Applied Mathematics Letters 25 (2012) 692–697

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Common fixed point theorems for the stronger Meir–Keeler cone-type function in cone ball-metric spaces

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ARTICLE INFO

Article history:
Received 15 June 2010
Received in revised form 17 January 2011
Accepted 26 September 2011

Keywords:
Cone ball-metric space
Common fixed point theorem
Stronger Meir–Keeler cone-type function

ABSTRACT

In this work, we introduce the concept of cone ball-metric spaces and we prove fixed point results on such spaces for mappings satisfying a contraction involving a stronger Meir–Keeler cone-type function.

1. Introduction and preliminaries

In 1997, Zabrejko [1] introduced the K-metric and K-normed linear spaces and showed the existence and uniqueness of fixed points for operators which act in K-metric or K-normed linear spaces. Later, Huang and Zhang [2] introduced the concept of a cone metric space by replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive-type mappings on cone metric spaces. The category of cone metric spaces is larger than that of metric spaces. Subsequently, many authors like Abbas and Jungck [3], and Ilić and Radenović [4] generalized the results of Huang and Zhang [2] and studied the existence of common fixed points of a pair of self-mappings satisfying a contractive-type condition in the framework of normal cone metric spaces. However, authors such as Janković et al. [5], Rezapour and Hamlbarani [6] studied the existence of common fixed points of pairs self-mappings and non-self-mappings satisfying a contractive-type condition in the situation in which the cone does not need be normal. Many authors studied this subject and many results on fixed point theory are proved [7–11].

We recall some definitions of the cone metric spaces and some of the properties [2], as follows:

Definition 1 (*[2]*). Let E be a real Banach space endowed with a norm $\|\cdot\|$ and P a subset of E. P is called a cone if and only if:

(i) P is nonempty, closed, and $p \neq \{0_E\}$, where 0_E is the zero vector of E,
(ii) $a, b \in \Re$, $a, b \geq 0_E$, $x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0_E$.

Given a cone $P \subset E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$ for all $x, y \in E$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

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doi:10.1016/j.aml.2011.09.047

The cone P is called normal if there exists a real number $\kappa > 0$ such that for all $x, y \in E$,

$$0_E \leq x \leq y \Rightarrow \|x\| \leq \kappa \|y\|.$$

The least positive number κ satisfying the above is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 2 ([2]). Let X be a nonempty set, and let E be a real Banach space endowed with a cone P in E with $\operatorname{int} P \neq \phi$ and $\leq$ be a partial ordering with respect to P. Suppose the mapping $d : X \times X \to E$ satisfies:

(i) $0_E < d(x, y)$ for all $x, y \in X, x \neq y$;
(ii) $d(x, y) = 0_E$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$;
(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Metric spaces play an important role in mathematics and the applied sciences. In 2003, Mustafa and Sims [12] introduced a more appropriate and robust notion of a generalized metric space as follows.

Definition 3 ([12]). Let X be a nonempty set, and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if and only if $x = y = z$;
(G2) $G(x, x, y) > 0$ for all $x \neq y$;
(G3) $G(x, y, z) \geq G(x, x, y)$ for all $x, y, z \in X$;
(G4) $G(x, y, z) = G(x, z, y) = G(z, y, x) = \cdots$ (symmetric in all three variables);
(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

This research subject is interesting and broad. But is so abstract that people find it hard to understand. So we introduce the concept of cone ball-metric spaces and we prove fixed point results on such spaces for mappings satisfying a contraction involving a stronger Meir–Keeler cone-type function.

In the following we always suppose that E is a real Banach space endowed with a cone P with apex at the origin 0_E, $\operatorname{int} P \neq \phi$ and a linear ordering $\leq$ with respect to P. We now introduce the concept of the cone ball-metric $\mathcal{B}$. Let (X, d) be a cone metric space, and $x, y, z \in X$. We define

$$B_\gamma(x) = B(x, \gamma) = \{y \in X : d(x, y) < \gamma\} \quad \text{for } x \in X;$$

this is a ball in X with the center x and the radius $\gamma \gg 0_E$, and we define the function $\mathcal{B} : X \times X \times X \to E$ by

$$\mathcal{B}(x, y, z) = \inf\{2\gamma : B_\gamma \text{ is a ball in } X, \text{ and } \{x, y, z\} \subset B_\gamma\},$$

where γ is the radius of the ball B_γ. Then we call $\mathcal{B}$ a cone ball-metric with respect to the cone metric d, and $(X, \mathcal{B})$ a cone ball-metric space. Moreover, we also define $\mathcal{B}(x, x, y) = d(x, y)$.

