\theta' - \theta)P_{mm}^{(MM)}(\theta, \theta')}\frac{d\theta}{2\pi}, \tag{11.9}$$

with $P_{mm}^{(MM)}$ given by (10.16). We choose $\beta_{mm} = (-1)^{m-1}$ in order to have agreement with TBA. Then, similarly to gluons, we obtain the meson bound-state measures

$$\frac{du}{2\pi}\mu_m^M(u) = \frac{\sqrt{\lambda}}{2\pi}(-1)^{m-1}\frac{d\theta}{\pi m^2\sinh^2 2\theta} = \frac{du}{2\pi}\frac{(-1)^{m-1}}{m^2}\mu^M(u) = \frac{d\theta}{2\pi}\frac{(-1)^{m-1}}{m^2}\mu^M(\theta), \tag{11.10}$$

where the single meson measure (at strong coupling)

$$\mu^M(u) = -1 + \dots . \tag{11.11}$$

We shall point out that, for compactness sake, the paths of integration are not explicitly written in what follows, so we need to describe now the choices we adopt throughout the rest of the section. The integrations over gluon rapidity u can be computed along the interval $[-\frac{\sqrt{\lambda}}{2\pi}, +\frac{\sqrt{\lambda}}{2\pi}]$, which turns to the whole real axis when the map to hyperbolic rapidities θ is performed $u = \frac{\sqrt{\lambda}}{2\pi}\tanh(2\theta)$. Otherwise, the integrations for meson u-rapidity is intended on a section of a straight line below the real axis, lying in the small fermion sheet [41,36] (we remind that large fermions in this regime behave as giant holes, hence their contributions are exponentially suppressed because of their dispersion laws [20] and scattering factors (4.50)), namely $(+\infty - i\epsilon, +\frac{\sqrt{\lambda}}{2\pi}) \cup (-\frac{\sqrt{\lambda}}{2\pi}, -\infty - i\epsilon)$, where $0 < \epsilon < 2$. When switching to meson θ rapidity according to the map $u = \frac{\sqrt{\lambda}}{2\pi}\coth(2\theta)$, the integration may be set to run along the straight line $(-\infty + i\hat{\varphi}, +\infty + i\hat{\varphi})$, where $\hat{\varphi}$ (according to [41]) can be fixed in terms of conformal ratios, as shown below (11.32). Eventually, the same straight line is chosen for the integrations on the gluonic θ rapidity.

11.2. One particle

Let us start from one particle contribution. With 'one particle contribution' we mean that in (11.3) we consider only one insertion, which can be a gluon, a meson or bound states of thereof. Therefore, we can write down

$$
W_{hex}^{(1)} = \sum_{\ell=1}^{+\infty} \int \frac{du}{2\pi} \mu_\ell^g(u) e^{-\tau E_\ell^g(u) + i\sigma p_\ell^g(u)} \left(e^{i\ell\phi} + e^{-i\ell\phi} \right)
$$
$$
+ \sum_{m=1}^{+\infty} \int \frac{du}{2\pi} \mu_m^M(u) e^{-\tau E_m^M(u) + i\sigma p_m^M(u)} \tag{11.12}
$$

which at strong coupling reads

$$
W_{hex}^{(1)} = + \frac{\sqrt{\lambda}}{2\pi} \sum_{\ell=1}^{+\infty} \int \frac{d\theta}{\pi \ell^2 \cosh^2 2\theta} (-1)^\ell e^{-\sqrt{2}\tau\ell\cosh\theta + i\sqrt{2}\sigma\ell\sinh\theta} \left(e^{i\ell\phi} + e^{-i\ell\phi} \right)
$$
$$
- \frac{\sqrt{\lambda}}{2\pi} \sum_{m=1}^{+\infty} \int \frac{d\theta}{\pi m^2 \sinh^2 2\theta} (-1)^m e^{-2\tau m\cosh\theta + 2i\sigma m\sinh\theta}
$$
$$
= \frac{\sqrt{\lambda}}{2\pi} \int \frac{d\theta}{\pi \cosh^2 2\theta} \left[\mathrm{Li}_2 \left(-e^{-\sqrt{2}\tau\cosh\theta + i\sqrt{2}\sigma\sinh\theta + i\phi} \right) \right.
$$
$$
\left. + \mathrm{Li}_2 \left(-e^{-\sqrt{2}\tau\cosh\theta + i\sqrt{2}\sigma\sinh\theta - i\phi} \right) \right]
$$
$$
- \frac{\sqrt{\lambda}}{2\pi} \int \frac{d\theta}{\pi \sinh^2 2\theta} \mathrm{Li}_2 \left(-e^{-2\tau\cosh\theta + 2i\sigma\sinh\theta} \right) \equiv W_{hex}^{(g)} + W_{hex}^{(M)}, \tag{11.13}
$$

where $\mathrm{Li}_2(z) = \sum_{m=1}^{+\infty} \frac{z^m}{m^2}$ is the dilogarithm function.

11.3. Two particles

Let us pass to two particle terms $W_{hex}^{(2)} = W_{hex}^{(gg)} + W_{hex}^{(MM)} + W_{hex}^{(Mg)}$, in which we distinguish three contributions: gluon–gluon, meson–meson and gluon–meson.

Gluon–gluon   Let us start from gluon–gluon:

$$
W_{hex}^{(gg)} = \frac{1}{2} \sum_{\ell_1=1}^{+\infty} \sum_{\ell_2=1}^{+\infty} \int \frac{du_1}{2\pi} \mu_{\ell_1}^g(u_1) \frac{du_2}{2\pi} \mu_{\ell_2}^g(u_2) e^{-\tau E_{\ell_1}^g(u_1) + i\sigma p_{\ell_1}^g(u_1)} e^{-\tau E_{\ell_2}^g(u_2) + i\sigma p_{\ell_2}^g(u_2)}
$$
$$
\cdot \left\{ \frac{e^{i(\ell_1+\ell_2)\phi} + e^{-i(\ell_1+\ell_2)\phi}}{P_{\ell_1\ell_2}^{(gg)}(u_1|u_2) P_{\ell_2\ell_1}^{(gg)}(u_2|u_1)} + \frac{e^{i(\ell_1-\ell_2)\phi} + e^{-i(\ell_1-\ell_2)\phi}}{P_{\ell_1\ell_2}^{(g\bar{g})}(u_1|u_2) P_{\ell_2\ell_1}^{(\bar{g}g)}(u_2|u_1)} \right\}. \tag{11.14}
$$

At strong coupling the symmetric product of P factors (10.7), (10.8), entering (11.14), enjoys the property

$$
\frac{1}{P_{\ell_1\ell_2}^{(gg)}(u_1|u_2) P_{\ell_2\ell_1}^{(gg)}(u_2|u_1)} = \frac{1}{P_{\ell_1\ell_2}^{(g\bar{g})}(u_1|u_2) P_{\ell_2\ell_1}^{(\bar{g}g)}(u_2|u_1)} = 1 - \frac{2\pi}{\sqrt{\lambda}} \ell_1\ell_2 K_{sym}^{(gg)}(\theta_1, \theta_2),
$$
$$
\tag{11.15}
$$

where

$$K_{sym}^{(gg)}(\theta_1, \theta_2) = \frac{\cosh 2\theta_1 \cosh 2\theta_2}{2 \cosh(\theta_1 - \theta_2)} . \tag{11.16}$$

We get

$$W_{hex}^{(gg)} = \left(-\frac{\sqrt{\lambda}}{2\pi}\right)^2 \frac{1}{2} \sum_{\ell_1=1}^{+\infty} \sum_{\ell_2=1}^{+\infty} \int \frac{d\theta_1}{\pi \ell_1^2 \cosh^2 2\theta_1} \frac{d\theta_2}{\pi \ell_2^2 \cosh^2 2\theta_2} (-1)^{\ell_1+\ell_2}$$
$$\cdot 4 \cos \ell_1 \phi \, \cos \ell_2 \phi \, e^{-\sqrt{2}\tau \ell_1 \cosh \theta_1 + i\sqrt{2}\sigma \ell_1 \sinh \theta_1} e^{-\sqrt{2}\tau \ell_2 \cosh \theta_2 + i\sqrt{2}\sigma \ell_2 \sinh \theta_2} \tag{11.17}$$

$$+ \left(-\frac{\sqrt{\lambda}}{2\pi}\right) \frac{1}{2} \sum_{\ell_1=1}^{+\infty} \sum_{\ell_2=1}^{+\infty} \int \frac{d\theta_1}{\pi \ell_1 \cosh^2 2\theta_1} \frac{d\theta_2}{\pi \ell_2 \cosh^2 2\theta_2} K_{sym}^{(gg)}(\theta_1, \theta_2)(-1)^{\ell_1+\ell_2}$$
$$\cdot 4 \cos \ell_1 \phi \, \cos \ell_2 \phi \, e^{-\sqrt{2}\tau \ell_1 \cosh \theta_1 + i\sqrt{2}\sigma \ell_1 \sinh \theta_1} e^{-\sqrt{2}\tau \ell_2 \cosh \theta_2 + i\sqrt{2}\sigma \ell_2 \sinh \theta_2} . \tag{11.18}$$

We can now perform the sums over ℓ_1, ℓ_2 and get

$$W_{hex}^{(gg)} = \frac{1}{2} \left[W_{hex}^{(g)} \right]^2 - \frac{1}{2} \frac{\sqrt{\lambda}}{2\pi} \int \frac{d\theta_1}{\pi \cosh^2 2\theta_1} \frac{d\theta_2}{\pi \cosh^2 2\theta_2} K_{sym}^{(gg)}(\theta_1, \theta_2) L^g(\theta_1) L^g(\theta_2), \tag{11.19}$$

where we used the short notation

$$L^g(\theta) = \ln \left[\left(1 + e^{i\phi - \sqrt{2}\tau \cosh \theta + i\sqrt{2}\sigma \sinh \theta} \right) \left(1 + e^{-i\phi - \sqrt{2}\tau \cosh \theta + i\sqrt{2}\sigma \sinh \theta} \right) \right]. \tag{11.20}$$

Meson–meson The meson–meson contribution is written in a completely analogous way:

$$W_{hex}^{(MM)} = \frac{1}{2} \sum_{m_1=1}^{+\infty} \sum_{m_2=1}^{+\infty} \int \frac{du_1}{2\pi} \mu_{m_1}^M(u_1) \frac{du_2}{2\pi} \mu_{m_2}^M(u_2) \frac{1}{P_{m_1 m_2}^{(MM)}(u_1|u_2) P_{m_2 m_1}^{(MM)}(u_2|u_1)}$$
$$\cdot e^{-\tau E_{m_1}^M(u_1) + i\sigma p_{m_1}^M(u_1)} e^{-\tau E_{m_2}^M(u_2) + i\sigma p_{m_2}^M(u_2)} \tag{11.21}$$

Expression (10.16) for the strong coupling limit of mesonic P factor implies the property

$$\frac{1}{P_{m_1 m_2}^{(MM)}(u_1|u_2) P_{m_2 m_1}^{(MM)}(u_2|u_1)} = 1 - \frac{2\pi}{\sqrt{\lambda}} m_1 m_2 K_{sym}^{(MM)}(\theta_1, \theta_2), \tag{11.22}$$

where

$$K_{sym}^{(MM)}(\theta_1, \theta_2) = -\frac{\sinh 2\theta_1 \sinh 2\theta_2}{\cosh(\theta_1 - \theta_2)} . \tag{11.23}$$

We get

$$W_{hex}^{(MM)} = \left(\frac{\sqrt{\lambda}}{2\pi}\right)^2 \frac{1}{2} \sum_{m_1=1}^{+\infty} \sum_{m_2=1}^{+\infty} \int \frac{d\theta_1}{\pi m_1^2 \sinh^2 2\theta_1} \frac{d\theta_2}{\pi m_2^2 \sinh^2 2\theta_2} (-1)^{m_1+m_2}$$
$$\cdot e^{-2\tau m_1 \cosh \theta_1 + 2i\sigma m_1 \sinh \theta_1} e^{-2\tau m_2 \cosh \theta_2 + 2i\sigma m_2 \sinh \theta_2}$$

$$+ \left(-\frac{\sqrt{\lambda}}{2\pi} \right) \frac{1}{2} \sum_{m_1=1}^{+\infty} \sum_{m_2=1}^{+\infty} \int \frac{d\theta_1}{\pi m_1 \sinh^2 2\theta_1} \frac{d\theta_2}{\pi m_2 \sinh^2 2\theta_2} (-1)^{m_1+m_2}$$

$$\cdot K_{sym}^{(MM)}(\theta_1, \theta_2) e^{-2\tau m_1 \cosh \theta_1 + 2i\sigma m_1 \sinh \theta_1} e^{-2\tau m_2 \cosh \theta_2 + 2i\sigma m_2 \sinh \theta_2}.$$

