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Combinatorics of open covers (V): Pixley–Roy spaces of sets of reals, and ω-covers

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Abstract

Daniels (1988) started an investigation of the duality between selection hypotheses for $X \subseteq \mathbb{R}$ and selection hypotheses for the Pixley–Roy space of X. Daniels, Kunen and Zhou (1994) introduced the "open–open game". We extend some results of Daniels (1988) by connecting the relevant selection hypotheses with game theory (Theorems 2, 3, 14 and 15) and Ramsey theory (Theorem 10, Corollary 11, Theorem 23 and Corollary 24). Our results give answers to some of the questions asked by Daniels et al. (1994). © 2000 Elsevier Science B.V. All rights reserved.

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In "Combinatorics of open covers", parts I–IV we investigated some of the combinatorial consequences associated with the fact that certain topologically significant families of sets satisfy certain selection hypotheses. We now develop this further.

The two selection hypotheses featured here are as follows: Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set. Then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the hypothesis: For every sequence $(O_n\colon n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(T_n\colon n \in \mathbb{N})$ such that for each n $T_n \in O_n$, and $\{T_n\colon n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ associated with this hypothesis is played as follows: ONE and TWO play an inning per positive integer. In the nth inning ONE first chooses a set $O_n \in \mathcal{A}$, and TWO responds with a $T_n \in O_n$. TWO wins a play $O_1, T_1, \ldots, O_n, T_n, \ldots$ if $\{T_n\colon n \in \mathbb{N}\}$ is in $\mathcal{B}$; otherwise, ONE wins.

The symbol $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the second selection hypothesis: for every sequence $(O_n\colon n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(T_n\colon n \in \mathbb{N})$ such that for each n T_n is a finite subset of O_n and $\bigcup_{n=1}^{\infty} T_n$ is in $\mathcal{B}$. The game $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ associated with this

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hypothesis is played as follows: ONE and TWO play an inning per positive integer. In the nth inning ONE first chooses an $O_n \in \mathcal{A}$, after which TWO chooses a finite set $T_n \subseteq O_n$. TWO wins a play $O_1, T_1, \ldots, O_n, T_n, \ldots$ if $\bigcup_{n=1}^{\infty} T_n \in \mathcal{B}$; otherwise, ONE wins.

Let (X, τ) be a topological space. An open cover $\mathcal{U}$ of X is an ω-cover if $X \notin \mathcal{U}$ and for each finite set $F \subseteq X$, there is a $U \in \mathcal{U}$ such that $F \subseteq U$. The examples of $\mathcal{A}$ and $\mathcal{B}$ that are studied in this paper are as follows:

- $\mathcal{O}$: the collection of all open covers of X;
- Ω: the collection of ω-covers of X;
- $\mathcal{D}$: the collection of $\mathcal{U} \subseteq \tau$ with $\bigcup \mathcal{U}$ dense in X;

and two more collections, $\mathcal{L}$ and $\mathcal{D}_\Omega$, which we define later. The most important other item regarding notation is that we use the symbol $\subset$ to mean "is a *proper* subset of".

The paper is organized as follows: First we study $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ for appropriate choices of $\mathcal{A}$ and $\mathcal{B}$ from the preceding list. Next we discuss $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$.

1. $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ and $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$

In [4] the authors introduce a game G_7 which is played as follows: ONE and TWO play an inning per positive integer. In the nth inning ONE chooses O_n, a maximal family of pairwise disjoint open sets. TWO responds with T_n, a finite subset of O_n. A play $O_1, T_1, \ldots, O_n, T_n, \ldots$ of G_7 is won by TWO if $\bigcup_{n=1}^{\infty} T_n \in \mathcal{D}$; otherwise, ONE wins. Since for each $\mathcal{U} \in \mathcal{D}$ there is a pairwise disjoint family $\mathcal{V} \in \mathcal{D}$ which refines $\mathcal{U}$, one can show:

Lemma 1. *ONE has a winning strategy in G_7 if, and only if, ONE has a winning strategy in $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$. TWO has a winning strategy in G_7 if, and only if, TWO has a winning strategy in $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$.*

Thus, the study of G_7 is the same as the study of $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$ which, according to the following theorem, amounts to a study of the selection hypothesis $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$.

Theorem 2. *A topological space satisfies $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ if, and only if, ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$.*

Proof. The implication that if the space satisfies $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$, then ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$ requires proof. The argument in the proof of Theorem 10 of [6] can be adapted to do this. We give only a brief description.

First, $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ implies that each element of $\mathcal{D}$ has a countable subset which is in $\mathcal{D}$ (incidentally, the latter statement is equivalent to saying that the cellularity of the space is countable). We may further restrict our attention to strategies of ONE which calls in each inning on ONE to play an ascending ω-sequence which is in $\mathcal{D}$: If ONE had a winning strategy, ONE would have one like this. Similarly we may restrict ourselves to strategies F for ONE which have the property that if $(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ is a finite sequence of finite families of open sets, then each element of $F(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ contains $\bigcup(\bigcup_{i=1}^{n} \mathcal{V}_i)$.

Let (X, τ) be a space with property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ and let F be a strategy for ONE which has the properties in the previous paragraph. Define a family $(U_\tau\colon\ \tau \in {}^{<\omega}\mathbb{N})$ of open subsets of X as follows: $U_\emptyset = \emptyset$. $(U_n\colon\ n \in \mathbb{N})$ enumerates $F(\emptyset)$, ONE's first move, in ascending order. For each n_1, $(U_{n_1,n}\colon\ n \in \mathbb{N})$ enumerates $F(U_{n_1})$ in ascending order, and so forth. Then this family of open sets has the following properties for each σ:

(1) if $m < n$ then $U_{\sigma\frown m} \subseteq U_{\sigma\frown n}$;

(2) for all n, $U_\sigma \subseteq U_{\sigma\frown n}$;

(3) $\{U_{\sigma\frown n}\colon\ n \in \mathbb{N}\}$ is in $\mathcal{D}$.

Then define for each n and k:

$$U_k^n = \begin{cases} U_k & \text{if } n = 1, \\ (\bigcap\{U_{\sigma\frown k}\colon\ \sigma \in {}^{n-1}\mathbb{N}\}) \cap U_k^{n-1} & \text{otherwise.} \end{cases}$$

One then shows by induction on n that for all $(i_1, \ldots, i_n)$ with $\max\{i_1, \ldots, i_n\} \geqslant k$, $U_k^n \subseteq U_{i_1,\ldots,i_n}$. This implies that each U_k^n is an open set, so that $\mathcal{U}_n = \{U_k^n\colon\ k \in \mathbb{N}\}$ is an ascending chain of open sets. Next one shows by induction on n that for each nonempty open set U there are only finitely many σ of length n with $U \cap U_\sigma = \emptyset$. This means that each $\mathcal{U}_n$ is in $\mathcal{D}$.

Apply $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ to the sequence $(\mathcal{U}_n\colon\ n \in \mathbb{N})$. Since each $\mathcal{U}_n$ is an ascending chain this gives for each n a $U_{k_n}^n \in \mathcal{U}_n$ such that $\{U_{k_n}^n\colon\ n \in \mathbb{N}\}$ is in $\mathcal{D}$.

Finally, observe that since for each n $U_{k_n}^n \subseteq U_{k_1,\ldots,k_n}$, the sequence of moves $U_{k_1}, U_{k_1,k_2}, \ldots$ by TWO defeats the strategy F. $\square$

For topological space (X, τ) $\mathsf{PR}(X)$ denotes the collection of nonempty finite subsets of X. For $S \in \mathsf{PR}(X)$ and an open set $V \subseteq X$, $[S, V]$ denotes $\{T \in \mathsf{PR}(X)\colon\ S \subseteq T \subseteq V\}$. Since the intersection of two sets of the form $[S, V]$ is again such a set, the collection of all such subsets of $\mathsf{PR}(X)$ is a basis for a topology, denoted $\mathsf{PR}(\tau)$, on $\mathsf{PR}(X)$. Then $(\mathsf{PR}(X), \mathsf{PR}(\tau))$ is the Pixley–Roy space of (X, τ). Most of the time τ will be clear from context and we shall omit both τ and $\mathsf{PR}(\tau)$. If X has a countable base, then $\mathsf{PR}(\tau)\setminus\{\emptyset\}$ is a union of countably many sets, each with the finite intersection property; this implies that $\mathsf{PR}(X)$ has countable cellularity. Having countable cellularity is equivalent to: each element of $\mathcal{D}$ has a countable subset which is in $\mathcal{D}$.

It is not in general true that if a space has property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$, then it has property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$: If X is an uncountable discrete space then its Stone–Čech compactification $\beta(X)$ satisfies $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ and thus $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$, but does not satisfy $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ as is witnessed by the element $\{\{x\}\colon\ x \in X\}$ of $\mathcal{D}$ for $\beta(X)$. In [3] Daniels identified exact circumstances under which, for $X \subseteq \mathbb{R}$ $\mathsf{PR}(X)$ would satisfy $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$.

Theorem 3. *If the Pixley–Roy space of a subset of* $\mathbb{R}$ *satisfies* $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$*, then it satisfies* $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$.

