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A sufficient condition for a polynomial to be stable

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ABSTRACT

A real polynomial is called Hurwitz (stable) if all its zeros have negative real parts. For a given $n \in \mathbb{N}$ we find the smallest possible constant $d_n > 0$ such that if the coefficients of $F(z) = a_0 + a_1 z + \cdots + a_n z^n$ are positive and satisfy the inequalities $a_k a_{k+1} > d_n a_{k-1} a_{k+2}$ for $k = 1, 2, \ldots, n-2$, then $F(z)$ is Hurwitz.

1. Introduction and statement of results

A real polynomial F is called Hurwitz (stable) if all its zeros have negative real parts, i.e. $F(z_0) = 0 \Rightarrow \operatorname{Re} z_0 < 0$. Polynomial stability problems of various types arise in a number of problems in mathematics and engineering. We refer to [5, Chapter 15] or [11, Chapter 9] for deep surveys on the stability theory.

The following statement (usually attributed to A. Stodola, see, for example, [12]) is the well-known necessary condition for a real polynomial to be stable.

Statement A. *$F(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{R}[z]$, $a_n > 0$, is stable $\Rightarrow a_j > 0$, $0 \leqslant j \leqslant n-1$.*

The following famous theorem gives the necessary and sufficient conditions for a polynomial to be stable.

The Routh–Hurwitz Criterion. *(See, for example, [5, pp. 225–230].) The polynomial $F(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n > 0$, is stable if and only if the first n principal minors of the corresponding Hurwitz matrix*

$$H(F) := \begin{Vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \ldots & 0 \\ a_n & a_{n-2} & a_{n-4} & \ldots & 0 \\ 0 & a_{n-1} & a_{n-3} & \ldots & 0 \\ 0 & a_n & a_{n-2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{Vmatrix}$$

are positive.

Note that the verification of positivity of principal minors is, in general, a very difficult problem. In [3] T. Craven and G. Csordas obtained the following useful and easily verified condition of positivity of all minors of a matrix.

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Theorem A. *Denote by* $\tilde{c}$ *the unique real root of* $x^3 - 5x^2 + 4x - 1 = 0$ ($\tilde{c} \approx 4.0796$). *Let* $M = (a_{ij})$ *be an* $n \times n$ *matrix with the properties*

(a) $a_{ij} > 0$ $(1 \leqslant i, j \leqslant n)$; *and*
(b) $a_{ij}a_{i+1,j+1} \geqslant \tilde{c}a_{i,j+1}a_{i+1,j}$ $(1 \leqslant i, j \leqslant n-1)$.

Then all minors of M are positive.

Using Theorem A and continuity reasonings D.K. Dimitrov and J.M. Peña proved the following theorem.

Theorem B. *(See [4].) Let* $\tilde{c}$ *be defined as in Theorem A. If the coefficients of* $F(z) = a_0 + a_1 z + \cdots + a_n z^n$ *are positive and satisfy the inequalities*

$$a_k a_{k+1} \geqslant \tilde{c} a_{k-1} a_{k+2} \quad \text{for } k = 1, 2, \ldots, n-2,$$

then $F(z)$ *is Hurwitz. In particular, the conclusion is true if*

$$a_k^2 \geqslant \sqrt{\tilde{c}} a_{k-1} a_{k+1} \quad \text{for } k = 1, 2, \ldots, n-1.$$

In [8] the authors of this note have proved that Theorem A remains valid if one replace the constant $\tilde{c}$ by the constant $c_n := 4\cos^2 \frac{\pi}{n+1}$. In [8] it is also shown that in the statement of Theorem A the constant c_n is the smallest possible not only in the class of matrices with positive entries but in the classes of Toeplitz matrices and of Hankel matrices. We recall that a matrix M is Toeplitz matrix if it is of the form $M = (a_{j-i})$ and a matrix M is Hankel matrix if it is of the form $M = (a_{j+i})$. In this paper we will show that the constant c_n is not the smallest possible in the class of Hurwitz matrices.

The natural problem is: for a given $n \in \mathbb{N}$ what is the smallest possible constant d_n such that if the coefficients of $F(z) = a_0 + a_1 z + \cdots + a_n z^n$ are positive and satisfy the inequalities $a_k a_{k+1} > d_n a_{k-1} a_{k+2}$ for $k = 1, 2, \ldots, n-2$, then $F(z)$ is Hurwitz? Our main result is the following theorem which solves this problem.

Theorem 1. *Let* x_0 *be the (unique) positive root of the polynomial* $x^3 - x^2 - 2x - 1$ ($x_0 \approx 2.1479$).