We can now perform the sums over m_1, m_2 and get

$$W_{hex}^{(MM)} = \frac{1}{2} \left[W_{hex}^{(M)} \right]^2 - \frac{1}{2} \frac{\sqrt{\lambda}}{2\pi} \int \frac{d\theta_1}{\pi \sinh^2 2\theta_1} \frac{d\theta_2}{\pi \sinh^2 2\theta_2} K_{sym}^{(MM)}(\theta_1, \theta_2) L^M(\theta_1) L^M(\theta_2),$$

$$(11.24)$$

where we used the short notation

$$L^M(\theta) = \ln \left(1 + e^{-2\tau \cosh \theta + 2i\sigma \sinh \theta} \right). \tag{11.25}$$

Meson–gluon Next step is to consider the meson–gluon contribution

$$W_{hex}^{(Mg)} = \frac{1}{2} \sum_{m=1}^{+\infty} \sum_{\ell=1}^{+\infty} \int \frac{du_1}{2\pi} \mu_m^M(u_1) \frac{du_2}{2\pi} \mu_\ell^g(u_2) e^{-\tau E_m^M(u_1) + i\sigma p_m^M(u_1)} e^{-\tau E_\ell^g(u_2) + i\sigma p_\ell^g(u_2)}$$

$$\cdot 2 \left[\frac{e^{i\ell\phi}}{P_{\ell m}^{(gM)}(u_2|u_1) P_{m\ell}^{(Mg)}(u_1|u_2)} + \frac{e^{-i\ell\phi}}{P_{\ell m}^{(\bar{g}M)}(u_2|u_1) P_{m\ell}^{(M\bar{g})}(u_1|u_2)} \right]. \tag{11.26}$$

Now, at strong coupling, with the redefinitions $u_1 = \sqrt{2}g \coth 2\theta_1$, $u_2 = \sqrt{2}g \tanh 2\theta_2$ expressions (10.20), (10.21) imply the property

$$\frac{1}{P_{\ell m}^{(gM)}(u_2|u_1) P_{m\ell}^{(Mg)}(u_1|u_2)} = \frac{1}{P_{\ell m}^{(\bar{g}M)}(u_2|u_1) P_{m\ell}^{(M\bar{g})}(u_1|u_2)}$$

$$= 1 - \frac{2\pi}{\sqrt{\lambda}} \ell m K_{sym}^{(Mg)}(\theta_1, \theta_2), \tag{11.27}$$

where

$$K_{sym}^{(Mg)}(\theta_1, \theta_2) = K^{(Mg)}(\theta_1, \theta_2) + K^{(gM)}(\theta_2, \theta_1) = K_{sym}^{(gM)}(\theta_2, \theta_1)$$

$$= \sqrt{2} \frac{\cosh 2\theta_2 \sinh 2\theta_1 \sinh(\theta_1 - \theta_2)}{\cosh(2\theta_2 - 2\theta_1)}. \tag{11.28}$$

Remembering the measures (11.7), (11.10) and the forms of energies and momenta (11.4), (11.8) and performing the sums, we arrive at the expression

$$W_{hex}^{(Mg)} = W_{hex}^{(g)} W_{hex}^{(M)} + \frac{\sqrt{\lambda}}{2\pi} \int \frac{d\theta_1}{\pi \sinh^2 2\theta_1} \frac{d\theta_2}{\pi \cosh^2 2\theta_2} K_{sym}^{(Mg)}(\theta_1, \theta_2) L^M(\theta_1) L^g(\theta_2)$$

$$(11.29)$$

11.4. Comparison and checks with TBA

We now compare our previous predictions at strong coupling with TBA outcome. We use (F.42–F.46) of [41]: these expressions depend on the functions $\epsilon(\theta - i\hat{\varphi})$, $\tilde{\epsilon}(\theta - i\hat{\varphi})$, which satisfy integral equations

$$\epsilon(\theta - i\hat{\varphi}) = E(\theta) - \int \frac{d\theta'}{\pi \cosh^2 2\theta'} K_{sym}^{(gg)}(\theta, \theta') L(\theta') + \int \frac{d\theta'}{\pi \sinh^2 2\theta'} K_{sym}^{(Mg)}(\theta', \theta) \tilde{L}(\theta')$$

$$\tilde{\epsilon}(\theta - i\hat{\varphi})$$
$$= \sqrt{2} E(\theta) - \int \frac{d\theta'}{\pi \cosh^2 2\theta'} K_{sym}^{(Mg)}(\theta, \theta') L(\theta') + \int \frac{d\theta'}{\pi \sinh^2 2\theta'} K_{sym}^{(MM)}(\theta, \theta') \tilde{L}(\theta'),$$
(11.30)

where the integrations are performed along the straight line $\mathbb{R} + i\hat{\varphi}$, and the functions $\tilde{L}(\theta)$, $L(\theta)$ are defined as

$$\tilde{L}(\theta) = \ln\left(1 + e^{-\tilde{\epsilon}(\theta - i\hat{\varphi})}\right), \quad L(\theta) = \ln\left[\left(1 + \mu e^{-\epsilon(\theta - i\hat{\varphi})}\right)\left(1 + \mu^{-1} e^{-\epsilon(\theta - i\hat{\varphi})}\right)\right]$$

while [41]

$$E(\theta) = -\frac{i}{\sqrt{2}} \sinh\theta \ln\left(\frac{u_1}{u_2 u_3}\right) + i \sinh\left(\theta - \frac{i\pi}{4}\right) \ln\left(\frac{1}{u_2} - 1\right).$$
(11.31)

The relation between $\hat{\varphi}$ and the cross-ratios can now be clarified by means of the functions ϵ, $\tilde{\epsilon}$, since

$$\ln\sqrt{\frac{u_1}{u_2 u_3}} = \epsilon(i\pi/4 - i\hat{\varphi}), \qquad \ln\left(\frac{1}{u_2} - 1\right) = \tilde{\epsilon}(-i\hat{\varphi}).$$
(11.32)

We expand (11.30) at large $E(\theta)$ and plug such expansions in (F.45, F.46) of [41], by using the Taylor expansion ($|f| \ll 1$)

$$\text{Li}_2(-e^{-F-f}) = \text{Li}_2(-e^{-F}) + f \ln(1 + e^{-F}) + O(f^2).$$
(11.33)

We obtain

$$(F.45) + (F.46) = -\int \frac{d\theta}{\pi \cosh^2 2\theta}\left[\text{Li}_2\left(-\mu e^{-E(\theta)}\right) + \text{Li}_2\left(-\mu^{-1} e^{-E(\theta)}\right)\right]$$
$$+ \int \frac{d\theta}{\pi \sinh^2 2\theta} \text{Li}_2\left(-e^{-\sqrt{2} E(\theta)}\right)$$
$$+ \int \frac{d\theta_1}{\pi \sinh^2 2\theta_1} \int \frac{d\theta_2}{\pi \sinh^2 2\theta_2} K_{sym}^{(MM)}(\theta_1, \theta_2) \tilde{L}_E(\theta_1) \tilde{L}_E(\theta_2)$$
$$+ \int \frac{d\theta_1}{\pi \cosh^2 2\theta_1} \int \frac{d\theta_2}{\pi \cosh^2 2\theta_2} K_{sym}^{(gg)}(\theta_1, \theta_2) L_E(\theta_1) L_E(\theta_2)$$
$$- 2 \int \frac{d\theta_1}{\pi \sinh^2 2\theta_1} \int \frac{d\theta_2}{\pi \cosh^2 2\theta_2} K_{sym}^{(Mg)}(\theta_1, \theta_2) \tilde{L}_E(\theta_1) L_E(\theta_2),$$
(11.34)

where we used the notations

$$\tilde{L}_E(\theta) = \ln\left(1 + e^{-\sqrt{2} E(\theta)}\right), \quad L_E(\theta) = \ln\left[\left(1 + \mu e^{-E(\theta)}\right)\left(1 + \mu^{-1} e^{-E(\theta)}\right)\right]. \quad (11.35)$$

Summing this result with (F.42), (F.43), (F.44) we get

$$\sum_{k=42}^{46} (F.k) = -\int \frac{d\theta}{\pi \cosh^2 2\theta}\left[\text{Li}_2\left(-\mu e^{-E(\theta)}\right) + \text{Li}_2\left(-\mu^{-1} e^{-E(\theta)}\right)\right]$$
$$+ \int \frac{d\theta}{\pi \sinh^2 2\theta} \text{Li}_2\left(-e^{-\sqrt{2} E(\theta)}\right)$$

$$+ \frac{1}{2} \int \frac{d\theta_1}{\pi \sinh^2 2\theta_1} \int \frac{d\theta_2}{\pi \sinh^2 2\theta_2} K_{sym}^{(MM)}(\theta_1, \theta_2) \tilde{L}_E(\theta_1) \tilde{L}_E(\theta_2)$$

$$+ \frac{1}{2} \int \frac{d\theta_1}{\pi \cosh^2 2\theta_1} \int \frac{d\theta_2}{\pi \cosh^2 2\theta_2} K_{sym}^{(gg)}(\theta_1, \theta_2) L_E(\theta_1) L_E(\theta_2)$$

$$- \int \frac{d\theta_1}{\pi \sinh^2 2\theta_1} \int \frac{d\theta_2}{\pi \cosh^2 2\theta_2} K_{sym}^{(Mg)}(\theta_1, \theta_2) \tilde{L}_E(\theta_1) L_E(\theta_2). \tag{11.36}$$

Now, in order to compare with our results, we parametrise the cross-ratios u_1, u_2, u_3 as in [35] (formula (157)), *i.e.*

$$\frac{1}{u_2} = 1 + e^{2\tau}, \quad \frac{1}{u_3} = 1 + (e^{-\tau} + e^{\sigma+i\phi})(e^{-\tau} + e^{\sigma-i\phi}), \quad \frac{u_1}{u_2 u_3} = e^{2\sigma+2\tau}. \tag{11.37}$$

Consequently, if we define $\mu = e^{i\phi}$, we obtain the relation

$$\mu + \mu^{-1} = \frac{1 - u_1 - u_2 - u_3}{\sqrt{u_1 u_2 u_3}}. \tag{11.38}$$

In addition, the function $E(\theta)$ may be written in a more transparent form as

$$E(\theta) = \sqrt{2}\tau \cosh \theta - i\sqrt{2}\sigma \sinh \theta. \tag{11.39}$$

Plugging (11.38), (11.39) into (11.36), the following relation

$$W_{hex} = W_{hex}^{(1)} + W_{hex}^{(2)} + \cdots = \exp\left(-\frac{\sqrt{\lambda}}{2\pi}[\sum_{k=42}^{46}(F.k)]\right)$$

$$\cong 1 - \frac{\sqrt{\lambda}}{2\pi}[\sum_{k=42}^{46}(F.k)] + \frac{1}{2}\left(\frac{\sqrt{\lambda}}{2\pi}\right)^2 [\sum_{k=42}^{46}(F.k)]^2 + \cdots \tag{11.40}$$

is in agreement with expressions for $W_{hex}^{(1)}$, $W_{hex}^{(2)}$ computed in last subsection.

11.5. Re-summation of the BSV series

The agreement displayed above between the series written in [34] for hexagonal Wilson loops and the TBA for scattering amplitudes [39,41,40] can be made even tighter, since it is not restricted to one and two particle contributions, but instead it does also extend to any number of particles. Even better, the BSV series for the hexagon (11.3) can be fully re-summed by exploiting some standard techniques: eventually we will reproduce (as for the strong coupling regime) the TBA (in the form elaborated in) [41]. In the following we will produce the main steps, but leave some further details and generalisations for an incoming publication [58].

The expression to sum up is the simple manipulation of the initial formula, (11.3), strictly speaking valid for singlets, which we re-call here for practical reasons

$$W_{hex} = \sum_{N=0}^{+\infty} \frac{1}{N!} \sum_{a_1} \cdots \sum_{a_N} \int \prod_{k=1}^{N} \left[\frac{du_k}{2\pi} \mu_{a_k}(u_k) e^{-\tau E_{a_k}(u_k) + i\sigma p_{a_k}(u_k) + im_{a_k}\phi} \right]$$

$$\cdot \prod_{i<j}^{N} \frac{1}{P_{a_i,a_j}(u_i|u_j) P_{a_j,a_i}(u_j|u_i)}, \tag{11.41}$$

where the indices a_k label the species of different particles (including bound states).