Proof. Let X be a set of real numbers such that $\mathsf{PR}(X)$ satisfies $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$, and let $(\mathcal{U}_n\colon\ n \in \mathbb{N})$ be a sequence of elements of $\mathcal{D}$ for $\mathsf{PR}(X)$. Since when replacing each element of $\mathcal{U}_n$ with the basis elements contained in it we still have an element of $\mathcal{D}$, and since

$\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$ for sequences of *such* elements of $\mathcal{D}$ implies $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$, we may assume that each $\mathcal{U}_n$ is of this form.

Since $\mathsf{PR}(X)$ has countable cellularity we may assume that each $\mathcal{U}_n$ is countable. Let $(B_n\colon n\in\mathbb{N})$ bijectively enumerate the set of unions of finitely many elements from a countable basis of X. If each element $[S,V]$ of a $\mathcal{U}_m$ is replaced by the countably many elements $[S,B_n]$ where $S\subseteq B_n\subset\overline{B_n}\subset V$, then we obtain once again an element of $\mathcal{D}$ which refines $\mathcal{U}_m$. Thus, we may further assume that each $\mathcal{U}_n$ is countable, that each element of each $\mathcal{U}_n$ is of the form $[S,B_m]$, and that for each $[S,B_m]\in\mathcal{U}_n$ the set $\{B_j\colon S\subseteq B_j\subset\overline{B}_j\subset B_m\}$ is an ω-cover of B_m. By Theorem 2A of [3] and Theorem 3.9 of [8] X has property $\mathsf{S}_{fin}(\Omega,\Omega)$, and thus every F_σ-subset (in particular each B_n) has property $\mathsf{S}_{fin}(\Omega,\Omega)$.

Enumerate each $\mathcal{U}_n$ bijectively as $([S^n_m,U^n_m]\colon m\in\mathbb{N})$, and choose a partition $(Y_n\colon n\in\mathbb{N})$ of $\mathbb{N}$ into pairwise disjoint infinite sets. For each n and for each $m\in Y_n$, define:

$$\mathcal{V}_m=\left\{U^m_k\colon U^m_k\subset\overline{U}^m_k\subset B_n\right\}.$$

Then $(\mathcal{V}_m\colon m\in Y_n)$ is a sequence of ω-covers of B_n. Applying $\mathsf{S}_{fin}(\Omega,\Omega)$, we find for each m a finite set $\mathcal{W}_m\subseteq\mathcal{V}_m$ such that for each n $\bigcup_{m\in Y_n}\mathcal{W}_m$ is an ω-cover of B_n. We may for each m write $\mathcal{W}_m=\{U^m_k\colon k\in F_m\}$ where F_m is a finite subset of $\mathbb{N}$. For each m put $\mathcal{G}_m=\{[S^m_k,U^m_k]\colon k\in F_m\}$, a finite subset of $\mathcal{U}_m$.

Then $\bigcup_{m=1}^\infty\mathcal{G}_m$ is in $\mathcal{D}$. To see this, consider any nonempty basic open subset $[S,V]$ of $\mathsf{PR}(X)$. Then choose B_n such that $S\subseteq B_n\subseteq V$. For this n, $\bigcup_{m\in Y_n}\mathcal{W}_m$ is an ω-cover of B_n. Choose an $m\in Y_n$ and a $k\in F_m$ such that $S\subseteq U^m_k\subset B_n$. Then $[S^m_k,U^m_k]\in\mathcal{G}_m$, and $[S,V]\cap[S^m_k,U^m_k]\neq\emptyset$. □

The symbol $\mathcal{D}_\Omega$ denotes the collection of $\mathcal{U}\in\mathcal{D}$ such that: no element of $\mathcal{U}$ is dense, but for every finite set $\mathcal{F}$ of nonempty open sets, there is a $U\in\mathcal{U}$ such that for each $F\in\mathcal{F}$, $U\cap F\neq\emptyset$. Each of the hypotheses $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$, $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D})$ and $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ implies that X has countable cellularity.

Theorem 4. *A topological space has property* $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$ *if, and only if, it has property* $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D})$.

Proof. We must show that if a space has property $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D})$, then it has property $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$. Thus, let X be such a space and let $(\mathcal{U}_n\colon n\in\mathbb{N})$ be a sequence from $\mathcal{D}$ for it.

We may assume that each $\mathcal{U}_n$ is countable; let $(U^n_k\colon k\in\mathbb{N})$ enumerate $\mathcal{U}_n$. For each n define

$$\mathcal{V}_n=\left\{\bigcup_{j\leqslant k}U^n_j\colon k\in\mathbb{N}\right\}.$$

If there is some n for which $\mathcal{V}_n$ contains a dense subset of X, nothing more is required. Thus, we may assume that each $\mathcal{V}_n$ is in $\mathcal{D}_\Omega$. Applying the fact that each $\mathcal{V}_n$ is an ascending chain and the property $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D})$, choose for each n a k_n such that the sequence $(\bigcup_{j\leqslant k_n}U^n_j\colon n\in\mathbb{N})$ is in $\mathcal{D}$. Then for each n $\mathcal{G}_n=\{U^n_j\colon j\leqslant k_n\}$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\mathcal{G}_n$ is in $\mathcal{D}$. □

Corollary 5. *If a space has property* $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$*, then it has property* $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$.

Through I do not know the exact relationship between the classes of spaces having property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ and of spaces having the property $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$, I have the following partial descriptions in Theorems 6 and 7.

Theorem 6. *If each finite power of a space has property* $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$*, then that space has property* $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

Proof. Let X be a space as in the hypothesis and let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence from $\mathcal{D}_\Omega$ for X. Let $(Y_m\colon m \in \mathbb{N})$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite sets.

Fix m. For each n in Y_m, put $\mathcal{V}_n = \{U^m\colon U \in \mathcal{U}_n\}$. Then $(\mathcal{V}_n\colon n \in Y_m)$ is a sequence from $\mathcal{D}$ for X^m. Apply $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ for X^m and select for each $n \in Y_m$ a finite set $\mathcal{H}_n \subseteq \mathcal{V}_n$ such that $\bigcup_{n\in Y_m} \mathcal{H}_n$ is in $\mathcal{D}$ for X^m. For each $n \in Y_m$, put $\mathcal{G}_n = \{U\colon U^m \in \mathcal{H}_n\}$.

Doing this for each m gives rise to a sequence $(\mathcal{G}_n\colon n \in \mathbb{N})$ of finite sets such that for each n $\mathcal{G}_n \subseteq \mathcal{U}_n$. To see that $\bigcup_{n\in\mathbb{N}} \mathcal{G}_n$ is in $\mathcal{D}_\Omega$ for X, consider the finitely many nonempty open subsets $V_1, \ldots, V_k$ of X. Then $V = V_1 \times \cdots \times V_k$ is an open subset of X^k. Since $\bigcup_{n\in Y_k} \mathcal{H}_n$ is in $\mathcal{D}$ for X^k, we find an $n \in Y_k$ and a $U^k \in \mathcal{H}_n$ with $V \cap U^k \neq \emptyset$. But then $U \in \mathcal{G}_n$ and for $1 \leqslant i \leqslant k$, $U \cap V_i \neq \emptyset$. $\square$

I do not know to what extent the converse of Theorem 6 is true. To formulate a partial converse we introduce the following notion: The sequence $(\mathcal{B}_n\colon n \in \mathbb{N})$ is a *discriminating* sequence for X if no finite union of elements of $\mathcal{B}_1$ is dense in X and for each n:

(1) $\mathcal{B}_n$ is a π-base for X and
(2) for each $U \in \mathcal{B}_n$ there is a $V \in \mathcal{B}_{n+1}$ such that for all $W \in \mathcal{B}_{n+2}$ with $W \cap V \neq \emptyset$, $W \subseteq U$.

Theorem 7. *If a space has a discriminating sequence and property* $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$*, then each finite power of the space has property* $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$.

Proof. Let X be a space as in the hypothesis and let $(\mathcal{B}_n\colon n \in \mathbb{N})$ be a discriminating sequence. We show that X^2 has property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$; the proof for higher powers is analogous.

First, we describe a procedure for associating with an element $\mathcal{U}$ of $\mathcal{D}_\Omega$ for X^2 and an $n \in \mathbb{N}$ an element $\Sigma(\mathcal{U}, n)$ of $\mathcal{D}_\Omega$ for X. This specific procedure is used below.

Thus, fix $\mathcal{U} \in \mathcal{D}_\Omega$ for X^2 and fix $n \in \mathbb{N}$. For $\mathcal{F}$ a finite set of nonempty open subsets of X, do the following:

> For each $F \in \mathcal{F}$ choose a $B_F \in \mathcal{B}_{2^n}$ such that $B_F \subset F$, and if $H \in \mathcal{B}_{2^n+1}$ has nonempty intersection with B_F, then $H \subseteq F$. Put $\phi_n(\mathcal{F}) = \{B_F\colon F \in \mathcal{F}\}$ and $\psi_n(\mathcal{F}) = \{B_F \times B_G\colon F, G \in \mathcal{F}\}$.
> Fix an $A \in \mathcal{U}$ such that for each $D \in \psi_n(\mathcal{F})$ $A \cap D \neq \emptyset$. Then choose for $R, S \in \phi_n(\mathcal{F})$ sets $C_1(R, S, A), C_2(R, S, A) \in \mathcal{B}_{2^{2^n}}$ such that $C_1(R, S, A) \subset R$, $C_2(R, S, A) \subset S$, and $C_1(R, S, A) \times C_2(R, S, A) \subseteq R \times S \cap A$.