Further, the cone ball-metric $\mathcal{B}$ has the following properties:

($\mathcal{B}$1) $\mathcal{B}(x, y, z) = 0_E$ if and only if $x = y = z$;
($\mathcal{B}$2) $\mathcal{B}(x, x, y) > 0_E$ for all $x \neq y$;
($\mathcal{B}$3) $\mathcal{B}(x, y, z) \geq \mathcal{B}(x, x, y)$ for all $x, y, z \in X$;
($\mathcal{B}$4) $\mathcal{B}(x, y, z) = \mathcal{B}(x, z, y) = \mathcal{B}(z, y, x) = \cdots$ (symmetric in all three variables);
($\mathcal{B}$5) $\mathcal{B}(x, y, z) \leq \mathcal{B}(x, w, w) + \mathcal{B}(w, y, z)$ for all $x, y, z, w \in X$;
($\mathcal{B}$6) $\mathcal{B}(x, x, y) = \mathcal{B}(x, y, y)$ for all $x, y \in X$.

Definition 4. Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is:

(a) A Cauchy sequence if for every $\varepsilon \in E$ with $0_E \ll \varepsilon$, there exists $n_0 \in \mathcal{N}$ such that for all $n, m, l > n_0$, $\mathcal{B}(x_n, x_m, x_l) \ll \varepsilon$.
(b) A convergent sequence if for every $\varepsilon \in E$ with $0_E \ll \varepsilon$, there exists $n_0 \in \mathcal{N}$ such that for all $n, m > n_0$, $\mathcal{B}(x_n, x_m, x) \ll \varepsilon$ for some $x \in X$. Here x is called the limit of the sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 5. Let $(X, \mathcal{B})$ be a cone ball-metric space. Then X is said to be complete if every Cauchy sequence is convergent in X.

Proposition 1. *Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}$ be a sequence in X. Then the following are equivalent:*

(i) $\{x_n\}$ *converges to* x;
(ii) $\mathcal{B}(x_n, x_n, x) \to 0_E$ *as* $n \to \infty$;
(iii) $\mathcal{B}(x_n, x, x) \to 0_E$ *as* $n \to \infty$;
(iv) $\mathcal{B}(x_n, x_m, x) \to 0_E$ *as* $n, m \to \infty$.

Proposition 2. *Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}$ be a sequence in X, $x, y \in X$. If $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$.*

Proof. Let $\varepsilon \in E$ with $0_E \ll \varepsilon$ be given. Since $x_n \to x$ and $x_n \to y$ as $n \to \infty$, there exists $n_0 \in \mathcal{N}$ such that for all $m, n > n_0$,

$$\mathcal{B}(x_n, x_m, x) \ll \frac{\varepsilon}{3} \quad \text{and} \quad \mathcal{B}(x_n, x_m, y) \ll \frac{\varepsilon}{3}.$$

Therefore,

$$\begin{aligned} \mathcal{B}(x, x, y) &\leq \mathcal{B}(x, x_n, x_n) + \mathcal{B}(x_n, x, y) \\ &= \mathcal{B}(x, x_n, x_n) + \mathcal{B}(y, x_n, x) \\ &\leq \mathcal{B}(x, x_n, x_n) + \mathcal{B}(y, x_m, x_m) + \mathcal{B}(x_m, x_n, x) \\ &\ll \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, $\mathcal{B}(x, x, y) \ll \frac{\varepsilon}{\alpha}$ for all $\alpha \geq 1$, and so $\frac{\varepsilon}{\alpha} - \mathcal{B}(x, x, y) \in P$ for all $\alpha \geq 1$. Since $\frac{\varepsilon}{\alpha} \to 0_E$ as $\alpha \to \infty$ and P is closed, we have that $-\mathcal{B}(x, x, y) \in P$. This implies that $\mathcal{B}(x, x, y) = 0_E$, since $\mathcal{B}(x, x, y) \in P$. So $x = y$. □