In general, not only at strong coupling, we may use a path integral trick of the type as in [60, 61], but then we should integrate eventually the extra ρ-field(s) [58]. Thus, we better perform on the above series (11.41) a similar trick without the ρ field(s), the Hubbard–Stratonovich transform [63]. The latter makes use of the well know identity of functional gaussian integration in the presence of a linear source term[23]

$$\prod_{i<j}^{N} e^{\langle X_{(a_i)}(u_i) \, X_{(a_j)}(u_j) \rangle} = \langle e^{X_{(a_1)}(u_1)} \cdots e^{X_{(a_N)}(u_N)} \rangle, \tag{11.42}$$

$(e^{\langle X_{(a_i)}(u_i) \, X_{(a_i)}(u_i) \rangle} \simeq 1)$ for allowing the summation to act on the single exponential of the r.h.s.[24] This means that we need also to relate the pentagonal amplitudes $P_{a,b}(u|v)$ to correlators (and then to the kinetic part) of the gaussian field $X_{(a)}$ in this way

$$\frac{1}{P_{a,b}(u|v) \, P_{b,a}(v|u)} = e^{\langle X_{(a)}(u) \, X_{(b)}(v) \rangle}. \tag{11.43}$$

For instance, we associate gluons to the (fluctuating) field $X_{(1)}^g(u) = X^g(u)$, whereas we denote their bound states as $X_{(\ell)}^g(u)$, where ℓ stands for the number of components. Thus, the 'linearisation' of the exponent is complete, namely we can recast the hexagonal Wilson loop (11.41) into a shape aiming at re-summing the different contributions:

$$W_{hex} = \langle \exp \left\{ \int \frac{du}{2\pi} \sum_a \left[\mu_a(u) \, e^{-\tau E_a(u) + i\sigma p_a(u) + i m_a \phi} \, e^{X_{(a)}(u)} \right] \right\} \rangle. \tag{11.44}$$

With this series (on any particle kind, a) inside, (11.44) can be interpreted as a Kac–Feynman path integral (partition function) for any value of the coupling, nevertheless it is at strong coupling $\lambda \longrightarrow \infty$ that a tangible simplification occurs since it can be summed up. For simplicity's sake, we will initially include only gluons and their bound states, then adapt our derivation easily to mesons and their bound states. Eventually, we will consider the general system (at strong coupling only), composed of gluons, meson and bound states. For, in this regime, the bound states enjoy a series of simple, useful properties: their energies and momenta (11.4) are simply additive, as so is the relation $X_{(a)}^g = a \, X_{(1)}^g$ implied by the peculiar limit form $\ell_1 \ell_2 K_{sym}^{(gg)}(u,v)$ of the (bound state) gluonic kernels in (11.15) via (11.43) on the bound state fields $X_{(a)}^g$; and finally the measures $\mu_\ell^g(u)$ (11.7) exhibit a peculiar square at denominator. Altogether these properties turn out to be crucial to re-sum the gluonic part of the hexagonal Wilson loop (11.44), $W_{hex}^{(g)}$, in a handy shape, and they bring up the dilogarithm function $\text{Li}_2(x)$ (tuned by the third property, *cf.* [58] for more details):

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[23] The following formula is the infinite dimensional $d \to \infty$ version of

$$\langle e^{X_1 J_1} e^{X_2 J_2} \cdots e^{X_d J_d} \rangle = \sqrt{\det T} \int \prod_{i=1}^{d} \frac{dX_i}{\sqrt{2\pi}} e^{-\frac{1}{2} \sum\limits_{i,j=1}^{d} X_i T_{ij} X_j} \, e^{\sum\limits_{i=1}^{d} X_i J_i} = e^{\frac{1}{2} \sum\limits_{i,j=1}^{d} J_i G_{ij} J_j},$$

with propagator $G = T^{-1}$, where we choose, for the continuum limit of the external field $J_i \to J(u)$, the configuration of point-like sources $J(u) = \sum_{i=1}^{N} a_i \delta(u - u_i)$, *cf.* [58] for details.

[24] We mention the talk held by B. Basso at IGST 2013 in Utrecht concerning one gluon (without bound states) and [62] for useful suggestions. We wish also to notice the possibility of interpreting the free boson $c = 1$ 2D CFT (Coulomb gas) correlation function formulæ by means of this one.

$$W_{hex}^{(g)} = \left\langle \exp\left\{-\int \frac{du}{2\pi}\, \mu^g(u) \left[\mathrm{Li}_2(-e^{-\tau E_1^g(u) + i\sigma p_1^g(u) + i\phi}\, e^{X^g(u)}) \right.\right.\right.$$

$$\left.\left.\left. + \mathrm{Li}_2(-e^{-\tau E_1^g(u) + i\sigma p_1^g(u) - i\phi}\, e^{X^g(u)}) \right] \right\} \right\rangle. \tag{11.45}$$

Now, we can make explicit the gaussian measure in (11.45) as a kinetic term so to read $W_{hex}^{(g)}$ as a quantum mechanics partition function for the field $X^g(u)$

$$W_{hex}^{(g)} = Z^{(g)}[X^g] = \int \mathcal{D}X^g\, e^{-\mathcal{S}^{(g)}[X^g]}, \tag{11.46}$$

where the action $\mathcal{S}^{(g)}[X^g]$, directly expressed in terms of the hyperbolic rapidity θ, has the form

$$\mathcal{S}^{(g)}[X^g] = \frac{1}{2}\int d\theta\, d\theta'\, X^g(\theta) T^g(\theta,\theta') X^g(\theta') + \int \frac{d\theta'}{2\pi}\, \mu^g(\theta')$$

$$\cdot \left[\mathrm{Li}_2(-e^{-E(\theta') + i\phi}\, e^{X^g(\theta')}) + \mathrm{Li}_2(-e^{-E(\theta') - i\phi}\, e^{X^g(\theta')}) \right], \tag{11.47}$$

with $E(\theta)$ coinciding with the derived (11.39). Of course the kinetic kernel $T^g(\theta',\theta'')$ is the inverse

$$\int d\theta'\, G^g(\theta,\theta') T^g(\theta',\theta'') = \delta(\theta - \theta'') \tag{11.48}$$

of the Green function[25]

$$G^g(\theta,\theta') = \langle X^g(\theta) X^g(\theta') \rangle. \tag{11.49}$$

Remarkably, the action (11.47) is proportional to g, which is going to $+\infty$, so making possible the applicability of the saddle point with classical equation of motion:

$$X^g(\theta) - \int \frac{d\theta'}{2\pi}\, G^g(\theta,\theta') \mu^g(\theta')$$

$$\cdot \log\left[(1 + e^{X^g(\theta')} e^{-E(\theta') + i\phi})(1 + e^{X^g(\theta')} e^{-E(\theta') - i\phi}) \right] = 0, \tag{11.50}$$

where the Green function, at strong coupling, can be easily related to (the symmetric part of) the gluonic pentagonal amplitude

$$G^g(\theta,\theta') = -\frac{2\pi}{\sqrt{\lambda}}\, K_{sym}^{(gg)}(\theta,\theta') + O(1/\lambda). \tag{11.51}$$

The introduction of the 'pseudo-energy' $\epsilon(\theta)$ via the relation $\epsilon(\theta - i\hat\varphi) = E(\theta) - X^g(\theta)$, leads us to the special version of the TBA equations for gluons only, the first of (11.30) in which we fully neglect $\tilde\epsilon(\theta - i\hat\varphi)$, i.e. the meson contribution. In other words we have found an action (a Yang–Yang functional) whose differentiation give rise to equations in TBA form [29,28], without thermodynamics. In this respect the generation of the $\mathrm{Li}_2(x)$ function via summation on bound states is of fundamental importance.

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[25] We could realise that directly $G^g(\theta,\theta')$ (instead of $T^g(\theta',\theta'')$) appears in the action by means of a Hubbard–Stratonovich transform which introduces some gaussian field ρ^g coupled to X^g: in this way we will end up with the usual form of the Yang–Yang potential of Nekrasov–Shatashvili [59] as it would be following *ab initio* the path integral trick contained in [60,61], *cf.* [58].

As for the mesonic sector, the reasonings outlined above can be easily repeated by substituting the gluon and bound states thereof with the meson and bound states thereof, respectively.[26] In first place, we associate the fields $X^M_{(\ell)}(\theta)$ to bound states of mesons, each represented by the single meson $X^M_{(1)}(\theta) = X^M(\theta)$ by means of the relation $X^M_{(\ell)}(\theta) = \ell\, X^M_{(1)}(\theta)$. From the identification

$$\frac{1}{P^M_{a,b}(u|v)\, P^M_{b,a}(v|u)} = e^{\langle X^M_{(a)}(\theta) X^M_{(b)}(\theta') \rangle} \tag{11.52}$$

it follows that the meson-only hexagonal Wilson loop $W^{(M)}_{hex}$ assumes a shape analogous to (11.44) and can be re-summed at all coupling, even though a remarkable simplification occur at strong coupling, owing to the properties of the mesonic kernel:

$$W^{(M)}_{hex} = \left\langle \exp\left\{ -\int \frac{du}{2\pi}\, \mu^M(u) \text{Li}_2(-e^{-\tau E^M_1(u) + i\sigma p^M_1(u)}\, e^{X^M(u)}) \right\} \right\rangle \tag{11.53}$$

Again, the meson hexagonal Wilson loop can be associated to a partition function, defined via the action $\mathcal{S}^{(M)}[X^M]$

$$\mathcal{S}^{(M)}[X^M] = \frac{1}{2} \int d\theta'\, d\theta''\, X^M(\theta') T^M(\theta', \theta'') X^M(\theta'')$$
$$+ \int \frac{d\theta''}{2\pi}\, \mu^M(\theta'') \text{Li}_2(-e^{-\sqrt{2}E(\theta'')}\, e^{X^M(\theta'')}) \tag{11.54}$$

which, under extremisation, gives the equation of motion:

$$X^M(\theta) - \int \frac{d\theta'}{2\pi}\, G^M(\theta, \theta')\, \mu^M(\theta') \log\left[1 + e^{X^M(\theta')} e^{-\sqrt{2}E(\theta')} \right] = 0 \tag{11.55}$$

where the mesonic Green function has been introduced

$$\int d\theta'\, G^M(\theta, \theta') T^M(\theta', \theta'') = \delta(\theta - \theta'')\,. \tag{11.56}$$

If we define the function $\tilde{\epsilon}(\theta - i\hat{\varphi}) = \sqrt{2}E(\theta) - X^M(\theta)$, we obtain uniquely the mesonic TBA equation, *i.e.* the second of (11.30) where we discard the gluonic contribution (considered above).

Complete system After the considerations outlined above for incomplete systems, made of a single type of particle (and relative bound states) at one time, we can now cope with the complete system, including gluons and mesons together, by arranging the gluonic and mesonic fields into a vector, and the measures as well:

$$X^a(\theta) = \begin{pmatrix} X^g(\theta) \\ X^M(\theta) \end{pmatrix} \qquad \mu_a(\theta) = \begin{pmatrix} \mu_1(\theta) \\ \mu_2(\theta) \end{pmatrix} \equiv \begin{pmatrix} \mu^g(\theta) \\ \mu^M(\theta) \end{pmatrix} \tag{11.57}$$

(the label a takes the values $a = 1, 2$; the sum convention on repeated indices is assumed). The complete hexagonal amplitude can thus be expressed as

$$W_{hex} = \int \mathcal{D}X_1\, \mathcal{D}X_2\, e^{-\mathcal{S}[X]}, \tag{11.58}$$

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[26] As anticipated about the bound state analysis in Section 9, a more mathematical understanding of the contributions of the mesons, as small fermion–antifermion state, and their bound states should be given in future [58] with a mechanism where the poles pinch the integration axis [59–62,36].

where the action reads

$$\mathcal{S}[X] = \frac{1}{2} \int d\theta\, d\theta'\, X^a(\theta)\, T_{ab}(\theta, \theta')\, X^b(\theta')$$
$$+ \int \frac{d\theta}{2\pi} \Big[\mu_1(\theta)\, \mathrm{Li}_2(-e^{-E(\theta)+i\phi}\, e^{X_1(\theta)})$$
$$+ \mu_1(\theta)\, \mathrm{Li}_2(-e^{-E(\theta)-i\phi}\, e^{X_1(\theta)}) + \mu_2(\theta)\, \mathrm{Li}_2(-e^{-\sqrt{2}E(\theta)}\, e^{X_2(\theta)}) \Big] , \qquad (11.59)$$

and the Green functions, now arranged in a matrix, can anew be associated to the pentagonal amplitudes

$$G^{ab}(\theta, \theta') = -\frac{2\pi}{\sqrt{\lambda}} \begin{pmatrix} K^{(gg)}_{sym}(\theta, \theta') & K^{(Mg)}_{sym}(\theta', \theta) \\ K^{(Mg)}_{sym}(\theta, \theta') & K^{(MM)}_{sym}(\theta, \theta') \end{pmatrix} , \qquad (11.60)$$

and allow us to define the kinetic 2×2 matrix as its inverse:

$$\int d\theta'\, G^{ab}(\theta, \theta')\, T_{bc}(\theta', \theta'') = \delta^a_c\, \delta(\theta - \theta'') . \qquad (11.61)$$

Upon extremising the action $\mathcal{S}[X]$, we obtain

$$X^a(\theta) - \int \frac{d\theta'}{2\pi}\, G^{ab}(\theta, \theta')\, \mu_b(\theta')\, L^b(\theta') = 0 , \qquad (11.62)$$

where we have assumed the definitions

$$L^1(\theta) = \log\Big[(1 + e^{X^1(\theta)} e^{-E(\theta)+i\phi})(1 + e^{X^1(\theta)} e^{-E(\theta)-i\phi})\Big] ,$$
$$L^2(\theta) \equiv \log\Big[1 + e^{X^2(\theta)} e^{-\sqrt{2}E(\theta)}\Big] . \qquad (11.63)$$