Then define

$$\Gamma_n(\mathcal{F}, A) = \bigcup\{C_i(R, S, A)\colon i \in \{1,2\},\ R, S \in \phi_n(\mathcal{F})\}.$$

Notice that for each $F \in \mathcal{F}$, $F \cap \Gamma_n(\mathcal{F}, A) \neq \emptyset$. Thus $\Sigma(\mathcal{U}, n)$, the collection of all sets $\Gamma_n(\mathcal{F}, A)$ generated like this, is in $\mathcal{D}_\Omega$ for X.

By Theorem 4 we may start with a sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ in $\mathcal{D}_\Omega$ for X^2. Let $(Y_n\colon n \in \mathbb{N})$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite sets such that for each n $\{1, \ldots, n\} \cap Y_n = \emptyset$.

With the procedure as described above, look for each n at the sequence $(\Sigma(\mathcal{U}_m, n)\colon m \in Y_n)$. Each of these sequences is a sequence from $\mathcal{D}_\Omega$ for X. Applying the property $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$ to each, we find for each $m \in Y_n$ a finite set $\mathcal{H}_m \subseteq \Sigma(\mathcal{U}_m, n)$ such that $\bigcup_{m \in Y_n} \mathcal{H}_m$ is in $\mathcal{D}_\Omega$ for X.

For each n and each $m \in Y_n$, $\mathcal{H}_m$ is of the form $\{\Gamma_n(\mathcal{F}_i^m, A_i^m)\colon i \in F_m\}$, where $F_m \subset \mathbb{N}$ is a finite set, and where

$$\Gamma_n(\mathcal{F}_i^m, A_i^m) = \bigcup\{C_j(R, S, A_i^m)\colon R, S \in \phi_n(\mathcal{F}_i^m),\ j \in \{1,2\}\}.$$

Now for each $i \in F_m$ choose $C_i^m \in \mathcal{U}_m$ such that for $j, \ell \in \{1,2\}$ and $R, S, T, U \in \mathcal{F}_i^m$, $C_i^m \cap (C_j(R, S, A_i^m) \times C_\ell(T, U, A_i^m)) \neq \emptyset$. Put

$$\mathcal{G}_m = \{C_i^m\colon i \in F_m\} (\subset \mathcal{U}_m), \quad m \in Y_n.$$

We show that $\bigcup_{m \in \mathbb{N}} \mathcal{G}_m$ is in $\mathcal{D}$ for X^2. Let $U \times V$ be a nonempty open subset of X^2. We may assume $U \cap V = \emptyset$ and $U, V \in \mathcal{B}_k$ for some fixed k. Choose $J_U, J_V \in \mathcal{B}_{2^k}$ so that $J_U \subseteq U$, $J_V \subseteq V$, and for all $W, R \in \mathcal{B}_{2^k+1}$ with $W \cap J_U \neq \emptyset$ and $R \cap J_V \neq \emptyset$, $W \subseteq U$ and $R \subseteq V$.

Choose $n > 2^k$. Since $\bigcup_{m \in Y_n} \mathcal{G}_m$ is in $\mathcal{D}_\Omega$ for X, choose $m \in Y_n$ and $T \in \mathcal{G}_m$ with $T \cap J_U \neq \emptyset$ and $T \cap J_V \neq \emptyset$. Then T is of the form $\Gamma_n(\mathcal{F}_j^m, A_j^m)$ for a $j \in F_m$. Thus, on account of the definition of $\Gamma_n(\mathcal{F}_j^m, A_j^m)$, choose $R_U, S_U, R_V, S_V \in \phi_n(\mathcal{F}_j^m)$ and $i_U, i_V \in \{1,2\}$ such that $C_{i_U}(R_U, S_U, A_j^m) \cap J_U \neq \emptyset$ and $C_{i_V}(R_V, S_V, A_j^m) \cap J_V \neq \emptyset$. Since $\phi_n(\mathcal{F}_j^m) \subseteq \mathcal{B}_{2^n}$ and $C_i(R, S, A_j^m) \in \mathcal{B}_{2^{2^n}}$ and $2^k < n < 2^n < 2^{2^n}$, we have

$$C_{i_U}(R_U, S_U, A_j^m) \subseteq U \quad \text{and} \quad C_{i_V}(R_V, S_V, A_j^m) \subseteq V.$$

But we have $C_j^m \in \mathcal{G}_m$ and

$$C_j^m \cap C_{i_U}(R_U, S_U, A_j^m) \times C_{i_V}(R_V, S_V, A_j^m) \neq \emptyset,$$

so that $C_j^m \cap U \times V \neq \emptyset$. $\square$

Problem 1. Find a space that satisfies $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ but not $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

Souslin lines are likely candidates for this, because they satisfy $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$, but (as is well known) their squares do not have countable cellularity, and thus do not have the property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$. To see that a Souslin line has property $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$, argue as follows: Let $(L, <)$ be a Souslin line. Then it has countably many maximal intervals which have countable dense subsets—these may be ignored when checking $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$ since the even-positioned terms from a given sequence of elements of $\mathcal{D}$ may be used to satisfy the selection hypothesis for these intervals. Thus assume that the line has no separable intervals, and thus has no isolated points. Let a sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of elements of $\mathcal{D}$ for L be given. We may assume that each $\mathcal{U}_n$ consists of countably many intervals, say $\mathcal{U}_n = \{(a^n_k, b^n_k)\colon k \in \mathbb{N}\}$. Define $\mathcal{F}$ to be the set of nonempty open intervals I with the property that for each n there is a k with $I \subseteq (a^n_k, b^n_k)$. Each nonempty interval from L contains an element of $\mathcal{F}$, since the set $\{a^n_k\colon k, n \in \mathbb{N}\} \cup \{b^n_k\colon n, k \in \mathbb{N}\}$ is nowhere dense in L. Thus, $\mathcal{F}$ is an element of D, as is any maximal pairwise disjoint subset of $\mathcal{F}$. Let $\mathcal{A}$ be a maximal infinite pairwise disjoint subset of $\mathcal{F}$. Then $\mathcal{A}$ is in $\mathcal{D}$. Since L is a Souslin line, $\mathcal{A}$ is countable, say $(A_n\colon n \in \mathbb{N})$ enumerates $\mathcal{A}$. Now recursively choose for each n a k_n such that $A_n \subseteq (a^n_{k_n}, b^n_{k_n})$. But then $((a^n_{k_n}, b^n_{k_n})\colon n \in \mathbb{N})$ is a selector for $(\mathcal{U}_n\colon n \in \mathbb{N})$, and is in $\mathcal{D}$.

Problem 2.[2] Is it true that if a space has property $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$, then each of its finite powers has property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$?

Theorem 8. *If the Pixley–Roy space for a set of real numbers has property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$, then it has property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ in each finite power.*

Proof. Let X be a set of real numbers for which $\mathsf{PR}(X)$ satisfies $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$. According to Theorem 2A of [3] and Theorem 3.9 of [8], every finite power of X satisfies $\mathsf{S}_{fin}(\Omega, \Omega)$. For finite powers of sets of reals open subsets inherit property $\mathsf{S}_{fin}(\Omega, \Omega)$. Since the topology of finite powers of $\mathsf{PR}(X)$ can be decomposed into countably many families, each with the finite intersection property, all finite powers of $\mathsf{PR}(X)$ have countable cellularity.

Let $(B_n\colon n \in \mathbb{N})$ enumerate the set of finite unions of elements of a countable basis for X. When determining if $\mathsf{PR}(X)^n$ has property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$, it suffices to consider elements $\mathcal{U}$ of $\mathcal{D}$ whose members are of the form

$$[S_1, B_{m_1}] \times \cdots \times [S_n, B_{m_n}]$$

and which have the additional property that for each $[S_1, B_{m_1}] \times \cdots \times [S_n, B_{m_n}]$ in $\mathcal{U}$, the set $\{B_{i_1} \times \cdots \times B_{i_n}\colon 1 \leqslant j \leqslant n \to S_j \subseteq B_{i_j} \subset \overline{B}_{i_j} \subset B_{m_j}\}$ is an ω-cover for $B_{m_1} \times \cdots \times B_{m_n}$. Countable cellularity allows us to further assume that those $\mathcal{U}$'s are countable sets.

Let $(\mathcal{U}_t\colon t \in \mathbb{N})$ be a sequence from $\mathcal{D}$ for $\mathsf{PR}(X)^n$. We may assume that each $\mathcal{U}_t$ has the properties just described. Enumerate each $\mathcal{U}_t$ bijectively as $([S^t_{k,1}, B^t_{k,1}] \times \cdots \times [S^t_{k,n}, B^t_{k,n}]\colon k \in \mathbb{N})$. Also let $(Y_{m_1,\dots,m_n}\colon m_1, \dots, m_n \in \mathbb{N})$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite subsets.