Proposition 3. *Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}, \{y_m\}, \{z_l\}$ be three sequences in X. If $x_n \to x$, $y_m \to y$, $z_l \to z$ as $n \to \infty$, then $\mathcal{B}(x_n, y_m, z_l) \to \mathcal{B}(x, y, z)$ as $n \to \infty$.*

Proof. Let $\varepsilon \in E$ with $0_E \ll \varepsilon$ be given. Since $x_n \to x$, $y_m \to y$, $z_l \to z$ as $n \to \infty$, there exists $n_0 \in \mathcal{N}$ such that for all $n, m, l > n_0$,

$$\mathcal{B}(x_n, x, x) \ll \frac{\varepsilon}{3}, \qquad \mathcal{B}(y_m, y, y) \ll \frac{\varepsilon}{3}, \qquad \mathcal{B}(z_l, z, z) \ll \frac{\varepsilon}{3}.$$

Therefore,

$$\begin{aligned} \mathcal{B}(x_n, y_m, z_l) &\leq \mathcal{B}(x_n, x, x) + \mathcal{B}(x, y_m, z_l) \\ &\leq \mathcal{B}(x_n, x, x) + \mathcal{B}(y_m, y, y) + \mathcal{B}(y, x, z_l) \\ &\leq \mathcal{B}(x_n, x, x) + \mathcal{B}(y_m, y, y) + \mathcal{B}(z_l, z, z) + \mathcal{B}(z, x, y) \\ &\ll \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \mathcal{B}(x, y, z), \end{aligned}$$

that is,

$$\mathcal{B}(x_n, y_m, z_l) - \mathcal{B}(x, y, z) \ll \varepsilon.$$

Similarly,

$$\mathcal{B}(x, y, z) - \mathcal{B}(x_n, y_m, z_l) \ll \varepsilon.$$

Therefore, for all $\alpha \geq 1$, we have

$$\mathcal{B}(x_n, y_m, z_l) - \mathcal{B}(x, y, z) \ll \frac{\varepsilon}{\alpha},$$

and

$$\mathcal{B}(x, y, z) - \mathcal{B}(x_n, y_m, z_l) \ll \frac{\varepsilon}{\alpha}.$$

These imply that

$$\begin{aligned} &\frac{\varepsilon}{\alpha} - \mathcal{B}(x_n, y_m, z_l) + \mathcal{B}(x, y, z) \in P, \\ &\frac{\varepsilon}{\alpha} + \mathcal{B}(x_n, y_m, z_l) - \mathcal{B}(x, y, z) \in P. \end{aligned}$$

Since P is closed and $\frac{\varepsilon}{\alpha} \to 0_E$ as $\alpha \to \infty$, we have that

$$\lim_{n,m,l\to\infty}[-\mathcal{B}(x_n, y_m, z_l) + \mathcal{B}(x, y, z)] \in P,$$
$$\lim_{n,m,l\to\infty}[\mathcal{B}(x_n, y_m, z_l) - \mathcal{B}(x, y, z)] \in P.$$

These results show that

$$\lim_{n,m,l\to\infty} \mathcal{B}(x_n, y_m, z_l) = \mathcal{B}(x, y, z).$$

So we complete the proof. □

2. The main results

In this section, we introduce the stronger Meir–Keeler cone-type function $\psi : \operatorname{int} P \cup \{0\} \to [0, 1)$ in cone ball-metric spaces, and prove the fixed point results on such spaces for mappings satisfying a contraction involving a stronger Meir–Keeler cone-type function.

Definition 6. Let $(X, \mathcal{B})$ be a cone ball-metric space with a regular cone P, and let

$$\psi : \operatorname{int} P \cup \{0_E\} \to [0, 1).$$

Then the function ψ is called a stronger Meir–Keeler-type function if for each $\eta \in P$ with $\eta \gg 0_E$, there exists $\delta \gg 0_E$ such that for $x, y, z \in X$ with $\eta \le \mathcal{B}(x, y, z) \ll \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that $\psi(\mathcal{B}(x, y, z)) < \gamma_\eta$.