These equations of motions (11.62) match the TB(ubble)A equations (11.30) of [41], provided we identify the pseudo-energies as $\epsilon(\theta - i\hat{\varphi}) = E(\theta) - X^1(\theta)$ and $\tilde{\epsilon}(\theta - i\hat{\varphi}) = \sqrt{2}E(\theta) - X^2(\theta)$, which simply translates into

$$L(\theta) = L^1(\theta), \quad \tilde{L}(\theta) = L^2(\theta). \qquad (11.64)$$

Since the action (11.59) possesses the diverging prefactor $\mathcal{S}[X] \propto \sqrt{\lambda}$, the hexagonal Wilson loop W_{hex} (11.58) is dominated by the classical configuration, achieved by imposing the equations of motion on the fields, and therefore with the aid of (11.62) we can rewrite the kinetic term in the action (11.59) as

$$\frac{1}{2} \int d\theta\, d\theta'\, X^a(\theta)\, T_{ab}(\theta, \theta')\, X^b(\theta')$$
$$= -\frac{\sqrt{\lambda}}{2\pi} \int \frac{d\theta\, d\theta'}{(2\pi)^2} \frac{L^1(\theta)\, L^1(\theta')}{\cosh 2\theta\, \cosh 2\theta'\, \cosh(\theta - \theta')}$$
$$+ \frac{\sqrt{\lambda}}{2\pi} \int \frac{d\theta\, d\theta'}{2\pi^2} \frac{L^2(\theta)\, L^2(\theta')}{\sinh 2\theta\, \sinh 2\theta'\, \cosh(\theta - \theta')}$$
$$+ \frac{\sqrt{\lambda}}{2\pi} \sqrt{2} \int \frac{d\theta\, d\theta'}{\pi^2} \frac{L^2(\theta)\, L^1(\theta')}{\sinh 2\theta\, \cosh 2\theta'} \frac{\sinh(\theta - \theta')}{\cosh(2\theta - 2\theta')} . \qquad (11.65)$$

Eventually, the sum of the kinetic term (11.65) and the potential part, given by the second and third lines of (11.59) computed on the solution on the saddle point equations, amounts to the

critical Yang–Yang functional $\frac{\sqrt{\lambda}}{2\pi} \, Y Y_{cr}$, which has been computed in [41] by adding together the right hand sides of the formulae from (F.42) to (F.46).

12. Conclusions in perspective

We have derived the complete set of Asymptotic Bethe Ansatz (ABA) equations referring to the GKP vacuum instead that to the half-BPS state (Beisert–Staudacher equations [2]). These describe the dynamics of all the elementary excitations over the GKP vacuum (gluons, fermions and scalars), but they also admit solutions in the form of complexes of Bethe and/or auxiliary roots, the so-called strings or stacks. The latter are the bound states, among whose the most important are the bound states of the elementary particles (the other are bound states of the auxiliary or isotopic roots, yet important for the spectrum TBA and so on). In this way, we have performed the 'fusion' of the fundamental (elementary and isotopic) excitations, which is in its whole an alternative way to perform the bootstrap of S-matrices (*cf.* for instance [64] for a review).

Moreover, we outlined this system of algebraic equations at all coupling values, also including weak and strong (in different dynamical regimes) coupling. Above all, we have mainly focused on the scattering phases between all kind of particles at any coupling, but also the new feature of two defects has arisen in the form of new scattering phases for any flavour. Then, we have devoted a meticulous care to the behaviour of the scattering factors in the three possible, – non-perturbative, perturbative and giant hole –, regimes which allow different large g expansions: for all these three, we obtained explicit expressions of all the scattering factors.

If the momentum of any particle enters the ABA equations, the energy/anomalous dimension is the final object expressed via a solution of these equations. And we could confirm for these first two conserved charges the achievements by [20], but also have been led to consider all the higher integrals of motion (which do play a so important rôle in the construction of the dressing factor in the usual ABA on the BMN vacuum).

A deeper look at the form of these new ABA equations brought to our attention an interesting property or identification for them: the $su(4)$ residual R-symmetry constraint the elementary particles to have as rapidities the inhomogeneities of an $su(4)$ symmetric spin chain of S-matrices which belong at any lattice site to the characteristic representation of the particle, *i.e.* **1**, **4**, **4̄**, **6** (for gluons, fermions, antifermions, scalars, respectively). Thus, as anticipated in [23], the matrix structure of the ABA equations could be inferred from the $SU(4)$ symmetry, but the specific form of the scalar factors and its g-dependence must be computed explicitly. For instance, in this perspective, the two defects are simply two purely transmitting impurities which still respect the $SU(4)$ symmetry. Moreover, the particular g-dependence shows explicitly the decoupling of the six scalars in the non-perturbative regime and their approach to the $O(6)$ non-linear sigma model S-matrix in [52], being, besides, the defects of no importance in this limit. More importantly, we have seen from the fusion of a fermion and an antifermion the formation of a new particle in the $g \to +\infty$ perturbative regime: a meson. Then we also identified bound states thereof.

At last, but not least we have been looking for confirmation and deep comparison of our careful strong coupling outcomes with the scattering amplitude/WL TBA [39,40] via the OPE or flux tube (BSV) series [41,34–38] (see, also, *e.g.* [66] for the multi-Regge limit). In fact, the basic object of the latter, the so-called pentagon amplitude, can be expressed via the aforementioned scattering factors as proposed for the gluons in [34]. The bound states of the latter, the meson and its bound states appear to be the only other relevant particles at leading order (the minimal area of classical string). Therefore, we have checked explicitly those features by re-summing the

BSV series [34] in case of a null hexagonal Wilson loop: we have used the saddle point method at large g to obtain the critical equations coinciding with the TBA equations of [39,41,40]. Then, we have computed the action on them and obtained the same (critical) Yang–Yang functional (or free energy) as in [39,41,40]. Interestingly, the same set-up should be easily applicable to the computation of the heptagon WL. Nevertheless, it would desirable to have a more direct understanding of the phase we dubbed *confinement* of the fermions, which disappear as free particles, inside the mesons and their bound states.

Acknowledgements

Appendix A. Functions

This appendix is devoted to the introduction of the functions we used throughout the paper. In the study of scalars we found convenient to use the following shorthand notations:

$$\Phi(u) = \Phi_0(u) + \Phi_H(u), \quad \phi(u,v) = \phi_0(u-v) + \phi_H(u,v), \tag{A.1}$$

with

$$\Phi_0(u) = -i\ln\frac{i+2u}{i-2u}, \quad \Phi_H(u) = -i\ln\left(\frac{1+\frac{g^2}{2x^-(u)^2}}{1+\frac{g^2}{2x^+(u)^2}}\right), \tag{A.2}$$

$$\phi_0(u-v) = i\ln\frac{i+u-v}{i-u+v}, \quad \phi_H(u,v) = -2i\left[\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(v)}}{1-\frac{g^2}{2x^-(u)x^+(v)}}\right) + i\theta(u,v)\right], \tag{A.3}$$

$\theta(u,v)$ being the dressing phase [10] and $x(u) = \frac{u}{2}\left[1+\sqrt{1-\frac{2g^2}{u^2}}\right]$, $x^\pm(u) = x(u\pm\frac{i}{2})$. We used also

$$\varphi(u,v) = \frac{1}{2\pi}\frac{d}{dv}\phi(u,v). \tag{A.4}$$

For what concerns gluon and their bound states, we used the function

$$\chi(v,u|l) = \chi_0(v-u|l+1) + \chi_H\left(v,u-\frac{il}{2}\right) + \chi_H\left(v,u+\frac{il}{2}\right),$$

where

$$\chi_0(u|l) = i\ln\frac{il+2u}{il-2u} = 2\arctan\frac{2u}{l}, \quad \chi_H(v,u) = i\ln\left(\frac{1-\frac{g^2}{2x^-(v)x(u)}}{1-\frac{g^2}{2x^+(v)x(u)}}\right), \tag{A.5}$$

which enjoys the expression

$$\chi(v, u|l) = i \ln \left(\frac{x^+(v) - x\left(u - \frac{il}{2}\right)}{x\left(u + \frac{il}{2}\right) - x^-(v)} \right) + i \ln \left(\frac{1 - \frac{g^2}{2x^-(v)x\left(u - \frac{il}{2}\right)}}{1 - \frac{g^2}{2x^+(v)x\left(u + \frac{il}{2}\right)}} \right). \tag{A.6}$$

Scattering factors involving gluons and their bound states are expressed in terms of the function

$$\chi(v, u|l) + \Phi(v) = i \ln \left(\frac{x^+(v) - x\left(u - \frac{il}{2}\right)}{x\left(u + \frac{il}{2}\right) - x^-(v)} \right) + i \ln \left(\frac{\frac{g^2}{2x\left(u - \frac{il}{2}\right)} - x^-(v)}{x^+(v) - \frac{g^2}{2x\left(u + \frac{il}{2}\right)}} \right). \tag{A.7}$$

Finally, for large fermions we introduced the function

$$\chi_F(u, v) = \chi_0(u - v|1) + \chi_H(u, v) = i \ln \left(\frac{x^+(u) - x(v)}{x(v) - x^-(u)} \right). \tag{A.8}$$

Scattering factors involving large fermions depend on the function

$$\chi_F(u, v) + \Phi(u) = i \ln \frac{x^+(u) - x(v)}{x(v) - x^-(u)} + i \ln \left(-\frac{x^-(u)}{x^+(u)} \right). \tag{A.9}$$

Scattering factors for small fermions are obtained from scattering factors for large fermions after the substitution

$$\chi_F(u, v) + \Phi(u) \to -\chi_H(u, v) = i \ln \left(\frac{1 - \frac{g^2}{2x^+(u)x(v)}}{1 - \frac{g^2}{2x^-(u)x(v)}} \right) = i \ln \left(\frac{1 - \frac{x_f(v)}{x^+(u)}}{1 - \frac{x_f(v)}{x^-(u)}} \right), \tag{A.10}$$

where

$$x_f(v) = \frac{g^2}{2x(v)} = \frac{v}{2} \left[1 - \sqrt{1 - \frac{2g^2}{v^2}} \right]. \tag{A.11}$$

Appendix B. Useful formulæ

B.1. Fourier transforms

We collect here some of the Fourier transforms

$$\hat{f}(k) = \int_{-\infty}^{+\infty} du \, e^{-iku} f(u) \tag{B.1}$$

of functions $f(u)$ we use in the main text.

For scalar we used

$$\Phi_0(u) = -i \ln \frac{i + 2u}{i - 2u} \quad \Rightarrow \quad \hat{\Phi}_0(k) = \int_{-\infty}^{+\infty} du \, e^{-iku} \Phi_0(u) = -\frac{2\pi}{ik} e^{-\frac{|k|}{2}}, \tag{B.2}$$

$$\Phi_H(u) = -i \ln \left(\frac{1 + \frac{g^2}{2x^-(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right) \quad \Rightarrow \quad \hat{\Phi}_H(k) = \frac{2\pi}{ik} e^{-\frac{|k|}{2}} [1 - J_0(\sqrt{2}gk)] \tag{B.3}$$

and also

$$\phi_0(u - v) = i \ln \frac{i + u - v}{i - u + v} \quad \Rightarrow \quad \hat{\phi}_0(k) = \frac{2\pi e^{-|k|}}{ik}, \tag{B.4}$$

$$\varphi_0(u - v) = \frac{1}{2\pi} \frac{d}{dv} \phi_0(u - v) = -\frac{1}{\pi} \frac{1}{1 + (u - v)^2} \quad \Rightarrow \quad \hat{\varphi}_0(k) = -e^{-|k|}, \tag{B.5}$$

$$\phi_H(u, v) = -2i \left[\ln \left(\frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right],$$

$$\hat{\phi}_H(k, t) = \int_{-\infty}^{+\infty} du\, e^{-iku} \int_{-\infty}^{+\infty} dv\, e^{-itv} \phi_H(u, v)$$

$$= -8i\pi^2 \frac{e^{-\frac{|t|+|k|}{2}}}{k|t|} \left[\sum_{r=1}^{\infty} r(-1)^{r+1} J_r(\sqrt{2}gk) J_r(\sqrt{2}gt) \frac{1 - \text{sgn}(kt)}{2} \right.$$

$$+ \text{sgn}(t) \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} c_{r,r+1+2\nu}(g)(-1)^{r+\nu} \left(J_{r-1}(\sqrt{2}gk) J_{r+2\nu}(\sqrt{2}gt) \right.$$

$$\left. - J_{r-1}(\sqrt{2}gt) J_{r+2\nu}(\sqrt{2}gk) \right) \right]. \tag{B.6}$$

We remark that in previous literature integral equations concerning the scalar sector are often written by using the 'magic kernel' $\hat{K}$ [10], related to $\hat{\phi}_H$ by

$$\hat{\phi}_H(k, t) + \hat{\phi}_H(k, -t) = 8i\pi^2 g^2 e^{-\frac{t+k}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt), \quad t, k > 0. \tag{B.7}$$