For each $(m_1, \dots, m_n)$ and for each $k \in Y_{m_1,\dots,m_n}$ put

$$\mathcal{V}_k = \left\{B^k_{j,1} \times \cdots \times B^k_{j,n}\colon 1 \leqslant i \leqslant n \Rightarrow \overline{B}^k_{j,i} \subset B_{m_i}\right\}.$$

Then $(\mathcal{V}_k\colon k \in Y_{m_1,\dots,m_n})$ is an ω-cover of $B_{m_1} \times \cdots \times B_{m_n}$. Apply $\mathsf{S}_{fin}(\Omega, \Omega)$ to the sequence $(\mathcal{V}_k\colon k \in Y_{m_1,\dots,m_n})$ of ω-covers of $B_{m_1} \times \cdots \times B_{m_n}$, and select for each k a finite

[2] Just showed in [7] that the answer is No: He uses $\Diamond$ to construct a Souslin line which has property $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

set $\mathcal{H}_k \subseteq \mathcal{V}_k$ such that $\bigcup_{k\in Y_{m_1,\dots,m_n}} \mathcal{H}_k$ is an ω-cover of $B_{m_1}\times\cdots\times B_{m_n}$. For each such k write:

$$\mathcal{H}_k = \{B^k_{j,1}\times\cdots\times B^k_{j,n}\colon\ j\in F_k\}$$

where F_k is a finite set, and then set

$$\mathcal{G}_k = \{[S^k_{j,1}, B^k_{j,1}]\times\cdots\times[S^k_{j,n}, B^k_{j,n}\colon\ j\in F_k]\}.$$

Then the sequence $[\mathcal{G}_k\colon\ k\in\mathbb{N}]$ is a sequence of finite sets such that for each k $\mathcal{G}_k\subseteq U_k$, and $\bigcup_{k\in\mathbb{N}}\mathcal{G}_k$ is in $\mathcal{D}$ for $\mathsf{PR}(X)^n$. □

If ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ then the space has property $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$; it is not clear if the converse is true (see Problem 3); here is a partial result:

Theorem 9. *If every finite power of X has property $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$, the ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$.*

Proof. We may assume that $X^m\cap V^n=\emptyset$ whenever $m\neq n$. Each X^m is clopen in $Y:=\sum_{m\in\mathbb{N}}X^m$, so that Y has property $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$. According to Theorem 2, ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{D},\mathcal{D})$ on Y. We now use this information to show that ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ on X.

Let F be a strategy for ONE in $\mathsf{G}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ on X, and define a strategy G for ONE in $\mathsf{G}_{fin}(\mathcal{D},\mathcal{D})$ on Y as follows. ONE's first move in $\mathsf{G}_{fin}(\mathcal{D},\mathcal{D})$ on Y is:

$$G(Y)=\{U^n\colon\ U\in F(X),\ n\in\mathbb{N}\};$$

since $F(X)$ is in $\mathcal{D}_\Omega$ for X, $G(Y)$ is in $\mathcal{D}$ for Y. If TWO responds with a finite set, $T_1\subseteq G(Y)$, then ONE responds as follows: first, set $S_1=\{U\in F(X)\colon\ (\exists n)(U^n\in T_1)\}$ and treat this as a move for TWO of $\mathsf{G}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ on X; then compute $F(S_1)$, and respond in $\mathsf{G}_{fin}(\mathcal{D},\mathcal{D})$ on Y with $G(T_1)=\{U^n\colon\ U\in F(S_1),\ n\in\mathbb{N}\}$. If TWO now responds with $T_2\subseteq G(T_1)$, then first compute $S_2=\{U\in F(S_1)\colon\ (\exists n)(U^n\in T_2)\}$; then compute $F(S_1,S_2)$, and play $G(T_1,T_2)=\{U^n\colon\ U\in F(S_1,S_2),\ n\in\mathbb{N}\}$, and so on.

Since G is not a winning strategy for ONE, fix a play

$$G(Y), T_1, G(T_1), T_2, G(T_1,T_2), T_3,\dots$$

which is lost by ONE; this means $\bigcup_{n=1}^\infty T_n\in\mathcal{D}$ for Y. For each n put $S_n=\{U\colon\ (\exists m)(U^m\in T_n)\}$. Then $F(X), S_1, F(S_1), S_2, F(S_1,S_2), S_3,\dots$ is a play lost by ONE in the game $\mathsf{G}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$ on X. □

Problem 3. Is it true that if a space satisfies $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$, the ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$?

For collections $\mathcal{A}$ and $\mathcal{B}_1,\dots,\mathcal{B}_m$ of subsets of a set S and for $n,k\in\mathbb{N}$, the symbol

$$\mathcal{A}\to(\mathcal{B}_1,\dots,\mathcal{B}_m)^n_k$$

denotes that for each $A \in \mathcal{A}$ and for each $f : [A]^n \to \{1, 2, \ldots, k\}$ there is an $i \in \{1, 2, \ldots, k\}$ and a subset B of A in $\mathcal{B}_i$ such that the value of f everywhere on $[B]^n$ is i. We say that *B is homogeneous of color i for f*.

For example, $\mathcal{D}_\Omega \to (\mathcal{D}_\Omega)^1_k$, which means that if an element of $\mathcal{D}_\Omega$ is partitioned into k pieces, then there is a piece which is a member of $\mathcal{D}_\Omega$, is true for each $k \in \mathbb{N}$. A second relevant partition relation is denoted by

$$\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$$

and means that for each $A \in \mathcal{A}$ and for each function $f : [A]^2 \to \{1, \ldots, k\}$ there is a subset B of A in $\mathcal{B}$, a finite-to-one function g with domain B, and an $i \in \{1, 2, \ldots, k\}$ such that $f(\{a, b\}) = i$ whenever $a, b \in B$ and $g(a) \neq g(b)$. This partition relation was introduced in [2].

In Theorem 10 below we use the hypothesis that every element of $\mathcal{D}_\Omega$ has a countable subset in $\mathcal{D}_\Omega$. One can show that if each element of $\mathcal{D}_\Omega$ has a countable subset in $\mathcal{D}$, then each element of $\mathcal{D}$ has a countable subset which is in $\mathcal{D}$. One can also show that if each finite power of a space has countable cellularity, then the space itself has the property that each element of $\mathcal{D}_\Omega$ has a countable subset which is in $\mathcal{D}_\Omega$. Thus, Martin's Axiom implies that countable cellularity is equivalent to each element of $\mathcal{D}_\Omega$ having a countable subset which is in $\mathcal{D}_\Omega$.

Problem 4.[3] If each element of $\mathcal{D}_\Omega$ for X has a countable subset which is an element of $\mathcal{D}_\Omega$, does it follow that each finite power of X has countable cellularity?

Theorem 10. *Let X be a space such that each element of $\mathcal{D}_\Omega$ has a countable subset in $\mathcal{D}_\Omega$. Then the following are equivalent*:

(1) *X satisfies $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$.*

(2) *For each $k \in \mathbb{N}$, X satisfies $\mathcal{D}_\Omega \to \lceil \mathcal{D} \rceil^2_k$.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{U}$ be an element of $\mathcal{D}_\Omega$. We may assume that $\mathcal{U}$ is countable. Enumerate it bijectively as $(U_n\colon n \in \mathbb{N})$. Let $k \in \mathbb{N}$ as well as a function $f : [\mathcal{U}]^2 \to \{1, \ldots, k\}$ be given. Construct a sequence $(\mathcal{U}_1, i_1), (\mathcal{U}_2, i_2), \ldots$ so that

(1) $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \cdots \supset \mathcal{U}_n \supset \cdots$ are in $\mathcal{D}_\Omega$;

(2) each i_n is in $\{1, 2, \ldots, k\}$;

(3) $\mathcal{U}_1 = \{U_n\colon n > 1 \text{ and } f(\{U_1, U_n\}) = i_1\}$, and for each n

$$\mathcal{U}_{n+1} = \bigl\{U_m \in \mathcal{U}_n\colon m > n+1 \text{ and } f(\{U_{n+1}, U_m\}) = i_{n+1}\bigr\}.$$

The partition relation $\mathcal{D}_\Omega \to (\mathcal{D}_\Omega)^1_k$ is used repeatedly to do this. Then, for $j \in \{1, \ldots, k\}$ put $\mathcal{C}_j = \{U_n\colon i_n = j\}$. Then partition each $\mathcal{U}_n$ as follows:

$$\mathcal{U}_n = (\mathcal{U}_n \cap \mathcal{C}_1) \cup \cdots \cup (\mathcal{U}_n \cap \mathcal{C}_k).$$

[3] In [7] Just shows that the answer is No: The counterexample he found for Problem 2 is also a counterexample for Problem 4.

For each n we find a j_n such that $\mathcal{U}_n \cap \mathcal{C}_{j_n}$ is in $\mathcal{D}_\Omega$. Fix j such that for infinitely many n we have $j_n = j$. Since the sequence of $\mathcal{U}_n$'s is descending this means that for each n we have $\mathcal{V}_n := \mathcal{U}_n \cap \mathcal{C}_j$ in $\mathcal{D}_\Omega$. Let $(U_{n_k}\colon k \in \mathbb{N})$ be the list, in the enumeration we chose earlier, of U_n's with $i_n = j$.

Look at the descending sequence $\mathcal{V}_{n_1} \supset \mathcal{V}_{n_2} \supset \cdots$. This is a sequence in $\mathcal{D}_\Omega$. Define a strategy F for ONE in the game $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$ as follows. With n_k minimal with $U_{n_k} \in \mathcal{V}_{n_1}$, play $F(X) = \mathcal{V}_{n_k}$. If TWO chooses the finite set $T_1 \subseteq F(X)$, compute $n_\ell = \max\{n\colon U_n \in T_1\}$, and then play $F(T_1) = \mathcal{V}_{n_\ell} \setminus T_1$. If TWO now chooses the finite set $T_2 \subset F(T_1)$, compute $n_{\ell_2} = \max\{n\colon U_n \in T_2\}$, and play $F(T_1, T_2) = \mathcal{V}_{n_{\ell_2}} \setminus (T_1 \cup T_2)$, and so on.

