The mean value theorem of Flett and divided differences

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Received 18 May 2003

Abstract

We obtain a new generalization of the Flett theorem and several new mean value theorems. We give condensed representations of the Flett and generalized Flett theorems in terms of divided differences. © 2004 Elsevier Inc. All rights reserved.

Keywords: Mean value theorems; Divided differences

1. Introduction and preliminary results

In [3] T.M. Flett gave a variation of the Lagrange mean value theorem:

**Theorem A (T.M. Flett).** Let \( f : [a, b] \rightarrow \mathbb{R} \) be differentiable on \([a, b]\) and \( f'(a) = f'(b) \). Then there exists a point \( c \in (a, b) \) such that

\[
 f(c) - f(a) = f'(c)(c - a).
\]

In recent years there has been renewed interest in Flett’s mean value theorem. We are particularly interested in two recent developments.
T. Riedel and P.K. Sahoo [14] removed the boundary assumption on the derivatives:

**Theorem B** (T. Riedel and P.K. Sahoo). Let \( f : [a, b] \to \mathbb{R} \) be differentiable on \([a, b]\). Then there exists a point \( c \in (a, b) \) such that

\[
f(c) - f(a) = f'(c)(c - a) - \frac{1}{2} \frac{f'(b) - f'(a)}{b - a} (c - a)^2.
\]

Among the many other extensions of the Flett theorem we focus on that of Iwona Pawlikovska [8]:

**Theorem C** (I. Pawlikovska). If \( f \) possesses a derivative of order \( n \) on \([a, b]\), then there exists a point \( c \in (a, b) \) such that

\[
f(c) - f(a) = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!} f^{(k)}(c)(c - a)^k + \frac{(-1)^n}{(n + 1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b - a} (c - a)^{n+1}.
\]

We emphasize the fact that Theorem C is an answer to a question raised by Zsolt Páles during the 35th International Symposium on Functional Equations held in Graz, Austria, 1997.

If \( f \) possesses a derivative of order \( n \) at a point point \( c \), we denote by \( T_n(f; c) \) the Taylor polynomial of degree \( n \) associated to \( f \) at \( c \),

\[
T_n(f; c)(x) := \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!} (x - c)^i.
\]

Note that Eq. (1) can be written in the form

\[
\frac{f(a) - T_n(f; c)(a)}{(a - c)^{n+1}} = \frac{1}{(n + 1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b - a}.
\]
We give a generalization of the Flett theorem in terms of divided differences and obtain a new form of Theorem C. To this end we need the following definitions and preliminary results.

Let \( f : [a, b] \to \mathbb{R} \) and \( t_0, \ldots, t_n \) be distinct points in \([a, b]\). We denote by \( L(t_0, \ldots, t_n; f) \) the Lagrange interpolating polynomial associated to \( f \) on the knots \( t_0, \ldots, t_n \).

The divided difference of \( f \) on the knots \( t_0, \ldots, t_n \) is defined to be the coefficient at \( x^n \) of \( L(t_0, \ldots, t_n; f) \) and denoted by \( \left[ t_0, \ldots, t_n; f \right] \). If the knots \( t_0, \ldots, t_n \) are not distinct, then the divided difference is defined by a limit process. Namely,

\[
\left[ t_0, \ldots, t_0, t_{k+1}, \ldots, t_n; f \right] := \lim_{t_1, \ldots, t_k \to t_0} \left[ t_0, t_1, \ldots, t_n; f \right],
\]

provided the limit exists. In particular,

\[
\left[ c, \ldots, c; f \right] := \lim_{t_1, \ldots, t_n \to c} \left[ c, t_1, \ldots, t_n; f \right].
\]

In this text, all points \( t_0, \ldots, t_n \) in the symbol \( \left[ t_0, \ldots, t_n; f \right] \) will be assumed to be pairwise distinct unless specified otherwise.

For repeated knots, we have the following result.