For what concerns gluon bound states, we introduced

$$\chi_0(u|l) = i \ln \frac{il + 2u}{il - 2u} = 2 \arctan \frac{2u}{l} \quad \Rightarrow$$

$$\hat{\chi}_0(k|l) = \int_{-\infty}^{+\infty} du\, e^{-iku} \chi_0(u|l) = \frac{2\pi}{ik} e^{-|k|\frac{l}{2}} \tag{B.8}$$

and for higher loops the function

$$\chi(v, u|l) = \chi_0(v - u|l + 1) + \chi_H \left(v, u - \frac{il}{2} \right) + \chi_H \left(v, u + \frac{il}{2} \right),$$

where

$$\chi_H(v, u) = i \ln \left(\frac{1 - \frac{g^2}{2x^-(v)x(u)}}{1 - \frac{g^2}{2x^+(v)x(u)}} \right), \tag{B.9}$$

whose Fourier transform reads

$$\int\limits_{-\infty}^{+\infty} du \int\limits_{-\infty}^{+\infty} dv e^{-ikv} e^{-itu} \chi(v, u|l)$$

$$= 2\pi \delta(t+k) \frac{2\pi}{ik} e^{-|k| \frac{l+1}{2}} + i \sum_{n=1}^{+\infty} n(-1)^n \frac{2\pi}{k} \frac{2\pi}{|t|} e^{-\frac{|k|}{2}} e^{-\frac{|t|l}{2}} J_n(\sqrt{2}gk) J_n(\sqrt{2}gt). \quad \text{(B.10)}$$

In getting (B.10) we used the Fourier transforms

$$\int\limits_{-\infty}^{+\infty} du e^{-iku} \frac{1}{x \left(u \pm i\frac{l}{2}\right)^n} = \pm n \left(\frac{\sqrt{2}}{ig}\right)^n \theta(\pm k) \frac{2\pi}{k} e^{\mp \frac{l}{2}k} J_n(\sqrt{2}gk). \quad \text{(B.11)}$$

It is useful to Fourier transform $\chi(v, u|l)$ and $\chi_H(v, u)$ with respect only to the variable v:

$$\int\limits_{-\infty}^{+\infty} dv e^{-ikv} \chi(v, u|l) = e^{-iku} \frac{2\pi}{ik} e^{-|k| \frac{l+1}{2}} + i \sum_{n=1}^{+\infty} \left(\frac{g}{\sqrt{2}i \; x \left(u - \frac{il}{2}\right)}\right)^n \frac{2\pi}{k} e^{-\frac{|k|}{2}} J_n(\sqrt{2}gk)$$

$$+ i \sum_{n=1}^{+\infty} \left(\frac{g}{\sqrt{2}i \; x \left(u + \frac{il}{2}\right)}\right)^n \frac{2\pi}{k} e^{-\frac{|k|}{2}} J_n(\sqrt{2}gk), \quad \text{(B.12)}$$

$$\int\limits_{-\infty}^{+\infty} dv e^{-ikv} \chi_H(v, u) = i \frac{2\pi}{k} e^{-\frac{|k|}{2}} \sum_{n=1}^{+\infty} \left(\frac{g}{\sqrt{2}i \; x(u)}\right)^n J_n(\sqrt{2}gk). \quad \text{(B.13)}$$

Finally, for large fermions we introduced the function

$$\chi_F(v, u) = \chi_0(v - u|1) + \chi_H(v, u) = i \ln \left(\frac{x^+(v) - x(u)}{x(u) - x^-(v)}\right), \quad \text{(B.14)}$$

whose Fourier transform with respect to v is easily extracted from (B.12), (B.13):

$$\int\limits_{-\infty}^{+\infty} dv e^{-ikv} \chi_F(v, u) = e^{-iku} \frac{2\pi}{ik} e^{-\frac{|k|}{2}} + i \frac{2\pi}{k} e^{-\frac{|k|}{2}} \sum_{n=1}^{+\infty} \left(\frac{g}{\sqrt{2}i \; x(u)}\right)^n J_n(\sqrt{2}gk). \quad \text{(B.15)}$$

B.2. BES and BES-like integral equations

The BES integral equation for the density $\hat{\sigma}_{BES}(k)$ in Fourier space reads as

$$\hat{\sigma}_{BES}(k) = -\frac{2ik}{1 - e^{-|k|}} \frac{\hat{\phi}_H(k, 0)}{\pi} + \frac{ik}{4(1 - e^{-|k|})} \int \frac{dt}{\pi^2} \hat{\phi}_H(k, t) \hat{\sigma}_{BES}(t). \quad \text{(B.16)}$$

Owing to the parity properties $\hat{\sigma}_{BES}(k) = \hat{\sigma}_{BES}(-k)$, we can restrict this equation in the region $k > 0$. Introducing the kernel $\hat{K}$

$$\hat{\phi}_H(k, t) + \hat{\phi}_H(k, -t) = 8i\pi^2 g^2 e^{-\frac{t+k}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt), \quad t, k > 0, \quad \text{(B.17)}$$

we have

$$\hat{\sigma}_{BES}(k) = \frac{4\pi g^2 k}{\sinh \frac{k}{2}} \hat{K}(\sqrt{2}gk, 0) - \frac{g^2 k}{\sinh \frac{k}{2}} \int\limits_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \hat{\sigma}_{BES}(t). \quad \text{(B.18)}$$

We can connect to quantities used in [20] by means of

$$\hat{\sigma}_{BES}(k) = \frac{\pi}{\sinh\frac{k}{2}}\left[\gamma_+^{\phi}(\sqrt{2}gk) + \gamma_-^{\phi}(\sqrt{2}gk)\right], \quad k > 0 \tag{B.19}$$

and

$$\gamma_+^{\phi}(\sqrt{2}gk) = 2\sum_{n\geq 1} 2n\gamma_{2n}^{\phi} J_{2n}(\sqrt{2}gk),$$

$$\gamma_-^{\phi}(\sqrt{2}gk) = 2\sum_{n\geq 1}(2n-1)\gamma_{2n-1}^{\phi} J_{2n-1}(\sqrt{2}gk). \tag{B.20}$$

The total density at order $\ln s$ is $\hat{\sigma}_{\ln s}(k) = -8\pi\delta(k) + \hat{\sigma}_{BES}(k)$ which satisfy the equation

$$\hat{\sigma}_{\ln s}(k) = -8\pi\delta(k) + \frac{ik}{4(1-e^{-|k|})}\int\frac{dt}{\pi^2}\hat{\phi}_H(k,t)\hat{\sigma}_{\ln s}(t). \tag{B.21}$$

The Fourier transform of the density associated to the first generalised scaling function [12] satisfies the equation

$$\hat{\sigma}^{(1)}(k) = \frac{\pi}{\sinh\frac{|k|}{2}}[e^{-\frac{|k|}{2}} - J_0(\sqrt{2}gk)] + \frac{ik}{1-e^{-|k|}}\int\limits_{-\infty}^{+\infty}\frac{dt}{4\pi^2}\hat{\phi}_H(k,t)\left[2\pi + \hat{\sigma}^{(1)}(t)\right]. \tag{B.22}$$

Eventually, the density 'all internal holes', which satisfies equation (3.8) of [21] with $L = 3$ is solution of

$$\hat{\sigma}(k;x) = \frac{2\pi e^{-|k|}}{1-e^{-|k|}}(\cos kx - 1)$$

$$+ \frac{ik}{1-e^{-|k|}}\int\limits_{-\infty}^{+\infty}\frac{dt}{4\pi^2}\hat{\phi}_H(k,t)\left[2\pi(\cos tx - 1) + \hat{\sigma}(t;x)\right]. \tag{B.23}$$

B.3. Integrals

In the one loop case we make use of the following integrals

$$-\int\limits_{-\infty}^{+\infty}\frac{dv}{2\pi}\ln\frac{ib+v}{ib-v}\frac{d}{dv}\ln\frac{\Gamma(a+iv-iu)}{\Gamma(a-iv+iu)} = i\ln\frac{\Gamma(a+b+iu)}{\Gamma(a+b-iu)}, \quad a,b > 0 \tag{B.24}$$

and

$$\int\limits_{-\infty}^{+\infty}\frac{dv}{2\pi}\ln\frac{ib-v+w}{ib+v-w}\frac{d}{dv}\ln\frac{ic-v+u}{ic+v-u} = i\ln\frac{i(c+b)-u+w}{i(c+b)+u-w}, \quad b,c > 0. \tag{B.25}$$

In order to show that (bound states of) gluons do not couple to (type b) isotopic roots, we used the following results

$$\bullet\int\limits_{-\infty}^{+\infty}\frac{dv}{2\pi}\ln\frac{x^+(v) - x\left(u-\frac{il}{2}\right)}{x\left(u+\frac{il}{2}\right) - x^-(v)}\frac{d}{dv}\ln\frac{\frac{i}{2}+v-u'}{\frac{i}{2}-v+u'} = i\ln\frac{x\left(u+\frac{il}{2}\right) - x(u'-i)}{x(u'+i) - x\left(u-\frac{il}{2}\right)}, \quad l \geq 1,$$

$$\tag{B.26}$$

$$\bullet \int\limits_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{x\left(u+\frac{il}{2}\right)-x^+(v)}{x^-(v)-x\left(u-\frac{il}{2}\right)} \frac{d}{dv} \ln \frac{\frac{i}{2}+v-u'}{\frac{i}{2}-v+u'}$$

$$= i \ln \frac{x(u'-i)-x\left(u-\frac{il}{2}\right)}{x\left(u+\frac{il}{2}\right)-x(u'+i)} + i \ln \frac{u-u'+\frac{il}{2}-i}{u-u'+\frac{il}{2}} \frac{u-u'-\frac{il}{2}}{u-u'-\frac{il}{2}+i}, \quad l \geq 2 \qquad \text{(B.27)}$$

$$\bullet \int\limits_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{x^+(u)-x^+(v)}{x^-(v)-x^-(u)} \frac{d}{dv} \ln \frac{\frac{i}{2}+v-u'}{\frac{i}{2}-v+u'}$$

$$= i \ln \frac{x(u'-i)-x^-(u)}{x^+(u)-x(u'+i)} + i \ln \frac{\frac{i}{2}-u+u'}{\frac{i}{2}+u-u'}. \qquad \text{(B.28)}$$

In calculations for the strong coupling limit of scattering factors, we used the following integrals

$$\bullet \int\limits_{-1}^{1} dk \frac{1}{u-k} \left(\frac{1+k}{1-k}\right)^{\frac{1}{4}} = -\pi\sqrt{2} \left[1 - \left(\frac{u+1}{u-1}\right)^{\frac{1}{4}}\right], \quad |u| > 1 \qquad \text{(B.29)}$$

$$\bullet \int\limits_{-1}^{1} dk \, P \frac{1}{u-k} \left(\frac{1+k}{1-k}\right)^{\frac{1}{4}} = -\pi\sqrt{2} + \pi \left(\frac{1+u}{1-u}\right)^{\frac{1}{4}}, \quad |u| < 1 \qquad \text{(B.30)}$$

$$\bullet \int\limits_{-1}^{1} dz \frac{1}{z-\bar{v}} \frac{1}{\sqrt{1-z^2}} = -\frac{\pi \, \text{sgn}(\bar{v})}{\sqrt{\bar{v}^2-1}}, \quad |\bar{v}| > 1, \qquad \text{(B.31)}$$

$$\bullet \int\limits_{-1}^{1} dz \, PV \frac{1}{z-\bar{v}} \frac{1}{\sqrt{1-z^2}} = 0, \quad |\bar{v}| < 1, \qquad \text{(B.32)}$$

$$\bullet \int\limits_{|\bar{w}|\geq 1} \frac{d\bar{w}}{2\pi} \frac{1}{\bar{w}-\bar{u}} PV \frac{1}{\bar{w}-\bar{z}} \left(\frac{\bar{w}+1}{\bar{w}-1}\right)^{\frac{1}{4}} = \frac{\frac{1}{2}\left(\frac{\bar{z}+1}{\bar{z}-1}\right)^{\frac{1}{4}} - \frac{1}{\sqrt{2}}\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}}}{\bar{z}-\bar{u}}, \quad |\bar{u}| \leq 1, \; |\bar{z}| \geq 1, $$

$$\text{(B.33)}$$

$$\bullet \int\limits_{|\bar{z}|\geq 1} \frac{d\bar{z}}{2\pi} \frac{1}{\bar{z}\sqrt{1-\frac{1}{\bar{z}^2}}} \frac{1}{\bar{x}_f(\bar{v})-\bar{x}(\bar{z})} \left(\frac{\bar{z}-1}{\bar{z}+1}\right)^{\frac{1}{4}} \frac{1}{\bar{u}-\bar{z}}$$

$$= \frac{\sqrt{\frac{1-2\bar{x}_f(\bar{v})}{1+2\bar{x}_f(\bar{v})}} + \frac{1}{\sqrt{2}}\left(\bar{x}_f(\bar{v})-\frac{1}{2}\right)\left[\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}}+\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}}\right]}{2\bar{x}_f(\bar{u})(\bar{u}-\bar{v})}$$

$$+ \frac{\frac{1}{\sqrt{2}}\left(\bar{x}_f(\bar{v})+\frac{1}{2}\right)\sqrt{\frac{1-\bar{u}}{1+\bar{u}}}\left[\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}}-\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}}\right]}{2\bar{x}_f(\bar{u})(\bar{u}-\bar{v})} \qquad \text{(B.34)}$$