By Theorem 2, F is not a winning strategy for ONE. Look at an F-play lost by ONE. It is of the form

$$\mathcal{V}_{n_{k_1}}, T_1, \mathcal{V}_{n_{k_2}}, T_2, \mathcal{V}_{n_{k_3}}, T_3, \ldots$$

where $n_{k_1} < n_{k_2} < n_{k_3} < \cdots$ and if $k < \ell$, then for all $U_m \in T_k$ and $U_n \in \mathcal{V}_{n_\ell}$, $f(\{U_m, U_n\}) = j$. The function g defined on $\bigcup_{n\in\mathbb{N}} T_n$ so that $g(U) = k$ only if $U \in T_k$ witnesses that $\bigcup_{n\in\mathbb{N}} T_n$ is eventually homogeneous for f.

(2) $\Rightarrow$ (1) We use a partition that has been used several times in part II and other related papers. It suffices to show that the partition relation implies that X has $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D})$. Thus, let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence from $\mathcal{D}_\Omega$. Each $\mathcal{U}_n$ may be assumed to be countable; let $(U^n_k\colon k \in \mathbb{N})$ enumerate $\mathcal{U}_n$ bijectively. Define

$$\mathcal{V} := \left\{U^1_n \cap U^n_k\colon n, k \in \mathbb{N}\right\} \setminus \{\emptyset\}.$$

Then $\mathcal{V}$ is in $\mathcal{D}_\Omega$. For each element of $\mathcal{V}$ choose a representation of the form $U^1_n \cap U^n_k$. Define $f : [\mathcal{V}]^2 \to \{1, 2\}$ by:

$$f\left(\left\{U^1_{n_1} \cap U^{n_1}_{k_1}, U^1_{n_2} \cap U^{n_2}_{k_2}\right\}\right) = \begin{cases} 1 & \text{if } n_1 = n_2, \\ 2 & \text{otherwise.} \end{cases}$$

Let $\mathcal{W} \subset \mathcal{V}$ be an element of $\mathcal{D}$ which is eventually homogeneous for f. A case analysis shows that $\mathcal{W}$ is eventually homogeneous of color 2, and this in turn implies that $\{U^n_k\colon U^1_n \cap U^n_k \in \mathcal{W}\}$ contains finitely many elements from each $\mathcal{U}_m$. $\square$

For a Tychonoff space X the set of continuous functions from X to $\mathbb{R}$ is a subset of the Tychonoff product space $\mathbb{R}^X$ of X copies of $\mathbb{R}$. $\mathsf{C}_p(X)$ denotes this set of continuous functions, endowed with the topology it inherits from this power of $\mathbb{R}$. Being a topological vector space, $\mathsf{C}_p(X)$ is homogeneous. This means that determining if a point $f \in \mathsf{C}_p(X)$ is in the closure of the subset A of $\mathsf{C}_p(X)$ is equivalent to determining if $\boldsymbol{o}$, the function which is zero everywhere on X, is in the closure of a corresponding set. Define

$$\Omega_{\boldsymbol{o}} := \left\{A \subseteq \mathsf{C}_p(X) \setminus \{\boldsymbol{o}\}\colon \boldsymbol{o} \in \overline{A}\right\}.$$

Corollary 11. *Let X be a set of real numbers. Then the following statements are equivalent*:

(1) *each finite power of X has property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$*;
(2) *X has property $\mathsf{S}_{fin}(\Omega, \Omega)$*;
(3) *ONE has no winning strategy in the game $\mathsf{G}_{fin}(\Omega, \Omega)$ on X*;

(4) *for X and each $k \in \mathbb{N}$, $\Omega \to \lceil \Omega \rceil^2_k$;*
(5) $\mathsf{C}_p(X)$ *has property* $\mathsf{S}_{fin}(\Omega_{\boldsymbol{o}}, \Omega_{\boldsymbol{o}})$;
(6) *ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\Omega_{\boldsymbol{o}}, \Omega_{\boldsymbol{o}})$ *on* $\mathsf{C}_p(X)$;
(7) *for each $k \in \mathbb{N}$, $\Omega_{\boldsymbol{o}} \to \lceil \Omega_{\boldsymbol{o}} \rceil^2_k$;*
(8) $\mathsf{PR}(X)$ *has property* $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$;
(9) $\mathsf{PR}(X)$ *has property* $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$;
(10) *ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$ *on* $\mathsf{PR}(X)$;
(11) *for each $k \in \mathbb{N}$,* $\mathsf{PR}(X)$ *satisfies* $\mathcal{D}_\Omega \to \lceil \mathcal{D} \rceil^2_k$;
(12) $\mathsf{PR}(X)$ *satisfies* $\mathsf{S}_{fin(\mathcal{D}_\Omega, \mathcal{D}_\Omega)}$;
(13) *ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$ *on* $\mathsf{PR}(X)$;
(14) *for each $k \in \mathbb{N}$,* $\mathsf{PR}(X)$ *satisfies* $\mathcal{D}_\Omega \to \lceil \mathcal{D}_\Omega \rceil^2_k$.

Proof. (1) $\Leftrightarrow$ (2) is proved in Theorem 3.9 of [8]. (1) $\Leftrightarrow$ (5) is proved in Theorem 4 of [1]. (1) $\Leftrightarrow$ (8) is proved in Theorems 2A and 2B of [3].

(2) $\Leftrightarrow$ (4) is proved in Theorem 6.2 of [8] and Theorem 10 of [11]. (2) $\Leftrightarrow$ (3) is given in Theorem 5 of [12].

The equivalence of (5), (6) and (7) were given in [12]. The equivalence of (8) and (9) follows from Theorem 3. The equivalence of (9) and (10) follows from Theorem 2. The equivalence of (10) and (11) follows from Theorem 10. The equivalence of (10) and (12) follows from Theorems 5, 8 and 6. The equivalence of (12) and (13) follows from Theorems 9 and 8. The equivalence of (14) and (12) are proved analogously to Theorem 10. □

2. The cardinal number $\mathfrak{d}$

Theorem 12. *For an infinite cardinal number κ the following statements are equivalent*:
(1) $\kappa < \mathfrak{d}$;
(2) *for each* T_1*-space of countable cellularity and π-weight κ, ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$.

Proof. (1) $\Rightarrow$ (2) Let F be a strategy for ONE of $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$ on the space X of countable cellularity and π-weight κ. Let $\mathcal{B}$ be a π-base of cardinality κ for X. By selecting in each inning from ONE's play a countable subset which is an element of $\mathcal{D}$, build the following array of open subsets of X: $(F_n\colon n \in \mathbb{N})$ enumerates the element of $\mathcal{D}$ selected from $F(\emptyset)$, ONE's first move. For each n_1, $(F_{n_1,n}\colon n \in \mathbb{N})$ enumerates the element of $\mathcal{D}$ selected from $F(\{F_j\colon j \leqslant n_1\})$. For each n_1 and n_2, $(F_{n_1,n_2,n}\colon n \in \mathbb{N})$ enumerates the element of $\mathcal{D}$ selected from $F(\{F_j\colon j \leqslant n_1\}, \{F_{n_1,j}\colon j \leqslant n_2\})$, and so on. The family of F_σ, $\sigma \in {}^{<\omega}\mathbb{N}$ has the property that for each σ, $\{F_{\sigma\frown n}\colon n \in \mathbb{N}\}$ is in $\mathcal{D}$.

For each $B \in \mathcal{B}$ define f_B so that $f_B(1) = \min\{k\colon B \cap F_k \neq \emptyset\} + 1$, and for each n, $f_B(n+1)$ is the least $m > f_B(n)$ such that for every σ in ${}^{\leqslant f_B(n)}\{1, \ldots, f_B(n)\}$ there is a $j \leqslant m$ with $B \cap F_{\sigma\frown j} \neq \emptyset$. Then each f_B is strictly increasing and $f_B(1) > 1$. Next, for each B define g_B by $g_B(1) = f_B(1)$ and for all n, $g_B(n+1) = f_B(g_B(n))$. On cardinality grounds $\{g_B\colon B\in\mathcal{B}\}$ is not cofinal in ${}^{\mathbb{N}}\mathbb{N}$. Choose a strictly increasing g such that for each B, $\{n\colon g_B(n)<g(n)\}$ is infinite. For each n let $h(n)$ be the nth iterate of g, computed at 1. Define the sets

$$T_1=\{F_j\colon j\leqslant h(1)\},\qquad T_{n+1}=\{F_{h(1),\dots,h(n),j}\colon j\leqslant h(n+1)\}.$$

Then $F(\emptyset), T_1, F(T_1), T_2, F(T_1,T_2),\dots$ is a play. We claim ONE lost it.