**Proposition 1** (T.J. Stieltjes [10, p. 36]). If \( f : [a, b] \to \mathbb{R} \) is bounded on \([a, b]\) and possesses a derivative of order \( n \) at \( c \in [a, b] \), then

\[
\lim_{x_1, \ldots, x_n \to c} [c, x_1, \ldots, x_n; f] = \frac{f^{(n)}(c)}{n!},
\]

i.e.,

\[
\left[ c, \ldots, c; f \right] = \frac{f^{(n)}(c)}{n!}. \tag{3}
\]

**Remark 2.** We point out that the conditions of Proposition 1 do not guarantee the existence of the limit

\[
\lim_{x_0, \ldots, x_n \to c} [x_0, x_1, \ldots, x_n; f]. \tag{4}
\]

However, we note that limit (4) exists if, during the limit process, the point \( c \) satisfies

\[
\min\{x_0, \ldots, x_n\} \leq c \leq \max\{x_0, \ldots, x_n\},
\]

since \( [x_0, \ldots, x_n; f] \) is a convex combination of \([c, x_0, \ldots, x_{n-1}; f]\) and \([c, x_1, \ldots, x_n; f]\). Indeed, in this case this limit equals \( f^{(n)}(c)/n! \) and thus Proposition 1 still holds.

**Definition 3** ([1, p. 258], see also [2]). Let \( f \) be defined in a neighbourhood of \( x_0 \). If the iterated limit

\[
\lim_{x_n \to x_0} \ldots \lim_{x_2 \to x_0} \lim_{x_1 \to x_0} n! [x_0, x_1, \ldots, x_n; f]
\]

exists (possibly infinite), then this limit is called the **generalized derivative of \( f \) of order \( n \) at \( x_0 \)** and is denoted by \( D_n f (x_0) \).
It should be remarked that if $D_n f(x_0)$ exists finitely, then it is equal to the $n$th Peano derivative $f^{(n)}(x_0)$ and, if $f^{(n)}(x_0)$ exists finitely then it is equal to $D_n f(x_0)$ (see, e.g., [1, §4]).

The next result is a well-known extension of Lagrange’s mean value theorem to the case of divided differences.

**Proposition 4** (Cauchy [10, p. 36]). Let $a \leq x_0 < \cdots < x_n \leq b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and has a derivative of order $n$ on $(a, b)$, then there exists $c \in (a, b)$ such that

$$[x_0, \ldots, x_n; f] = \frac{f^{(n)}(c)}{n!}.$$

**Proposition 5** [7,10–12]. Let $a \leq x_0 < \cdots < x_n \leq b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that in any neighborhood of the point $c$ there exist equidistant points $c_0 < \cdots < c_n$, $c_0 < c < c_n$, with

$$[x_0, \ldots, x_n; f] = [c_0, \ldots, c_n; f].$$

Note that Proposition 5 is a generalization of Proposition 4.

Let $a_0, \ldots, a_m, x_0, \ldots, x_n$ be pairwise distinct points in $[a, b]$. We will need the following well-known identities:

$$[a_0, \ldots, a_m, x_0, \ldots, x_n; (t-a_0) \cdots (t-a_m)f(t)] = [x_0, \ldots, x_n; f],$$

(5)

$$[a_0, \ldots, a_m, x_0, \ldots, x_n; f] = [x_0, \ldots, x_n; [a_0, \ldots, a_m, t; f]],$$

(6)

We present a lemma, which relates the left-hand side of Eq. (2) to a divided difference.

**Lemma 6.** 1 If $f : [a, b] \to \mathbb{R}$ is $n$ times differentiable on $[a, b]$, then for any $c \in (a, b)$,

$$\frac{[a, c, \ldots, c; f]}{n!} = \frac{f(a) - T_n(f, c)(a)}{(a-c)^{n+1}}$$

(7)

$$= \frac{1}{n!} \left( \frac{d}{dt} \right)^n [a, t; f] \bigg|_{t=c}.$$  

(8)