Appendix C. Collection of scattering factors

C.1. One loop: explicit expressions

We list here the scattering factors at one loop:
- Scalar–scalar

$$S_0^{(ss)}(u_h, u_{h'}) = -\frac{\Gamma\left(\frac{1}{2} - iu_h\right)\Gamma\left(\frac{1}{2} + iu_{h'}\right)\Gamma(1 + iu_h - iu_{h'})}{\Gamma\left(\frac{1}{2} + iu_h\right)\Gamma\left(\frac{1}{2} - iu_{h'}\right)\Gamma(1 - iu_h + iu_{h'})}. \tag{C.1}$$

Formula (C.1) does agree with result (3.8) of Basso–Belitsky [54], but seems to be the inverse of (2.13) of Dorey–Zhao [53].
- Gluon–gluon

$$S_0^{(gg)}(u, v) = -\frac{\Gamma\left(1 + iu - iv\right)}{\Gamma\left(1 - iu + iv\right)}\frac{\Gamma\left(\frac{3}{2} - iu\right)}{\Gamma\left(\frac{3}{2} + iu\right)}\frac{\Gamma\left(\frac{3}{2} + iv\right)}{\Gamma\left(\frac{3}{2} - iv\right)}. \tag{C.2}$$

In addition, we have

$$S_0^{(gg)}(u, v) = S_0^{(\bar{g}\bar{g})}(u, v) = S_0^{(g\bar{g})}(u, v)\frac{u - v + i}{u - v - i}, \quad S_0^{(\bar{g}g)}(u, v) = [S_0^{(g\bar{g})}(v, u)]^{-1}. \tag{C.3}$$

- (Large) fermion–(large) fermion

$$S_0^{(FF)}(u, v) = \frac{\Gamma(1 + iu - iv)}{\Gamma(1 - iu + iv)}\frac{\Gamma(1 - iu)}{\Gamma(1 + iu)}\frac{\Gamma(1 + iv)}{\Gamma(1 - iv)} \tag{C.4}$$

and when antifermions get involved

$$S_0^{(FF)}(u, v) = S_0^{(F\bar{F})}(u, v) = S_0^{(\bar{F}F)}(u, v) = S_0^{(\bar{F}\bar{F})}(u, v). \tag{C.5}$$

- Gluon–scalar

$$S_0^{(gs)}(u, u_h) = [S_0^{(sg)}(u_h, u)]^{-1} = S_0^{(\bar{g}s)}(u, u_h) = [S_0^{(s\bar{g})}(u_h, u)]^{-1}$$
$$= \frac{\Gamma\left(1 + iu - iu_h\right)}{\Gamma\left(1 - iu + iu_h\right))}\frac{\Gamma\left(\frac{1}{2} + iu_h\right)}{\Gamma\left(\frac{1}{2} - iu_h\right)}\frac{\Gamma\left(\frac{3}{2} - iu\right)}{\Gamma\left(\frac{3}{2} + iu\right)}. \tag{C.6}$$

- (Large) fermion–scalar

$$S_0^{(Fs)}(u, u_h) = S_0^{(\bar{F}s)}(u, u_h) = [S_0^{(sF)}(u_h, u)]^{-1} = [S_0^{(s\bar{F})}(u_h, u)]^{-1}$$
$$= \frac{\Gamma(\frac{1}{2} + iu - iu_h)}{\Gamma(\frac{1}{2} - iu + iu_h)}\frac{\Gamma(1 - iu)}{\Gamma(1 + iu)}\frac{\Gamma(\frac{1}{2} + iu_h)}{\Gamma(\frac{1}{2} - iu_h)}. \tag{C.7}$$

- Gluon–(large) fermion

$$S_0^{(gF)}(u, v) = [S_0^{(Fg)}(v, u)]^{-1} = S_0^{(\bar{g}\bar{F})}(u, v) = [S_0^{(\bar{F}\bar{g})}(v, u)]^{-1}$$
$$= -\frac{\Gamma\left(\frac{1}{2} + iu - iv\right)}{\Gamma\left(\frac{1}{2} - iu + iv\right)}\frac{\Gamma\left(\frac{3}{2} - iu\right)}{\Gamma\left(\frac{3}{2} + iu\right)}\frac{\Gamma\left(1 + iv\right)}{\Gamma\left(1 - iv\right)} \tag{C.8}$$

and

$$
\begin{aligned}
S_0^{(\bar{g}F)}(u,v) &= [S_0^{(F\bar{g})}(v,u)]^{-1} = S_0^{(g\bar{F})}(u,v) = [S_0^{(\bar{F}g)}(v,u)]^{-1} \\
&= S_0^{(gF)}(u,v)\frac{u-v-i/2}{u-v+i/2}\,.
\end{aligned}
\tag{C.9}
$$

C.2. All loops: expressions in terms of solutions of integral equations

We list here the factors found in [23]. We start from the 'direct' S factors:

$$
S^{(ss)}(u,v) = -\exp[-i\,\Theta(u,v)],
\tag{C.10}
$$

$$
\begin{aligned}
S^{(FF)}(u,v) = \exp\Big\{ & i \int_{-\infty}^{+\infty} \frac{dw}{2\pi}[\chi_F(w,u)+\Phi(w)]\frac{d}{dw}[\chi_F(w,v)+\Phi(w)] \\
& -i\int \frac{dw}{2\pi}\frac{dz}{2\pi}[\chi_F(w,u)+\Phi(w)]\frac{d^2}{dwdz}\Theta(w,z)[\chi_F(z,v)+\Phi(z)]\Big\},
\end{aligned}
\tag{C.11}
$$

$$
S^{(FF)}(u,v) = S^{(F\bar{F})}(u,v) = S^{(\bar{F}F)}(u,v) = S^{(\bar{F}\bar{F})}(u,v),
\tag{C.12}
$$

$$
\begin{aligned}
S^{(gg)}(u,v) = -\exp\Big\{ & -i\,\chi_0(u-v|2) \\
& +i\int_{-\infty}^{+\infty} \frac{dw}{2\pi}[\chi(w,u|1)+\Phi(w)]\frac{d}{dw}[\chi(w,v|1)+\Phi(w)] \\
& -i\int \frac{dw}{2\pi}\frac{dz}{2\pi}[\chi(w,u|1)+\Phi(w)]\frac{d^2}{dwdz}\Theta(w,z)[\chi(z,v|1)+\Phi(z)]\Big\} \\
= & \frac{u-v+i}{u-v-i}\,S_{red}^{(gg)}(u,v) = -\exp\{-i\,\chi_0(u-v|2)\}\,S_{red}^{(gg)}(u,v),
\end{aligned}
\tag{C.13}
$$

$$
\begin{aligned}
S_{red}^{(gg)}(u,v) = \exp\Big\{ & i \int_{-\infty}^{+\infty} \frac{dw}{2\pi}[\chi(w,u|1)+\Phi(w)]\frac{d}{dw}[\chi(w,v|1)+\Phi(w)] \\
& -i\int \frac{dw}{2\pi}\frac{dz}{2\pi}[\chi(w,u|1)+\Phi(w)]\frac{d^2}{dwdz}\Theta(w,z)[\chi(z,v|1)+\Phi(z)]\Big\},
\end{aligned}
\tag{C.14}
$$

$$
S^{g\bar{g}}(u,v) = [S^{\bar{g}g}(v,u)]^{-1} = S_{red}^{(gg)}(u,v)\,.
\tag{C.15}
$$

The 'mixed' S factors are:

$$
\begin{aligned}
S^{(sF)}(u,v) = [S^{(Fs)}(v,u)]^{-1} = \exp\Big\{ & -i[\chi_F(u,v)+\Phi(u)] \\
& +i\int \frac{dw}{2\pi}\frac{d\Theta}{dw}(u,w)[\chi_F(w,v)+\Phi(w)]\Big\},
\end{aligned}
\tag{C.16}
$$

$$S^{(sF)}(u, v) = S^{(s\bar{F})}(u, v), \quad S^{(Fs)}(u, v) = S^{(\bar{F}s)}(u, v), \tag{C.17}$$

$$S^{(gs)}(u, v) = [S^{(sg)}(v, u)]^{-1} = S^{(\bar{g}s)}(u, v) = [S^{(s\bar{g})}(v, u)]^{-1}$$
$$= \exp\left\{i[\chi(v, u|1) + \Phi(v)] - i\int \frac{dw}{2\pi}\frac{d\Theta}{dw}(v, w)[\chi(w, u|1) + \Phi(w)]\right\}, \tag{C.18}$$

$$S^{(gF)}(u, v) = [S^{(Fg)}(v, u)]^{-1} = -\exp\left\{-i\chi_0(u - v|1)\right.$$
$$+ i\int\limits_{-\infty}^{+\infty} \frac{dw}{2\pi}[\chi(w, u|1) + \Phi(w)]\frac{d}{dw}[\chi_F(w, v) + \Phi(w)]$$
$$- i\int \frac{dw}{2\pi}\frac{dz}{2\pi}[\chi(w, u|1) + \Phi(w)]\frac{d^2}{dwdz}\Theta(w, z)[\chi_F(z, v) + \Phi(z)]\right\}$$
$$= \frac{u - v + \frac{i}{2}}{u - v - \frac{i}{2}}S^{(gF)}_{red}(u, v) = -\exp\{-i\chi_0(u - v|1)\}S^{(gF)}_{red}(u, v) \tag{C.19}$$

$$S^{(gF)}(u, v) = S^{(\bar{g}\bar{F})}(u, v), \tag{C.20}$$

$$S^{(\bar{g}F)}(u, v)$$
$$= [S^{(F\bar{g})}(v, u)]^{-1} = \exp\left\{i\int\limits_{-\infty}^{+\infty} \frac{dw}{2\pi}[\chi(w, u|1) + \Phi(w)]\frac{d}{dw}[\chi_F(w, v) + \Phi(w)]\right.$$
$$- i\int \frac{dw}{2\pi}\frac{dz}{2\pi}[\chi(w, u|1) + \Phi(w)]\frac{d^2}{dwdz}\Theta(w, z)[\chi_F(z, v) + \Phi(z)]\right\}, \tag{C.21}$$

$$S^{(g\bar{F})}(u, v) = [S^{(\bar{F}g)}(v, u)]^{-1} = S^{(\bar{g}F)}(u, v). \tag{C.22}$$

The S matrices involving small fermions are obtained from the corresponding ones for large fermions by means of the replacement

$$\chi_F(v, u) + \Phi(v) \longrightarrow -\chi_H(v, u). \tag{C.23}$$

All the scalar factors are expressed in terms of known functions listed in Appendix A and the 'dynamical' function $\Theta(u, v)$ [23], which equals

$$\Theta(u, v) = \Theta'(u, v) + \tilde{P}(v), \tag{C.24}$$

where $\Theta'(u, v)$ and $\tilde{P}(v)$ are found as solutions of the linear integral equations

$$\Theta'(u, v) = \phi(u, v) + \Phi(u) - \int\limits_{-\infty}^{+\infty} dw\varphi(u, w)\Theta'(w, v), \tag{C.25}$$

$$\tilde{P}(v) = -\Phi(v) - \int\limits_{-\infty}^{+\infty} \frac{dw}{2}[\varphi(v, w) - \varphi(v, -w)]\tilde{P}(w). \tag{C.26}$$

Scattering factors involving bound states of gluons If bound states of gluons are present, the factors involving the gauge field should be generalised as follows.

The scattering factor between a bound state of gluons with length m and center u and a scalar is

$$S_m^{(gs)}(u, v) = [S_m^{(sg)}(v, u)]^{-1} = S_m^{(\bar{g}s)}(u, v) = [S_m^{(s\bar{g})}(v, u)]^{-1}$$
$$= \exp\left\{i[\chi(v, u|m) + \Phi(v)] - i \int \frac{dw}{2\pi} \frac{d\Theta}{dw}(v, w)[\chi(w, u|m) + \Phi(w)]\right\}.$$
(C.27)

The scattering factor between a bound state of gluons with length m and center u and large fermions is

$$S_m^{(gF)}(u, v) = [S_m^{(Fg)}(v, u)]^{-1} = -\exp\left\{-i\chi_0(u - v|m)\right.$$
$$+ i \int_{-\infty}^{+\infty} \frac{dw}{2\pi} [\chi(w, u|m) + \Phi(w)] \frac{d}{dw} [\chi_F(w, v) + \Phi(w)]$$
$$\left. - i \int \frac{dw}{2\pi} \frac{dz}{2\pi} [\chi(w, u|m) + \Phi(w)] \frac{d^2}{dwdz} \Theta(w, z)[\chi_F(z, v) + \Phi(z)]\right\},$$
(C.28)

$$S_m^{(gF)}(u, v) = S_m^{(\bar{g}\bar{F})}(u, v),$$
(C.29)

$$S_m^{(\bar{g}F)}(u, v) = [S_m^{(F\bar{g})}(v, u)]^{-1}$$
$$= \exp\left\{i \int_{-\infty}^{+\infty} \frac{dw}{2\pi} [\chi(w, u|m) + \Phi(w)] \frac{d}{dw} [\chi_F(w, v) + \Phi(w)]\right.$$
$$\left. - i \int \frac{dw}{2\pi} \frac{dz}{2\pi} [\chi(w, u|m) + \Phi(w)] \frac{d^2}{dwdz} \Theta(w, z)[\chi_F(z, v) + \Phi(z)]\right\},$$
(C.30)

$$S_m^{(g\bar{F})}(u, v) = [S_m^{(\bar{F}g)}(v, u)]^{-1} = S_m^{(\bar{g}F)}(u, v).$$
(C.31)

The replacement $\chi_F(w, v) + \Phi(w) \to -\chi_H(w, v)$ gives the corresponding quantities for small fermions.