For look at $B\in\mathcal{B}$. Choose m minimal with $g_B(m)<h(m)$. If $m=1$, then $f_B(1)=g_B(1)<h(1)=g(1)$, and so $B\cap(\bigcup T_1)\neq\emptyset$. Thus, assume that m is larger than 1, say $m=k+1$. Then we have

$$k<h(k)\leqslant f_B^k(1)=g_B(k)<g_B(k+1)=f_B^{k+1}(1)<h(k+1)$$

which means that $(h(1),\dots,h(k))$ was one of the sequences considered when $f_B(f_B^{k-1}(1)+1)$ was defined. Since $f_B(f_B^{k-1}(1)+1)\leqslant f_B(f_B^k(1))$, we see that $B\cap(\bigcup T_m)\neq\emptyset$.

(2) $\Rightarrow$ (1) Let X be a set of real numbers of cardinality κ. Then $\mathsf{PR}(X)$ has π-weight κ and has countable cellularity. It follows that ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{D},\mathcal{D})$. By Corollary 11 X has property $\mathsf{S}_{fin}(\Omega,\Omega)$. We showed that every set of reals of cardinality κ has property $\mathsf{S}_{fin}(\Omega,\Omega)$. By Theorem 4.6 of [8] this means $\kappa<\mathfrak{d}$. $\square$

3. $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ and $\mathsf{S}_1(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$

In [4] the authors study a game G_1 which is played as follows: ONE and TWO play an inning per positive integer. In the nth inning ONE choose O_n, a maximal family of pairwise disjoint open sets. TWO responds with $T_n\in O_n$. A play $O_1, T_1,\dots,O_n,T_n,\dots$ of G_1 is won by TWO if $\{T_n\colon n\in\mathbb{N}\}\in\mathcal{D}$; otherwise, ONE wins. One can show:

Lemma 13. *ONE has a winning strategy in G_1 if, and only if, ONE has a winning strategy in $\mathsf{G}_1(\mathcal{D},\mathcal{D})$. TWO has a winning strategy in G_1 if, and only if, TWO has a winning strategy in $\mathsf{G}_1(\mathcal{D},\mathcal{D})$.*

Theorem 14. *A topological space satisfies $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ if, and only if, ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{D},\mathcal{D})$.*

Proof. We must show that if a space has property $\mathsf{S}_1(\mathcal{D},\mathcal{D})$, the ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{D},\mathcal{D})$. The proof is a minor variation of that of Lemma 2 of [9]—we give an outline for the reader's convenience. Let X be a space with property $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$. Let F be a strategy for ONE in the game $\mathsf{G}_1(\mathcal{D},\mathcal{D})$. We may assume that in each inning F calls on ONE to play a countable element of $\mathcal{D}$.

Define the array U_σ, σ in ${}^{<\omega}\mathbb{N}$, as follows: $(U_n\colon n\in\mathbb{N})$ enumerates ONE's first move, $F(\emptyset)$. For n_1, $(U_{n_1,n}\colon n\in\mathbb{N})$ enumerates $F(U_{n_1})$. For n_1, n_2, $(U_{n_1,n_2,n}\colon n\in\mathbb{N})$ enumerates $F(U_{n_1},U_{n_1,n_2})$, and so on. This array has the property that for each σ the set $\{U_{\sigma\frown n}\colon n\in\mathbb{N}\}$ is in $\mathcal{D}$.

For fixed m and $j \in \mathbb{N}$ and ρ a function from $\{1, \ldots, j^m\}$ to $\mathbb{N}$, define the set

$$U_\rho(m, j) := \bigcap_{\sigma \in^m \{1,\ldots,j\}} \left(\bigcup \{U_{\sigma \frown \rho\lceil_i} \colon i \leqslant j^m\} \right)$$

and then for fixed m and j define

$$\mathcal{U}(m, j) := \{U_\rho(m, j)\colon \rho \text{ a function from } \{1, \ldots, j^m\} \text{ to } \mathbb{N}\}.$$

Then each $\mathcal{U}(m, j)$ is in $\mathcal{D}$.

There exist increasing sequences $(j_n\colon n \in \mathbb{N})$ and $(m_n\colon n \in \mathbb{N})$ such that for each nonempty open set $U \subseteq X$ and for each n there is a function σ from $\{1, \ldots, m_{n+1} - m_n\}$ to j_{n+1} for which $U \cap U_\sigma(m_n, j_n)$ is nonempty. To see this, let ONE play the game $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$ using the following strategy, G. For a first move ONE puts $j_1 = m_1 = 1$, and plays $G(\emptyset) = \mathcal{U}(m_1, j_1)$. For a response $T_1 \subset \mathcal{U}(m_1, j_1)$ by TWO, ONE first does the following computations: $m_2 = m_1 + j_1^{m_1}$, and $j_2 > j_1$ is at least the maximum of all values of σ's for which $U_\sigma(m_1, j_1)$ is in T_1. Then ONE plays $G(T_1) = \mathcal{U}(m_2, j_2)$. For a response $T_2 \subseteq G(T_1)$ by TWO, ONE again first computes the numbers m_3 and j_3 according to the rules that $m_3 = m_2 + j_2^{m_2}$, and $j_3 > j_2$ is at least the maximum of all values of σ's for which $U_\sigma(m_2, j_2)$ is in T_2, and so on. Since X has property $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$, it has property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ and by Theorem 2 G is not a winning strategy for ONE. Look at a G-play $G(\emptyset), T_1, G(T_1), T_2, G(T_1, T_2), \ldots$ which is lost by ONE. Then $\bigcup_{n\in\mathbb{N}} T_n$ is in $\mathcal{D}$, and we find increasing sequences $(j_n\colon n \in \mathbb{N})$ and $(m_n\colon n \in \mathbb{N})$ such that for each n:

(1) $m_{n+1} = m_n + j_n^{m_n}$;

(2) $G(T_1, \ldots, T_n) = \mathcal{U}(m_{n+1}, j_{n+1})$;

(3) j_{n+1} is at least as large as the value of an σ for which $U_\sigma(m_n, j_n)$ is in T_n.

It follows that the m_n's and j_n's have the required properties.

With the sequences $(j_n\colon n \in \mathbb{N})$ and $(m_n\colon n \in \mathbb{N})$ fixed, define next for each n the family $\mathcal{W}_n$ as follows: For every sequence $k_1 < \cdots < k_n$ from $\mathbb{N}$, and for any $\sigma_1, \ldots, \sigma_n$ where each σ_i is an $\{1, \ldots, j_{k_i+1}\}$-valued function with domain $m_{k_i+1} - m_{k_i}$, define

$$W(k_1, \ldots, k_n; \sigma_1, \ldots, \sigma_n) := \bigcap_{i \leqslant n} U_{\sigma_i}(m_{k_i}, j_{k_i}).$$

$\mathcal{W}_n$ consists of all sets of the form $W(k_1, \ldots, k_n; \sigma_1, \ldots, \sigma_n)$.

Since each $\mathcal{W}_n$ is in $\mathcal{D}$, the selection hypothesis $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$ applied to $(\mathcal{W}_n\colon n \in \mathbb{N})$ gives for each n a set $S_n := W(k_1^n, \ldots, k_n^n; \sigma_1^n, \ldots, \sigma_n^n)$ such that $\{S_n\colon n \in \mathbb{N}\}$ is in $\mathcal{D}$. Recursively choose for each n an $\ell_n \in \{k_1^n, \ldots, k_n^n\} \setminus \{\ell_i\colon i < n\}$. For each n define $\rho_n = \sigma_{i_n}^n$ where i_n is such that $\ell_n = k_{i_n}^n$.

From the definitions we see that for each n $S_n \subseteq U_{\rho_n}(m_{\ell_n}, j_{\ell_n})$. If we now define $f : \mathbb{N} \to \mathbb{N}$ so that for each n $f(m_{\ell_n} + i) = \rho_n(i)$ whenever $i \leqslant m_{\ell_n+1} - m_{l_n}$, we find that the play

$$F(\emptyset), U_{f(1)}, F(U_{f(1)}), U_{f(1),f(2)}, F(U_{f(1)}, U_{f(1),f(2)}), U_{f(1),f(2),f(3)}, \ldots$$

is won by TWO. $\square$

A minor variation of the proof of Theorem 3 gives:

Theorem 15. *If the Pixley–Roy space of a set of real numbers satisfies* $\mathsf{S}_1(\mathcal{O}, \mathcal{D})$*, then it satisfies* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$.

In Theorem 21 we shall further strengthen Theorem 15. To see that this is at least formally a strengthening, we need an analogue of Theorem 4. To prove the analogue of Theorem 4 in the present context takes a little bit more work. To this end we introduce the subfamily $\mathcal{L}$ of $\mathcal{D}$:

$\mathcal{L}$ consists of those $\mathcal{U} \in \mathcal{D}$ with the property that for each nonempty open subset U of the space, $\{V \in \mathcal{U}\colon\ U \cap V \neq \emptyset\}$ is infinite. The next theorem, an analogue of Theorem 15 of [11], shows that $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$ implies a certain "splitting property" that was introduced in [11]:

Theorem 16. *If a space satisfies* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$*, then for every* $\mathcal{U}_1, \mathcal{U}_2$ *in* $\mathcal{L}$ *there are* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *in* $\mathcal{L}$ *such that* $\mathcal{V}_1 \subseteq \mathcal{U}_1$, $\mathcal{V}_2 \subseteq \mathcal{U}_2$, *and* $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$.