**Proof.** Let $c \in (a, b)$ and $a < x_1 < \cdots < x_n \leq b$, such that $c \neq x_i$, $i = 1, \ldots, n$. The remainder in the Lagrange approximation formula gives

$$f(a) = f(c) + (a-c)[c, x_1; f]$$

$$+ \sum_{k=2}^{n} (a-c)(a-x_1) \cdots (a-x_{k-1})[c, x_1, \ldots, x_k; f]$$

$$+ (a-c)(a-x_1) \cdots (a-x_n)[a, c, x_1, \ldots, x_n; f].$$

1 See also [1, Lemma (4.1)], a similar result which is proved by induction.
For $x_i \to c$, $i = 1, \ldots, n$, by using Proposition 1, we obtain

$$f(a) = \sum_{k=0}^{n} (a-c)^k \frac{f^{(k)}(c)}{k!} + (a-c)^{n+1} \left[ a, c, \ldots, c; f \right]_{n+1},$$

i.e.,

$$\frac{f(a) - T_n(f; c)(a)}{(a-c)^{n+1}} = \left[ a, c, \ldots, c; f \right]_{n+1}. $$

On the other hand, by Eq. (6), we have

$$\left[ a, c, x_1, \ldots, x_n; f \right] = \left[ c, x_1, \ldots, x_n; [a, t; f] \right]_{t=1}.$$

The function $t \mapsto \left[ a, t; f \right]$ is continuous on $[a, b]$ and $n$-times differentiable on $(a, b)$. For $x_i \to c$, $i = 1, \ldots, n$, by Proposition 1, we obtain

$$\left[ a, c, \ldots, c; f \right]_{n+1} = \left[ c, \ldots, c; [a, t; f] \right]_{t=1}.$$

Connected results can be found in [4–6,9,13].

2. Main results

**Theorem 7.** If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and the generalized derivatives $D_n f(a) = D_n f(b)$ exist, then there exists a system of knots $a < x_0 < \cdots < x_n < b$ such that

$$\left[ a, x_0, \ldots, x_n; f \right] = 0.$$ 

**Proof.** Suppose that $\left[ a, x_0, \ldots, x_n; f \right] \neq 0$ on the convex set

$$D = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | a < x_0 < \cdots < x_n < b \right\}.$$ 

Since $f$ is continuous on $[a, b]$ the function $g(t_0, \ldots, t_n) = \left[ a, t_0, \ldots, t_n; f \right]$ is continuous on $D$. It follows that $g$ has a constant sign on $D$. For definiteness, suppose that

$$\left[ a, x_0, \ldots, x_n; f \right] > 0 \quad \text{on } D. \quad (9)$$

This implies

$$\left[ a, x_0, \ldots, x_{n-1}; f \right] < \left[ a, x_1, \ldots, x_n; f \right]$$

$$< \left[ a, x_2, \ldots, x_n, y_1; f \right]$$

$$< \left[ a, x_3, \ldots, x_n, y_1, y_2; f \right]$$

$$\vdots$$

$$< \left[ a, t_1, \ldots, t_n; f \right]$$

provided
\[
\begin{align*}
&\quad a < x_0 < \cdots < x_n < y_1 < \cdots < y_n < z_1 < \cdots < z_n < t_1 < \cdots < t_n = b.
\end{align*}
\]

From
\[
[a, x_0, \ldots, x_{n-1}; f] < [a, y_1, \ldots, y_n; f] < [a, z_1, \ldots, z_n; f] < [a, t_1, \ldots, t_n; f],
\]
for \(x_i \to a\) and \(t_i \to b\), by Proposition 1, we deduce
\[
D_n f(a) < [a, b, \ldots, b; f],
\]
(10)

Eq. (9) implies also
\[
[a, x_0, \ldots, x_{n-1}; f] < [x_0, \ldots, x_{n-1}, b; f],
\]
hence, for \(x_i \to b\) \((i = 0, \ldots, n-1)\), we obtain
\[
[a, b, \ldots, b; f] \leq D_n f(b).
\]
(11)

Equations (10) and (11) imply \(D_n f(a) < D_n f(b)\), and the proof is completed. \(\Box\)

Proposition 5 and Theorem 7 imply

**Theorem 8.** If \(f : [a, b] \to \mathbb{R}\) is continuous on \([a, b]\) and the generalized derivatives \(D_n f(a) = D_n f(b)\) exist, then there exists \(c \in (a, b)\) such that in any neighborhood of the point \(c\) there exist equidistant points \(c_0 < \cdots < c_n, c_0 < c < c_n, \) such that

\[
[a, c_0, \ldots, c_n; f] = 0.
\]