The scattering factors between a bound state of gluons with length m and center u and a bound state of gluons with length l and center v are

$$S_{ml}^{(gg)}(u, v)$$
$$= \exp\left\{-i\tilde{\chi}(u, v|m, l) + i \int_{-\infty}^{+\infty} \frac{dw}{2\pi} [\chi(w, u|m) + \Phi(w)] \frac{d}{dw} [\chi(w, v|l) + \Phi(w)]\right.$$
$$\left. - i \int \frac{dw}{2\pi} \frac{dz}{2\pi} [\chi(w, u|m) + \Phi(w)] \frac{d^2}{dwdz} \Theta(w, z)[\chi(z, v|l) + \Phi(z)]\right\},$$
(C.32)

$$S_{ml}^{(g\bar{g})}(u, v) = \exp\left\{ i \int_{-\infty}^{+\infty} \frac{dw}{2\pi}[\chi(w, u|m) + \Phi(w)]\frac{d}{dw}[\chi(w, v|l) + \Phi(w)] \right.$$

$$\left. - i \int \frac{dw}{2\pi}\frac{dz}{2\pi}[\chi(w, u|m) + \Phi(w)]\frac{d^2}{dwdz}\Theta(w, z)[\chi(z, v|l) + \Phi(z)] \right\},$$

$$(C.33)$$

where

$$\tilde{\chi}(u, v|m, l)$$

$$= \chi_0(u - v|m + l) - \chi_0(u - v|m - l) + 2\sum_{\gamma=1}^{m-1} \chi_0(u - v|m + l - 2\gamma). \quad (C.34)$$

In the particular case $m = l = 1$, since $e^{i\chi_0(u-v|0)} = -1$, one recovers from (C.32) the gluon–gluon scattering factor (C.13).

Remark. The following relations hold, for u real and $u^2 < 2g^2$,

$$\chi(v, u|m) + \Phi(v) = \sum_{l=-\frac{m-1}{2}}^{\frac{m-1}{2}} [\chi(v, u + il|1) + \Phi(v)], \quad m \geq 1, \quad (C.35)$$

where $\chi(v, u + il|1)$ has to be understood as analytical continuation of $\chi(v, u|1)$.

To prove this statement, we refer to (A.7) and remember the following properties

$$\lim_{\epsilon \to 0^+} x(u - i\epsilon) = \lim_{\epsilon \to 0^+} \frac{g^2}{2x(u + i\epsilon)}, \quad (C.36)$$

which are valid for u real and $u^2 < 2g^2$. Therefore, when the complex variable u crosses the real axis in the region $-\sqrt{2}g < \text{Re } u < \sqrt{2}g$, the function $x(u)$ is analytically continued in $g^2/2x(u)$. With the help of this property, relation (C.35) is easily shown.

C.3. Strong coupling and mirror in hyperbolic rapidities

In the strong coupling perturbative regime, the scattering matrices for gluons, fermions and mesons can be suitably recast in term of hyperbolic rapidities, according to the following identities (written up to $O(1/g^2)$ corrections):
Gluons:

$$u^g = \sqrt{2}g\bar{u}^g, \quad \bar{u}^g = \tanh(2\theta), \quad (C.37)$$

Fermions:

$$u_f = \sqrt{2}g\bar{u}_f, \quad \bar{u}_f = \coth(2\theta) \qquad \text{or else} \qquad 2\bar{x}_f = \tanh\theta, \quad (C.38)$$

Mesons:

$$u_M = \sqrt{2}g\bar{u}_M, \quad \bar{u}_M = \coth(2\theta). \quad (C.39)$$

Explicitly, we obtain:

$$S^{(gg)}(\theta, \theta') = \exp\left\{\frac{i}{\sqrt{2}g}\left[\frac{1}{\tanh 2\theta - \tanh 2\theta'} + \frac{\cosh 2\theta \cosh 2\theta'}{2\sinh(\theta - \theta')}\right] + O\left(\frac{1}{g^2}\right)\right\}, \quad \text{(C.40)}$$

$$S^{(g\bar{g})}(\theta, \theta') = \left(1 - \frac{1}{\sqrt{2}g}\frac{2i}{\tanh 2\theta - \tanh 2\theta'} + O\left(\frac{1}{g^2}\right)\right) S^{(gg)}(\theta, \theta'), \quad \text{(C.41)}$$

$$S^{(ff)}(\theta, \theta') = \exp\left\{-\frac{i}{2\sqrt{2}g}\frac{\sinh 2\theta \sinh 2\theta'}{\sinh(2\theta - 2\theta')}(\cosh(\theta - \theta') - 1) + O\left(\frac{1}{g^2}\right)\right\}, \quad \text{(C.42)}$$

$$S^{(gf)}(\theta, \theta_f) = \exp\left\{\frac{i}{4g}\frac{2\cosh(\theta_f - \theta) + \sqrt{2}}{\tanh 2\theta - \coth 2\theta_f} + O\left(\frac{1}{g^2}\right)\right\}, \quad \text{(C.43)}$$

$$S^{(\bar{g}f)}(\theta, \theta_f) = \exp\left\{\frac{i}{4g}\frac{2\cosh(\theta_f - \theta) - \sqrt{2}}{\tanh 2\theta - \coth 2\theta_f} + O\left(\frac{1}{g^2}\right)\right\}, \quad \text{(C.44)}$$

$$S^{(MM)}(\theta, \theta') = \exp\left[-\frac{i}{\sqrt{2}g}\frac{\sinh 2\theta \sinh 2\theta'}{\sinh(\theta - \theta')}\right], \quad \text{(C.45)}$$

$$S^{(Mg)}(\theta, \theta') = S^{(M\bar{g})}(\theta, \theta') = \exp\left[-\frac{i}{g}\frac{\cosh(\theta - \theta')}{\tanh 2\theta' - \coth 2\theta}\right]. \quad \text{(C.46)}$$

Mirror transformations:

The mirror rotation should be implemented in different ways on the scattering phases, depending on the kind of particle the transformation is acting on. For instance, in the scalar case it is achieved by means of a shift $u \longrightarrow u^\gamma = u + i$. For gluons, the mirror transform is performed via a closed path across the complex rapidity plane ($u^\gamma = u$), passing through a cut, so that actually the initial and final points do not lie on the same sheet. Defining a procedure for the mirror rotation on fermions is more involved, and for this purpose we refer to [36]. A complete all coupling study of all the mirror S matrices can be found in [65]. Nevertheless, as long as the perturbative strong coupling regime is concerned, the mirror rotation gets simplified: for gluons, scalars and mesons it amounts to an imaginary shift in the hyperbolic rapidities $\theta^\gamma = \theta + i\frac{\pi}{2}$. For instance, we have:

$$S_{mir}^{(gg)}(\theta, \theta') = S^{(gg)}(\theta + i\frac{\pi}{2}, \theta')$$
$$= \exp\left\{\frac{i}{\sqrt{2}g}\left[\frac{1}{\tanh 2\theta - \tanh 2\theta'} - \frac{\cosh 2\theta \cosh 2\theta'}{2i\cosh(\theta - \theta')}\right] + O\left(\frac{1}{g^2}\right)\right\}, \quad \text{(C.47)}$$

$$S_{mir}^{(g\bar{g})}(\theta, \theta') = S^{(g\bar{g})}(\theta + i\frac{\pi}{2}, \theta') = \left(1 - \frac{1}{\sqrt{2}g}\frac{2i}{\tanh 2\theta - \tanh 2\theta'} + O\left(\frac{1}{g^2}\right)\right)$$
$$\cdot \exp\left\{\frac{i}{\sqrt{2}g}\left[\frac{1}{\tanh 2\theta - \tanh 2\theta'} - \frac{\cosh 2\theta \cosh 2\theta'}{2i\cosh(\theta - \theta')}\right] + O\left(\frac{1}{g^2}\right)\right\}, \quad \text{(C.48)}$$

$$S_{mir}^{(ff)}(\theta, \theta') = \exp\left\{-\frac{i}{2\sqrt{2}g}\frac{\sinh 2\theta \sinh 2\theta'}{\sinh(2\theta - 2\theta')}(i\sinh(\theta - \theta') - 2) + O\left(\frac{1}{g^2}\right)\right\}, \quad \text{(C.49)}$$

$$S_{mir}^{(f\bar{f})}(\theta, \theta') = \exp\left\{\frac{1}{4\sqrt{2}g}\frac{\sinh 2\theta \sinh 2\theta'}{\cosh(\theta - \theta')} + O\left(\frac{1}{g^2}\right)\right\}, \quad \text{(C.50)}$$

$$S_{mir}^{(MM)}(\theta, \theta') = \exp\left[\frac{1}{\sqrt{2}g}\frac{\sinh 2\theta \sinh 2\theta'}{\cosh(\theta - \theta')}\right], \tag{C.51}$$

$$S_{mir}^{(Mg)}(\theta, \theta') = S_{mir}^{(M\bar{g})}(\theta, \theta') = \exp\left[\frac{1}{g}\frac{\sinh(\theta - \theta')}{\tanh 2\theta' - \coth 2\theta}\right]. \tag{C.52}$$

C.4. On the factor Θ

Since the scalar–scalar factor Θ is the building block for all the scattering factors, we give here some alternative constructions for it. Following what was done in [23], we define the function

$$M(u, v) = \frac{\Theta(u, v) + \Theta(u, -v)}{2}, \tag{C.53}$$

which stores all the information on Θ, since (see (2.23) of [23])

$$\Theta(u, v) = M(u, v) - M(v, u). \tag{C.54}$$

Then, we use (2.21) of [23] to express $M(u, v)$ in terms of densities $\sigma^{(1)}(u)$ and $\sigma(u; v)$. Formulæ (3.7), (3.8) and Neumann expansions (3.9) allow to arrive at

$$\begin{aligned}
M(u, v) &= \frac{i}{2}\ln\frac{\Gamma(1 + iu + iv)\Gamma(1 + iu - iv)}{\Gamma(1 - iu + iv)\Gamma(1 - iu - iv)} + i\ln\frac{\Gamma\left(\frac{1}{2} - iu\right)}{\Gamma\left(\frac{1}{2} + iu\right)} \\
&+ \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\frac{1}{ip}\left(\frac{ig}{\sqrt{2}}\right)^p[S_p^{(1)} + S_p'(v)] \\
&\cdot\left[\left(\frac{1}{x\left(u + \frac{i}{2}(1 + 2n)\right)}\right)^p - \left(-\frac{1}{x\left(u - \frac{i}{2}(1 + 2n)\right)}\right)^p\right].
\end{aligned} \tag{C.55}$$

The first line of (C.55) is the one loop contribution; the second and third lines are the higher than one loop correction. We can manipulate (C.55) in order to get alternative expressions for $M(u, v)$: for instance,

$$\begin{aligned}
M(u, v) &= \frac{i}{2}\ln\frac{\Gamma(1 + iu + iv)\Gamma(1 + iu - iv)}{\Gamma(1 - iu + iv)\Gamma(1 - iu - iv)} + i\ln\frac{\Gamma\left(\frac{1}{2} - iu\right)}{\Gamma\left(\frac{1}{2} + iu\right)} \\
&+ \sum_{p=1}^{\infty}[S_p^{(1)} + S_p'(v)]\int_{-\pi}^{+\pi}\frac{d\theta}{2\pi i}e^{-ip\theta}\ln\frac{\Gamma\left(\frac{1}{2} - i\sqrt{2}g\sin\theta - iu\right)}{\Gamma\left(\frac{1}{2} - i\sqrt{2}g\sin\theta + iu\right)}
\end{aligned} \tag{C.56}$$

and

$$\begin{aligned}
M(u, v) &= \frac{i}{2}\ln\frac{\Gamma(1 + iu + iv)\Gamma(1 + iu - iv)}{\Gamma(1 - iu + iv)\Gamma(1 - iu - iv)} + i\ln\frac{\Gamma\left(\frac{1}{2} - iu\right)}{\Gamma\left(\frac{1}{2} + iu\right)} \\
&+ \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\frac{(-1)^{n+p}}{in!(n + p)!}\left(\frac{g}{\sqrt{2}}\right)^{2n+p}[S_p^{(1)} + S_p'(v)] \\
&\cdot\left[\psi^{(2n+p-1)}\left(\frac{1}{2} - iu\right) - \psi^{(2n+p-1)}\left(\frac{1}{2} + iu\right)\right].
\end{aligned} \tag{C.57}$$