Proof. The proof is just like that of Theorem 15 of [11]. □

Then, using the idea of the proof of Theorem 16 of [11], one obtains from the preceding theorem:

Corollary 17. *If a space has property* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$*, then for every sequence* $(\mathcal{U}_n\colon\ n \in \mathbb{N})$ *of elements of* $\mathcal{L}$*, there is a sequence* $(\mathcal{V}_n\colon\ n \in \mathbb{N})$ *of elements of* $\mathcal{L}$ *such that for each* n $\mathcal{V}_n \subseteq \mathcal{U}_n$ *and* $\mathcal{V}_m \cap \mathcal{V}_n = \emptyset$ *whenever* $m \neq n$.

Theorem 18. *The following selection hypotheses are equivalent*:

(1) $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$;
(2) $\mathsf{S}_1(\mathcal{L}, \mathcal{L})$;
(3) $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{L})$;
(4) $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D})$.

Proof. We must show that (1) implies (2), and (4) implies (1).

(1) ⇒ (2) Let $(\mathcal{U}_n\colon\ n \in \mathbb{N})$ be a sequence from $\mathcal{L}$. By Corollary 17 we may assume that $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$ whenever $m \neq n$. Let $(Y_n\colon\ n \in \mathbb{N})$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite sets. For each n apply $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D})$ to the sequence $(\mathcal{U}_m\colon\ m \in Y_n)$. The result is a selector for $(\mathcal{U}_j\colon\ j \in \mathbb{N})$ which is moreover in $\mathcal{L}$.

(4) ⇒ (1) Let $(\mathcal{U}_n\colon\ n \in \mathbb{N})$ be sequence from $\mathcal{D}$. Let $(Y_n\colon\ n \in \mathbb{N})$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite sets. For each n, let $\mathcal{V}_n$ consist of sets of the form $U_{i_1} \cup \cdots \cup U_{i_m}$ where $m \in \mathbb{N}$, $U_{i_j} \in \mathcal{U}_{i_j}$, and $i_1 < \cdots < i_m$ are in Y_n. If some such $\mathcal{V}_n$ contains a dense subset of the space, then nothing more is to be done. Thus, we may assume that each $\mathcal{V}_n$ is in $\mathcal{D}_\Omega$. Now apply $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D})$ to the sequence $(\mathcal{V}_n\colon\ n \in \mathbb{N})$. The selector for this sequence can be modified to an appropriate selector for the original sequence of $\mathcal{U}_n$'s. □

Corollary 19. *Every space with property* $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$ *has property* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$.

Problem 5. Find a space which has property $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$ but not $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

The method of proof of Theorems 6 and 7 also work to prove

Theorem 20. *If each finite power of a space has property* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$*, then the space has property* $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$*. Conversely, if the space has a discriminating sequence and property* $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$*, then it has* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$ *in all finite powers.*

Analogous to Theorem 8 we have:

Theorem 21. *If the Pixley–Roy space for a set of real numbers has property* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$*, then it has property* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$ *in each finite power, and thus has property* $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

The method of proof of Theorem 9 also gives:

Theorem 22. *If every finite power of* X *has property* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$*, then ONE has no winning strategy in the game* $\mathsf{G}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

By slightly adjusting the methods in the proof of Theorem 10, one obtains:

Theorem 23. *Let* X *be a space such that each element of* $\mathcal{D}_\Omega$ *has a countable subset in* $\mathcal{D}_\Omega$*. Then the following are equivalent*:
(1) X *satisfies* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$.
(2) X *satisfies*: *For each* $k \in \mathbb{N}$, $\mathcal{D}_\Omega \to (\mathcal{D})^2_k$.

This brings us now to our second summary of how the preceding results, when applied to the Pixley–Roy spaces of sets of reals, fit in with the work from parts I–III.

Corollary 24. *For* $X \subseteq \mathbb{R}$ *the following statements are equivalent*:
(1) *each finite power of* X *has property* $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$;
(2) X *has property* $\mathsf{S}_1(\Omega, \Omega)$;
(3) *ONE has no winning strategy in the game* $\mathsf{G}_1(\Omega, \Omega)$ *on* X;
(4) *for* X *and each,* $n, k \in \mathbb{N}$, $\Omega \to (\Omega)^n_k$;
(5) $\mathsf{C}_p(X)$ *has property* $\mathsf{S}_1(\Omega_{\boldsymbol{o}}, \Omega_{\boldsymbol{o}})$;
(6) *ONE has no winning strategy in the game* $\mathsf{G}_1(\Omega_{\boldsymbol{o}}, \Omega_{\boldsymbol{o}})$ *on* $\mathsf{C}_p(X)$;
(7) *for each* $k, n \in \mathbb{N}$, $\Omega_{\boldsymbol{o}} \to (\Omega_{\boldsymbol{o}})^n_k$;
(8) $\mathsf{PR}(X)$ *has property* $\mathsf{S}_1(\mathcal{O}, \mathcal{D})$;
(9) $\mathsf{PR}(X)$ *has property* $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$;
(10) *ONE has no winning strategy in the game* $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$ *in* $\mathsf{PR}(X)$;
(11) *for each* $k \in \mathbb{N}$, $\mathsf{PR}(X)$ *satisfies* $\mathcal{D}_\Omega \to (\mathcal{D})^2_k$;
(12) $\mathsf{PR}(X)$ *satisfies* $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$;

(13) *ONE has no winning strategy in the game* $\mathsf{G}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$ *on* $\mathsf{PR}(X)$;
(14) *for each* $k, n \in \mathbb{N}$, $\mathsf{PR}(X)$ *satisfies* $\mathcal{D}_\Omega \to (\mathcal{D}_\Omega)^n_k$.

Proof. (1) ⇔ (2) is proved in the Lemma of [10]. (1) ⇔ (5) is proved in Theorem 1 of [10]. (1) ⇔ (8) is proved in Theorems 5A and 5B of [3].

(2) ⇔ (3) is given in Theorem 2 of [12]. (2) ⇔ (4) is proved by combining Theorem 6.1 of [8] and Theorems 23 and 24 of [11].

The equivalence of (5), (6) and (7) was given in [12]. The equivalence of (8) and (9) follows from Theorem 15. The equivalence of (9) and (10) follows from Theorem 14. The equivalence of (10) and (11) follows from Theorem 23. The equivalence of (10) and (12) follows from Theorem 21. The equivalence of (12) and (13) follows from Theorems 22 and 21. The equivalence of (12) and (14) is proved similarly to the analogous fact for $\mathsf{S}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$. □

4. The cardinal number $\mathsf{cov}(\mathcal{M})$

Let $\mathsf{cov}(\mathcal{M})$ denote the minimum number of first category sets required to cover $\mathbb{R}$. Since the space ${}^{\mathbb{N}}\mathbb{N}$, considered as a countable power of the discrete space $\mathbb{N}$, is homeomorphic to the set of irrational numbers, $\mathsf{cov}(\mathcal{M})$ is equal to the analogous covering number for this space.

Theorem 25. *For an infinite cardinal number* κ *the following are equivalent*:
(1) $\kappa < \mathsf{cov}(\mathcal{M})$;
(2) *for each* T_1*-space of countable cellularity and* π*-weight* κ*, ONE has no winning strategy in the game* $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$.

Proof. (1) ⇒ (2) Let F be a strategy for ONE in $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$. In each inning, TWO may restrict attention to a countable subset in $\mathcal{D}$ from ONE's selected set. Build the following array of sets: $(U_n\colon n \in \mathbb{N})$ enumerates a countable element of $\mathcal{D}$ contained in $F(\emptyset)$; $(U_{n_1,n}\colon n \in \mathbb{N})$ enumerates a countable element of $\mathcal{D}$ contained in $F(U_{n_1})$, $(U_{n_1,n_2,n}\colon n \in \mathbb{N})$ enumerates a countable elements of $\mathcal{D}$ contained in $F(U_{n_1}, U_{n_1,n_2})$, and so on. Let $\mathcal{B}$ be a π-base of cardinality κ. For each $B \in \mathcal{B}$ define

$$S_B := \left\{ f \in {}^{\mathbb{N}}\mathbb{N}\colon\ (B \cap U_{f\lceil_{k+1}} = \emptyset) \right\}.$$

Since each S_B is closed and nowhere dense we find, by cardinality considerations, an element f of ${}^{\mathbb{N}}\mathbb{N}$ not in any S_B. Then the play

$$F(\emptyset), U_{f(1)}, F(U_{f(1)}), U_{f(1),f(2)}, F(U_{f(1)}, U_{f(1),f(2)}), \ldots$$

is lost by ONE.

(2) ⇒ (1) Let X be a set of real numbers of cardinality κ. Then $\mathsf{PR}(X)$ has π-weight κ and countable cellularity. Then ONE does not have a winning strategy in $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$. Since $\mathcal{O} \subseteq \mathcal{D}$ this implies that $\mathsf{PR}(X)$ has property $\mathsf{S}_1(\mathcal{O}, \mathcal{D})$, so that by Theorem 5A of [3] X

has property $\mathsf{S}_1(\Omega, \Omega)$. We see that each set of real numbers of cardinality κ has property $\mathsf{S}_1(\Omega, \Omega)$. Theorem 4.8 of [8] implies that κ is less than $\mathsf{cov}(\mathcal{M})$. $\square$

One can also show that each of the clauses of this theorem is equivalent to the statement that for each T_1-space of countable cellularity and π-weight κ, ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

We are now in a position to answer some questions from [4]:

(A) [4, p. 207] The authors ask if it is true for X a set of real numbers that if ONE has a winning strategy in the game $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$ on $\mathsf{PR}(X)$, then ONE has a winning strategy in $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ on X. The answer is No. For ONE to have a winning strategy in $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$ on $\mathsf{PR}(X)$ is equivalent to X not having property $\mathsf{S}_1(\Omega, \Omega)$ (Corollary 24); For ONE to have a winning strategy in $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ on X is equivalent to X not having property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ [9, Lemma 2]. The Continuum Hypothesis implies that there is a set X of reals which has property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$, but does not have property $\mathsf{S}_1(\Omega, \Omega)$ [8, Theorem 2.8].