**Proof.** By Proposition 5 and Theorem 7 there exist distinct points \(x_0, \ldots, x_n\) and equidistant points \(c_0, \ldots, c_n\) such that

\[
0 = [a, x_0, \ldots, x_n; f] = [x_0, \ldots, x_n; a, t; f] = [c_0, \ldots, c_n; a, t; f] = [a, c_0, \ldots, c_n; f]. \quad \Box
\]

In the same way, using Proposition 1, we obtain

**Theorem 9.** If \(f : [a, b] \to \mathbb{R}\) is continuous on \([a, b]\) and possesses derivatives of order \(n\) at \(a\) and \(b\) such that \(f^{(n)}(a) = f^{(n)}(b)\), then there exists \(c \in (a, b)\) such that in any neighborhood of the point \(c\) there exist equidistant points \(c_0 < \cdots < c_n, c_0 < c < c_n, \) with

\[
[a, c_0, \ldots, c_n; f] = 0.
\]

In the special case \(n = 1\), we obtain

**Corollary 10.** If \(f : [a, b] \to \mathbb{R}\) is continuous on \([a, b]\) and differentiable at \(a\) and \(b\) with \(f'(a) = f'(b)\), then there exist two distinct points \(a_1, b_1 \in (a, b)\) such that

\[
[a, a_1, b_1; f] = 0 \quad \text{(see Fig. 2)}.
\]
Theorem 11. If $f : [a, b] \to \mathbb{R}$ possesses a derivative of order $n$ on $[a, b]$, then there exists $c \in (a, b)$ such that in any neighborhood of the point $c$ there exist equidistant points $c_0 < \cdots < c_n$, with $c_0 < c < c_n$, with

$$[a, c_0, \ldots, c_n; f] = \frac{1}{(n+1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b - a}.$$ 

Proof. We use the relation

$$[a, c_0, \ldots, c_n; (t - a)^{n+1}] = 1$$

and apply Theorem 9 to the function $h : [a, b] \to \mathbb{R}$,

$$h(t) = f(t) - \frac{1}{(n+1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b - a} (t - a)^{n+1}.$$ 

From Theorem 11, taking $c_i \to c$, $i = 0, \ldots, n$, we get

**Corollary 12** (A new form of Pawlikovska’s theorem). If $f : [a, b] \to \mathbb{R}$ is $n$ times differentiable on $[a, b]$, then there exists $c \in (a, b)$ such that

$$[a, c, \ldots, c; f] = \frac{1}{(n+1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b - a}.$$ 

From Corollary 12, with $n = 1$, we obtain

**Corollary 13** (A new form of Flett’s theorem). If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(a) = f'(b)$, then there exists $c \in (a, b)$ such that

$$[a, c, f] = 0.$$ 

**Corollary 14** (A Cauchy–Flett type theorem). If $f : [a, b] \to \mathbb{R}$ possesses a derivative of order $n$ on $[a, b]$, and $f^{(n)}(a) = f^{(n)}(b)$, then there exists $c \in (a, b)$ such that

$$[a, c, \ldots, c; f] = \frac{f^{(n)}(c)}{n!}$$

(see Fig. 3).
Fig. 3. The Taylor polynomial $T_n(f; c)$ intersects the graph of $f$ at $(a, f(a))$.

**Proof.** By Theorem 9, there exists $c \in (a, b)$ such that in any neighborhood of the point $c$ there exist *equidistant* points $c_0 < \cdots < c_n$, $c_0 < c < c_n$, with

$$[a, c_0, \ldots, c_n; f] = 0.$$  

It follows

$$[a, c_1, \ldots, c_n; f] = [c_0, \ldots, c_n; f] = 0,$$

and hence, for $c_i \to c$, $i = 0, \ldots, n$, by Eq. (3), we get

$$[a, c, \ldots, c; f] = [c, \ldots, c; f] = \frac{f^{(n)}(c)}{n!}.$$  

\[\square\]

**Acknowledgment**

The authors gratefully acknowledge the referee’s helpful remarks on the previous version of the manuscript.

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