Appendix D. Some calculations on how to get scattering factors

D.1. Scalars vs others

The 'fermionic' contribution to $Z_4(v) - P$ reads

$$Z_4(v) - P|_F = F^F(v, u_{F,j}) + \int \frac{dw}{2\pi} \frac{d}{dw} \Phi(w) F^F(w, u_{F,j}). \tag{D.1}$$

Inserting the integral equation (2.63) for F^F, we arrive at

$$Z_4(v) - P|_F = \chi_F(v, u_{F,j}) + \Phi(v) + \int \frac{dw}{2\pi} \frac{d}{dw} [\phi(w, v) + \Phi(w)] F^F(w, u_{F,j}). \tag{D.2}$$

Remembering the integral equation (2.62) for Θ' and then, after an integration by parts, the equation (2.63) for F^F, we get

$$\begin{aligned}
Z_4(v) - P|_F &= \chi_F(v, u_{F,j}) + \Phi(v) + \int \frac{dw}{2\pi} \frac{d}{dw} \Theta'(w, v)[\chi_F(w, u_{F,j}) + \Phi(w)] \\
&= \chi_F(v, u_{F,j}) + \Phi(v) - \int \frac{dw}{2\pi} \frac{d}{dw} \Theta(v, w)[\chi_F(w, u_{F,j}) + \Phi(w)] \\
&= i \ln S^{(sf)}(v, u_{F,j}).
\end{aligned} \tag{D.3}$$

The same calculation can be done for large antifermions. For gluons we can repeat the same reasonings after the substitution $\chi_F(v, u) \to \chi(v, u|1)$. For small (anti)fermions we perform the substitution $\chi_F(v, u) + \Phi(v) \to -\chi_H(v, u)$.

For scalar–scalar factor we have to consider

$$Z_4(v) - P|_s = \Theta'(v, u_h) - \Phi(u_h) + \int \frac{dw}{2\pi} \frac{d}{dw} \Phi(w) \Theta'(w, u_h). \tag{D.4}$$

Now, we remember the integral equation (2.62) for Θ', which allows to write the identity

$$\int \frac{dw}{2\pi} \frac{d}{dw} \Phi(w) \Theta'(w, u_h) = - \int \frac{dw}{2\pi} \frac{\varphi(u_h, w) - \varphi(u_h, -w)}{2} \tilde{P}(w). \tag{D.5}$$

Therefore, using the integral equation (2.67) for $\tilde{P}$ we get

$$Z_4(v) - P|_s = \Theta'(v, u_h) + \tilde{P}(u_h) = \Theta(v, u_h) = i \ln[-S^{(ss)}(v, u_h)]. \tag{D.6}$$

D.2. Non-scalars vs others

Taking as a prototypical example the case of fermions, after multiplication by e^{iP} we have to cope with

$$\exp\left[i \int \frac{dv}{2\pi} [\chi_F(v, u_{F,k}) + \Phi(v)] \frac{d}{dv} Z_4(v) \right], \tag{D.7}$$

which is the master relation from which the various scattering factors originate.

For instance, the fermion–fermion factor is

$$\exp\left[i \int \frac{dv}{2\pi} [\chi_F(v, u_{F,k}) + \Phi(v)] \frac{d}{dv} F^F(v, u_{F,j}) \right]. \tag{D.8}$$

Mixing equations (2.62) and (2.63), we get

$$F^F(v, u) = \chi_F(v, u) + \Phi(v) - \int \frac{dw}{2\pi} \frac{d}{dw} \Theta'(v, w)[\chi_F(w, u) + \Phi(w)], \tag{D.9}$$

which, inserted in (D.8) gives the fermion–fermion factor as reported in Appendix C:

$$\exp\left\{ i \int\limits_{-\infty}^{+\infty} \frac{dw}{2\pi} [\chi_F(w, u_{F,k}) + \Phi(w)] \frac{d}{dw} [\chi_F(w, u_{F,j}) + \Phi(w)] \right.$$
$$\left. - i \int \frac{dw\,dx}{2\pi\,2\pi} [\chi_F(w, u_{F,k}) + \Phi(w)] \frac{d^2}{dw\,dx} \Theta(w, x)[\chi_F(x, u_{F,j}) + \Phi(x)] \right\}. \tag{D.10}$$

For what concerns the fermion–scalar factor, it receives contribution from

$$\exp\left[i \int \frac{dv}{2\pi} [\chi_F(v, u_{F,k}) + \Phi(v)] \frac{d}{dv} \Theta'(v, u_h) \right]$$
$$= \exp\left[i \int \frac{dv}{2\pi} [\chi_F(v, u_{F,k}) + \Phi(v)] \frac{d}{dv} \Theta(v, u_h) \right] \tag{D.11}$$

Expression (D.11) has to be multiplied to the factor

$$\frac{x_{F,k} - x_h^-}{x_h^+ - x_{F,k}} = e^{i\chi_F(u_h, u_{F,k})} \tag{D.12}$$

present in equation (2.80) and to the factor $e^{i\Phi(u_h)}$ present in e^{iP} to get the full fermion–scalar factor.

Appendix E. Bethe equations

We list the complete set of Bethe Ansatz equations we found in this paper:
• Scalars

$$1 = e^{iRP^{(s)}(u_h) + 2iD^{(s)}(u_h)} \prod_{j=1}^{K_b} \frac{u_h - u_{b,j} + \frac{i}{2}}{u_h - u_{b,j} - \frac{i}{2}} \prod_{\substack{h'=1 \\ h' \neq h}}^{H} S^{(ss)}(u_h, u_{h'}) \prod_{j=1}^{N_g} S^{(sg)}(u_h, u_j^g)$$
$$\cdot \prod_{j=1}^{N_{\bar{g}}} S^{(s\bar{g})}(u_h, u_j^{\bar{g}}) \prod_{j=1}^{N_F} S^{(sF)}(u_h, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(s\bar{F})}(u_h, u_{\bar{F},j})$$
$$\cdot \prod_{j=1}^{N_f} S^{(sf)}(u_h, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(s\bar{f})}(u_h, u_{\bar{f},j}) \tag{E.1}$$

• Large fermions

$$1 = e^{iRP^{(F)}(u_{F,k}) + 2iD^{(F)}(u_{F,k})} \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + i/2}{u_{F,k} - u_{a,j} - i/2} \prod_{h=1}^{H} S^{(Fs)}(u_{F,k}, u_h)$$
$$\cdot \prod_{j=1}^{N_F} S^{(FF)}(u_{F,k}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(F\bar{F})}(u_{F,k}, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(Ff)}(u_{F,k}, u_{f,j})$$

$$\cdot \prod_{j=1}^{N_{\bar{f}}} S^{(F\bar{f})}(u_{F,k}, u_{\bar{f},j}) \prod_{j=1}^{N_g} S^{(Fg)}(u_{F,k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(F\bar{g})}(u_{F,k}, u_j^{\bar{g}}) \tag{E.2}$$

- Large antifermions

$$1 = e^{iRP^{(F)}(u_{\bar{F},k}) + 2iD^{(F)}(u_{\bar{F},k})} \prod_{j=1}^{K_c} \frac{u_{\bar{F},k} - u_{c,j} + i/2}{u_{\bar{F},k} - u_{c,j} - i/2} \prod_{h=1}^{H} S^{(\bar{F}s)}(u_{\bar{F},k}, u_h)$$

$$\cdot \prod_{j=1}^{N_F} S^{(\bar{F}F)}(u_{\bar{F},k}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{F}\bar{F})}(u_{\bar{F},k}, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(\bar{F}f)}(u_{\bar{F},k}, u_{f,j})$$

$$\cdot \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{F}\bar{f})}(u_{\bar{F},k}, u_{\bar{f},j}) \prod_{j=1}^{N_g} S^{(\bar{F}g)}(u_{\bar{F},k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(\bar{F}\bar{g})}(u_{\bar{F},k}, u_j^{\bar{g}}) \tag{E.3}$$

- Small fermions

$$1 = e^{iRP^{(f)}(u_{f,k}) + 2iD^{(f)}(u_{f,k})} \prod_{j=1}^{K_a} \frac{u_{f,k} - u_{a,j} + i/2}{u_{f,k} - u_{a,j} - i/2} \prod_{h=1}^{H} S^{(fs)}(u_{f,k}, u_h)$$

$$\cdot \prod_{j=1}^{N_F} S^{(fF)}(u_{f,k}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(f\bar{F})}(u_{f,k}, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(ff)}(u_{f,k}, u_{f,j})$$

$$\cdot \prod_{j=1}^{N_{\bar{f}}} S^{(f\bar{f})}(u_{f,k}, u_{\bar{f},j}) \prod_{j=1}^{N_g} S^{(fg)}(u_{f,k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(f\bar{g})}(u_{f,k}, u_j^{\bar{g}}) \tag{E.4}$$

- Small antifermions

$$1 = e^{iRP^{(f)}(u_{\bar{f},k}) + 2iD^{(f)}(u_{\bar{f},k})} \prod_{j=1}^{K_c} \frac{u_{\bar{f},k} - u_{c,j} + i/2}{u_{\bar{f},k} - u_{c,j} - i/2} \prod_{h=1}^{H} S^{(\bar{f}s)}(u_{\bar{f},k}, u_h)$$

$$\cdot \prod_{j=1}^{N_F} S^{(\bar{f}F)}(u_{\bar{f},k}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{f}\bar{F})}(u_{\bar{f},k}, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(\bar{f}f)}(u_{\bar{f},k}, u_{f,j})$$

$$\cdot \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{f}\bar{f})}(u_{\bar{f},k}, u_{\bar{f},j}) \prod_{j=1}^{N_g} S^{(\bar{f}g)}(u_{\bar{f},k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(\bar{f}\bar{g})}(u_{\bar{f},k}, u_j^{\bar{g}}) \tag{E.5}$$

- Gluons

$$1 = e^{iRP^{(g)}(u_k^g) + 2iD^{(g)}(u_k^g)} \prod_{j=1, j\neq k}^{N_g} S^{(gg)}(u_k^g, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(g\bar{g})}(u_k^g, u_j^{\bar{g}}) \prod_{h=1}^{H} S^{(gs)}(u_k^g, u_h)$$

$$\cdot \prod_{j=1}^{N_F} S^{(gF)}(u_k^g, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(g\bar{F})}(u_k^g, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(gf)}(u_k^g, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(g\bar{f})}(u_k^g, u_{\bar{f},j})$$

$$\tag{E.6}$$

• Barred gluons

$$1 = e^{i R P^{(g)}(u_k^{\bar{g}}) + 2i D^{(g)}(u_k^{\bar{g}})} \prod_{j=1}^{N_g} S^{(\bar{g}g)}(u_k^{\bar{g}}, u_j^g) \prod_{j=1, j \neq k}^{N_{\bar{g}}} S^{(\bar{g}\bar{g})}(u_k^{\bar{g}}, u_j^{\bar{g}}) \prod_{h=1}^{H} S^{(\bar{g}s)}(u_k^{\bar{g}}, u_h)$$

$$\cdot \prod_{j=1}^{N_F} S^{(\bar{g}F)}(u_k^{\bar{g}}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{g}\bar{F})}(u_k^{\bar{g}}, u_{\bar{F},j}) \prod_{j=1}^{N_f} S^{(\bar{g}f)}(u_k^{\bar{g}}, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{g}\bar{f})}(u_k^{\bar{g}}, u_{\bar{f},j})$$

$$(E.7)$$

• Isotopic roots

$$\prod_{j=1}^{N_F} \frac{u_{a,k} - u_{F,j} + i/2}{u_{a,k} - u_{F,j} - i/2} \prod_{j=1}^{N_f} \frac{u_{a,k} - u_{f,j} + i/2}{u_{a,k} - u_{f,j} - i/2}$$

$$= \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - i/2}{u_{a,k} - u_{b,j} + i/2} \qquad (E.8)$$

$$\prod_{h=1}^{H} \frac{u_{b,k} - u_h + i/2}{u_{b,k} - u_h - i/2}$$

$$= \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - i/2}{u_{b,k} - u_{a,j} + i/2} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - i/2}{u_{b,k} - u_{c,j} + i/2} \prod_{\substack{j=1 \\ j \neq k}}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \qquad (E.9)$$

$$\prod_{j=1}^{N_{\bar{F}}} \frac{u_{c,k} - u_{\bar{F},j} + i/2}{u_{c,k} - u_{\bar{F},j} - i/2} \prod_{j=1}^{N_{\bar{f}}} \frac{u_{c,k} - u_{\bar{f},j} + i/2}{u_{c,k} - u_{\bar{f},j} - i/2}$$

$$= \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - i/2}{u_{c,k} - u_{b,j} + i/2} \qquad (E.10)$$

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