Let $(X, \mathcal{B})$ be a cone ball-metric space and $T, F : X \to X$ be two single-valued mappings. The point ν is called a coincidence point of T and F if $\nu = T\mu = F\mu$ for some $\mu \in X$. Maps T and F are said to be weakly compatible if they commute at coincidence points. That is, $F\nu = FT\mu = TF\mu = T\nu$.

Theorem 1. *Let $(X, \mathcal{B})$ be a cone ball-metric space with a regular cone P and $T, F, S, f : X \to X$ be four single-valued mappings. Suppose that there exists a stronger Meir–Keeler-type function $\psi : \operatorname{int} P \cup \{0_E\} \to [0, 1)$ such that:*

(1) $\mathcal{B}(Tx, Fy, Sz) \le \psi(\mathcal{B}(fx, fy, fz)) \cdot \mathcal{B}(fx, fy, fz)$ *for all* $x, y, z \in X$.

If

$$TX \cup FX \cup SX \subset fX,$$

and fX is a complete subspace of X, then S, T, F and f have a unique point of coincidence.

Moreover, if (T, f), (F, f) and (S, f) are weakly compatible, then T, F, S and f have a unique common fixed point ν in X.

Proof. Given $x_0 \in X$, define the sequence $\{fx_n\}$ recursively as follows:

$$fx_{3n+1} = Tx_{3n}, \qquad fx_{3n+2} = Fx_{3n+1}, \qquad fx_{3n+3} = Sx_{3n+2}.$$

Then for each $n \in \mathcal{N}$, we have

$$\begin{aligned}\mathcal{B}(fx_{3n+1}, fx_{3n+2}, fx_{3n+3}) &= \mathcal{B}(Tx_{3n}, Fx_{3n+1}, Sx_{3n+2}) \\ &\le \psi(\mathcal{B}(fx_{3n}, fx_{3n+1}, fx_{3n+2})) \cdot \mathcal{B}(fx_{3n}, fx_{3n+1}, fx_{3n+2}) \\ &\ll \mathcal{B}(fx_{3n}, fx_{3n+1}, fx_{3n+2}).\end{aligned}$$

Hence the sequence $\{\mathcal{B}(fx_n, fx_{n+1}, fx_{n+2})\}$ is decreasing and bounded below. Let $\lim_{n\to\infty} \mathcal{B}(fx_n, fx_{n+1}, fx_{n+2}) = \eta \ge 0_E$. Then there exists $\kappa_0 \in \mathcal{N}$ and $\delta \gg 0_E$ such that for all $n > \kappa_0$,

$$\eta \le \mathcal{B}(fx_n, fx_{n+1}, fx_{n+2}) \ll \eta + \delta.$$

For each $n \in \mathcal{N}$, since $\psi : \operatorname{int} P \cup \{0_E\} \to [0, 1)$ is a stronger Meir–Keeler-type mapping, for these $\eta \gg 0$ and $\delta \gg 0$ we have that for $fx_{\kappa_0+n}, fx_{\kappa_0+n+1}, fx_{\kappa_0+n+2} \in X$ with

$$\eta \le \mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+n+1}, fx_{\kappa_0+n+2}) \ll \delta + \eta,$$

there exists $\gamma_\eta \in [0, 1)$ such that

$$\psi(\mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+n+1}, fx_{\kappa_0+n+2})) \ll \gamma_\eta.$$

Thus, by (1), we can deduce

$$\mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+n+1}, fx_{\kappa_0+n+2}) \ll \gamma_\eta \cdot \mathcal{B}(fx_{\kappa_0+n-1}, fx_{\kappa_0+n}, fx_{\kappa_0+n+1}),$$

and it follows that for each $n \in \mathcal{N}$,

$$\begin{aligned}\mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+n+1}, fx_{\kappa_0+n+1}) &\le \mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+n+1}, fx_{\kappa_0+n+2})\\ &\ll \gamma_\eta \cdot \mathcal{B}(fx_{\kappa_0+n-1}, fx_{\kappa_0+n}, fx_{\kappa_0+n+1})\\ &\ll \cdots\\ &\ll \gamma_\eta^n \cdot \mathcal{B}(fx_{\kappa_0}, fx_{\kappa_0+1}, fx_{\kappa_0+2}).\end{aligned}$$