(B) [4, p. 214, Question 2.4] On p. 213 the authors prove the implication (1) $\Rightarrow$ (2) of Theorem 25. Question 2.4 (not as stated, but as intended) asks if the converse implication is true. As shown in Theorem 25, the answer is Yes.

(C) [4, p. 220, Question 4.3] The authors ask if a player has a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$ if, and only if, the same player has a winning strategy in the game $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$. The answer is No. This can be seen as follows: It is well known that $\mathsf{cov}(\mathcal{M}) \leqslant \mathfrak{d}$, and that it is consistent that inequality between these two cardinal numbers is strict (for the latter, see for example [5, Theorem 3.8]). Let X be a set of real numbers of minimal cardinality which does not have property $\mathsf{S}_1(\Omega, \Omega)$. Then ONE has a winning strategy in the game $\mathsf{G}_1(\mathcal{D}, \mathcal{D})$ on $\mathsf{PR}(X)$ (by Corollary 24). However, if we have $\mathsf{cov}(\mathcal{M}) < \mathfrak{d}$, then ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{D}, \mathcal{D})$ on $\mathsf{PR}(X)$ (Corollary 11 and [8, Theorem 4.6]).

5. Closing remarks

In parts I–III additional properties, all motivated by analogous properties that have been studied for ultrafilters on $\mathbb{N}$, were considered in connection with these selection hypotheses. Here is a partial list of these:

- $\mathsf{K}(\mathcal{A}, \mathcal{B})$: For every first-countable compact T_2-space Z, for each $A \in \mathcal{A}$ and for every $f: A \to Z$ such that for some $a \in Z$ and for each neighborhood U of a $\{x \in A:\ f(x) \in U\}$ is in $\mathcal{A}$, there is a $B \subseteq A$ such that $B \in \mathcal{B}$ and a is the unique limit point of $\{f(x):\ x \in B\}$.
- $\mathsf{P}(\mathcal{A}, \mathcal{B})$: For every descending sequence $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ in $\mathcal{A}$, there is a $B \in \mathcal{B}$ such that for each n, $B \setminus A$ is finite.
- $\mathsf{Q}(\mathcal{A}, \mathcal{B})$: For every countable $A \in \mathcal{A}$, for each partition of A into pairwise disjoint finite sets, there is a $B \in \mathcal{B}$ which meets each element of the partition in at most one point.

- $\mathsf{B}_{linear}(\mathcal{A},\mathcal{B})$: For each $A\in\mathcal{A}$, for each linear ordering R of A, there is a $B\in\mathcal{B}$ such that $B\subseteq A$ and the order type of B relative to R is ω or ω^*.
- $\mathsf{B}_{tree}(\mathcal{A},\mathcal{B})$: For each $A\in\mathcal{A}$ and for each tree ordering R of A, there is a $B\in\mathcal{B}$ such that $B\subseteq A$ and B is a chain, or an antichain, in the tree on A.
- $\mathsf{Ind}_{fin}(\mathcal{A},\mathcal{B})$: For every descending sequence $A_1\supseteq A_2\supseteq\cdots\supseteq A_n\supseteq\cdots$ of countable sets in $\mathcal{A}$ and for every bijective enumeration $(a_n\colon n\in\mathbb{N})$ of A_1, there is a function $H:\mathbb{N}\to[\mathbb{N}]^{<\aleph_0}$ such that:
  (1) if $m<n$, then $\sup(H(m))<\sup(H(n))$ and $|H(m)|<|H(n)|$;
  (2) $\bigcup_{n=1}^{\infty}\{a_j\colon j\in H(n)\}\in\mathcal{B}$; and
  (3) for each n, $\{a_j\colon j\in H(n+1)\}\subset A_{\sup(H(n))}$.
- $\mathsf{Ind}_1(\mathcal{A},\mathcal{B})$: For every descending sequence $A_1\supseteq A_2\supseteq\cdots\supseteq A_n\supseteq\cdots$ of countable sets in $\mathcal{A}$ and for every bijective enumeration $(a_n\colon n\in\mathbb{N})$ of A_1, there is a function $g:\mathbb{N}\to\mathbb{N}$ such that:
  (1) if $m<n$, then $g(m)<g(n)$;
  (2) $\{a_{g(n)}\colon n\in\mathbb{N}\}\in\mathcal{B}$; and
  (3) for each n, $a_{g(n+1)}\in A_{g(n)}$.
- $\mathsf{C}_{fin}(\mathcal{A},\mathcal{B})$: For each $A\in\mathcal{A}$ and for each $f:A\to\mathbb{N}$ there is a $B\in\mathcal{B}$ such that $B\subseteq A$, and on B f is finite-to-one, or constant.
- $\mathsf{C}_1(\mathcal{A},\mathcal{B})$: For each $A\in\mathcal{A}$ and for each $f:A\to\mathbb{N}$ there is a $B\in\mathcal{B}$ such that $B\subseteq A$, and on B f is one-to-one, or constant.
- $\mathsf{BT}_{fin}(\mathcal{A},\mathcal{B})$: For each $A\in\mathcal{A}$, for each partition $A=\bigcup_{n=1}^{\infty}A_n$ of A into disjoint sets, none in $\mathcal{A}$, there is a $B\in\mathcal{B}$ such that $B\subseteq\mathcal{B}$ and for each n $B\cap A_n$ is finite.
- $\mathsf{BT}_1(\mathcal{A},\mathcal{B})$: For each $A\in\mathcal{A}$, for each partition $A=\bigcup_{n=1}^{\infty}A_n$ of A into disjoint sets, none in $\mathcal{A}$, there is a $B\in\mathcal{B}$ such that $B\subseteq\mathcal{B}$ and for each n $B\cap A_n$ has at most one element.

Theorem 26. *If X is a set of real numbers, then for $\mathsf{PR}(X)$ the following statements are equivalent*:
(a) $\mathsf{S}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$;
(b) *ONE has no winning strategy in* $\mathsf{G}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$;
(c) $\mathsf{Ind}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$;
(d) $\mathsf{K}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$;
(e) $\mathsf{P}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$;
(f) *for each k, $\mathcal{D}_\Omega\to\lceil\mathcal{D}\rceil_k^2$*;
(g) $\mathsf{B}_{linear}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$;
(h) $\mathsf{C}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$;
(i) $\mathsf{BT}_{fin}(\mathcal{D}_\Omega,\mathcal{D}_\Omega)$.

For arbitrary spaces of countable cellularity each of the statements in Theorem 26 implies the next. Using the combinatorial structure of Pixley–Roy spaces one can prove that i implies a, as follows: First, we may assume that X is uncountable. Let $(\mathcal{U}_n\colon n\in\mathbb{N})$ be a sequence from $\mathcal{D}_\Omega$. We may assume that each element of each $\mathcal{U}_n$ is a finite union of sets of the form $[S,U]$, and that each $\mathcal{U}_n$ is countable. Enumerate $\mathcal{U}_n$ bijectively as

$([S^n_{m,1}, U^n_{m,1}] \cup \cdots \cup [S^n_{m,k^n_m}, U^n_{m,k^n_m}]$: $m \in \mathbb{N})$, and pick $x_n \in X \setminus \bigcup_{m=1}^{\infty}(S^n_{m,1} \cup \cdots \cup S^n_{m,k^n_m})$. Then let $\mathcal{U}^*_n$ consist of the sets $[S^n_{m,1}, U^n_{m,1} \setminus \{x_n\}] \cup \cdots \cup [S^n_{m,k^n_m}, U^n_{m,k^n_m} \setminus \{x_n\}]$, $m \in \mathbb{N}$.

No $\mathcal{U}^*_n$ is in $\mathcal{D}$, but $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{U}^*_n$ is in $\mathcal{D}_\Omega$. Apply $\mathsf{BT}_{fin}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$ to find for each n a finite set $\mathcal{F}_n \subseteq \mathcal{U}^*_n$ such that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is in $\mathcal{D}_\Omega$. By restoring elements of $\mathcal{F}_n$ to being elements of $\mathcal{U}_n$ we then find for each n a finite set $\mathcal{G}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is in $\mathcal{D}_\Omega$.

Similar remarks apply to the next theorem, and a similar argument shows that h implies a.

Theorem 27. *If X is a set of real numbers, then for* $\mathsf{PR}(X)$ *the following statements are equivalent*:

(a) $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$;
(b) *ONE has no winning strategy in* $\mathsf{G}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$;
(c) $\mathsf{Ind}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$;
(d) $\mathsf{P}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$ *and* $\mathsf{Q}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$;
(e) *for all n and k*, $\mathcal{D}_\Omega \to (\mathcal{D}_\Omega)^n_k$;
(f) $\mathsf{B}_{tree}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$;
(g) $\mathsf{C}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$;
(h) $\mathsf{BT}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.

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