So,

$$\lim_{n\to\infty} \mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+n+1}, fx_{\kappa_0+n+1}) = 0_E, \quad \text{since } \gamma_\eta < 1.$$

We next claim that $\lim_{m,n\to\infty} \mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+m}, fx_{\kappa_0+m}) = 0_E$. For $m, n \in \mathcal{N}$ with $m > n$, we have

$$\begin{aligned}\mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+m}, fx_{\kappa_0+m}) &\le \sum_{i=n}^{m-1} \mathcal{B}(fx_{\kappa_0+i}, fx_{\kappa_0+i+1}, fx_{\kappa_0+i+1})\\ &\ll \frac{\gamma_\eta^{m-1}}{1-\gamma_\eta} \mathcal{B}(fx_{\kappa_0+1}, fx_{\kappa_0+2}, fx_{\kappa_0+2}),\end{aligned}$$

and hence $\mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+m}, fx_{\kappa_0+m}) \to 0_E$ as $m, n \to \infty$, since $0 < \gamma_\eta < 1$.

By the property ($\mathcal{B}5$) of the cone ball-metric, we obtain

$$\mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+m}, fx_{\kappa_0+l}) \le \mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+m}, fx_{\kappa_0+m}) + \mathcal{B}(fx_{\kappa_0+m}, fx_{\kappa_0+m}, fx_{\kappa_0+l}),$$

and taking the limit as $m, n, l \to \infty$, we get $\mathcal{B}(fx_{\kappa_0+n}, fx_{\kappa_0+m}, fx_{\kappa_0+l}) \to 0_E$. So $\{fx_n\}$ is a Cauchy sequence. Since fX is a complete subspace of X, there exists $\mu \in X$ such that $\lim_{n\to\infty} fx_n = \mu$, that is, $\mathcal{B}(fx_n, fx_n, \mu) \to 0_E$ as $n \to \infty$.

Since fX is a complete subspace of X, there exists $\nu, \mu \in X$ such that $\lim_{n\to\infty} fx_n = \nu$ and $f\mu = \nu$. So,

$$\begin{aligned}&\mathcal{B}(fx_{3n}, fx_{3n}, \nu) \to 0_E, \qquad \mathcal{B}(Tx_{3n}, Tx_{3n}, \nu) \to 0_E,\\ &\mathcal{B}(Fx_{3n+1}, Fx_{3n+1}, \nu) \to 0_E, \qquad \mathcal{B}(Sx_{3n+2}, Sx_{3n+2}, \nu) \to 0_E.\end{aligned}$$

Thus we can choose $\epsilon \gg 0$ and $n \in \mathcal{N}$ such that

$$\begin{aligned}&\mathcal{B}(fx_{3n}, fx_{3n}, \nu) \ll \frac{\epsilon}{6};\\ &\mathcal{B}(Tx_{3n}, Tx_{3n}, \nu) \ll \frac{\epsilon}{6};\\ &\mathcal{B}(Fx_{3n+1}, Fx_{3n+1}, \nu) \ll \frac{\epsilon}{6};\\ &\mathcal{B}(Sx_{3n+2}, Sx_{3n+2}, \nu) \ll \frac{\epsilon}{6},\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{B}(f\mu, f\mu, T\mu) &\le \mathcal{B}(f\mu, f\mu, fx_{3n}) + \mathcal{B}(fx_{3n}, fx_{3n}, T\mu)\\ &= \mathcal{B}(f\mu, f\mu, fx_{3n}) + \mathcal{B}(fx_{3n}, T\mu, fx_{3n-1}) + \mathcal{B}(fx_{3n-1}, fx_{3n-1}, fx_{3n})\\ &= \mathcal{B}(f\mu, f\mu, fx_{3n}) + \mathcal{B}(Sx_{3n-1}, T\mu, Fx_{3n-2}) + \mathcal{B}(fx_{3n-1}, fx_{3n-1}, \nu) + \mathcal{B}(\nu, \nu, fx_{3n})\\ &\le \mathcal{B}(f\mu, f\mu, fx_{3n}) + \psi(\mathcal{B}(f\mu, fx_{3n-2}, fx_{3n-1})) \cdot \mathcal{B}(f\mu, fx_{3n-2}, fx_{3n-1})\\ &\quad + \mathcal{B}(fx_{3n-1}, fx_{3n-1}, \nu) + \mathcal{B}(\nu, \nu, fx_{3n})\\ &\le \mathcal{B}(f\mu, f\mu, fx_{3n}) + \gamma_\eta \cdot [\mathcal{B}(fx_{3n-2}, f\mu, f\mu) + \mathcal{B}(f\mu, f\mu, fx_{3n-1})]\\ &\quad + \mathcal{B}(fx_{3n-1}, fx_{3n-1}, \nu) + \mathcal{B}(\nu, \nu, fx_{3n})\\ &\le \frac{\epsilon}{6} + \gamma_\eta \cdot \left[\frac{\epsilon}{6} + \frac{\epsilon}{6}\right] + \frac{\epsilon}{6} + \frac{\epsilon}{6} \ll \epsilon.\end{aligned}$$

Therefore, $\mathcal{B}(f\mu, f\mu, T\mu) = 0_E$, that is, $f\mu = \nu = T\mu$. Similarly, by the same process, we can deduce that $f\mu = \nu = F\mu$ and $f\mu = \nu = S\mu$. So ν is a point of coincidence of T, F, F and f, that is,

$$\nu = f\mu = T\mu = F\mu = S\mu.$$

Now we show that T, F, S and f have a unique point of coincidence. Let ν^* be another coincidence point of S, T and f, that is,

$$\nu^* = f\mu^* = T\mu^* = F\mu^* = S\mu^* \quad \text{for some } \mu^* \in X.$$

Then

$$\begin{aligned}\mathcal{B}(\nu,\nu,\nu^*) &= \mathcal{B}(T\mu,F\mu,S\mu^*)\\ &\le \psi(\mathcal{B}(f\mu,f\mu,f\mu^*))\cdot\mathcal{B}(f\mu,f\mu,f\mu^*)\\ &= \psi(\mathcal{B}(\nu,\nu,\nu^*))\cdot\mathcal{B}(\nu,\nu,\nu^*)\\ &\ll \gamma_\eta\mathcal{B}(\nu,\nu,\nu^*),\end{aligned}$$

which implies $\nu=\nu^*$. Hence ν is the unique coincidence point of S,T and f.

By the weak compatibility of (T,f), (F,f), and (S,f), we have

$$\begin{aligned}&T\nu = Tf\mu = fT\mu = f\nu;\\ &F\nu = Ff\mu = fF\mu = f\nu;\\ &S\nu = Sf\mu = fS\mu = f\nu.\end{aligned}$$

Hence there exists $w\in X$ such that $w=T\nu=F\nu=S\nu=f\nu$ and w is a point of coincidence of f,S,F and T. Therefore, by the uniqueness of the point of coincidence, we have $\nu=w$. Thus, ν is a unique common fixed point of f,S,F and T. □

Bari and Vetro [13] defined a pair of ϕ-maps and studied common fixed points in cone metric spaces, while Shatanawi [14] studied several fixed point theorems for contractive mappings satisfying ϕ-maps in G-metric spaces. Applying Theorem 1, we immediate get the following corollary.

Corollary 1. *Let (X,d) be a cone rectangular metric space with regular cone P such that $d(x,y)\in \operatorname{int}P$ for all $x,y\in X$ with $x\neq y$. Let $\phi:\operatorname{int}P\cup\{0_E\}\to\operatorname{int}P\cup\{0_E\}$ be a ϕ-mapping and let $\xi:\operatorname{int}P\cup\{0_E\}\to[0,1)$ be a stronger Meir–Keeler-type function. Suppose that $S,T,f:X\to X$ are three single-valued functions such that for all $x,y\in X$,*

$$\phi(d(Sx,Ty))\le\xi(\phi(d(fx,fy)))\cdot\phi(d(fx,fy)).$$

If

$$SX\cup TX\subset fX,$$

and fX is a complete subspace of X, then S,T and f have a unique point of coincidence.

Moreover, if (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point z in X.

Acknowledgments

The authors thank the referees for useful comments and suggestions for the improvement of the work. First author's research was supported by the National Science Council of the Republic of China.

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