1. *If the coefficients of* $F(z) = \sum_{k=0}^{4} a_k z^k$ *are positive and satisfy the inequalities* $a_k a_{k+1} > 2a_{k-1}a_{k+2}$ *for* $k = 1, 2$, *then* $F(z)$ *is Hurwitz. In particular, the conclusion is true if* $a_k^2 > \sqrt{2} a_{k-1} a_{k+1}$ *for* $k = 1, 2, 3$.
2. *If the coefficients of* $F(z) = \sum_{k=0}^{5} a_k z^k$ *are positive and satisfy the inequalities* $a_k a_{k+1} > x_0 a_{k-1} a_{k+2}$ *for* $k = 1, 2, 3$, *then* $F(z)$ *is Hurwitz. In particular, the conclusion is true if* $a_k^2 > \sqrt{x_0} a_{k-1} a_{k+1}$ *for* $k = 1, 2, 3, 4$.
3. *If the coefficients of* $F(z) = \sum_{k=0}^{n} a_k z^k$, $n > 5$, *are positive and satisfy the inequalities* $a_k a_{k+1} \geqslant x_0 a_{k-1} a_{k+2}$ *for* $k = 1, 2, \ldots, n-2$, *then* $F(z)$ *is Hurwitz. In particular, the conclusion is true if* $a_k^2 \geqslant \sqrt{x_0} a_{k-1} a_{k+1}$ *for* $k = 1, 2, \ldots, n-1$.

Note that

$$\frac{a_k a_{k+1}}{a_{k-1} a_{k+2}} = \frac{a_k^2}{a_{k-1} a_{k+1}} \frac{a_{k+1}^2}{a_k a_{k+2}},$$

and thus the following theorem demonstrates that the constants in Theorem 1 are the smallest possible for every n.

Theorem 2.

1. *For every* $d \leqslant \sqrt{2}$ *there exists a polynomial* $F(z) = \sum_{k=0}^{4} a_k z^k$ *with positive coefficients under condition* $a_k^2 = d a_{k-1} a_{k+1}$ *for* $k = 1, 2, 3$, *such that* $F(z)$ *is not Hurwitz.*
2. *For every* $d \leqslant \sqrt{x_0}$ *there exists a polynomial* $F(z) = \sum_{k=0}^{5} a_k z^k$ *with positive coefficients under condition* $a_k^2 = d a_{k-1} a_{k+1}$ *for* $k = 1, 2, 3, 4$, *such that* $F(z)$ *is not Hurwitz.*
3. *For every* $n > 5$ *and every* $\varepsilon > 0$ *there exists a polynomial* $F(z) = \sum_{k=0}^{n} a_k z^k$ *with positive coefficients under condition* $a_k^2 > (\sqrt{x_0} - \varepsilon) a_{k-1} a_{k+1}$ *for* $k = 1, 2, \ldots, n-1$, *such that* $F(z)$ *is not Hurwitz.*

Theorem 1 may be generalized for entire functions as follows.

Theorem 3. *If the coefficients of* $G(z) = \sum_{k=0}^{\infty} a_k z^k$ *are positive and satisfy the inequalities* $a_k a_{k+1} \geqslant x_0 a_{k-1} a_{k+2}$ *for* $k \in \mathbb{N}$, *then all zeros of* $G(z)$ *have negative real parts. In particular, the conclusion is true if* $a_k^2 \geqslant \sqrt{x_0} a_{k-1} a_{k+1}$ *for* $k \in \mathbb{N}$.

As we will show in the proof of Theorem 2 the constant in Theorem 3 is the smallest possible.

To prove Theorem 1 we use the famous Hermite–Biehler Criterion. The following statement is a version of the Hermite–Biehler theorem.

The Hermite–Biehler Criterion of stability. *(See [2] and [6], or [10, Chapter VII].) Let* $F(z)=\sum_{k=0}^{n} a_k z^k$ *be a polynomial with positive coefficients. The polynomial F is stable if and only if the following two polynomials* $f(z)=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m a_{2m} z^m$ *and* $g(z)=z\sum_{m=0}^{\lfloor (n-1)/2\rfloor}(-1)^m a_{2m+1} z^m$ *have simple real interlacing zeros.*

We will use also the following result by Hutchinson [7, p. 327].

Theorem C. *Let* $f(z)=\sum_{k=0}^{\infty} a_k z^k$, $a_k>0$, $\forall k$. *Inequality* $a_n^2 \geqslant 4a_{n-1}a_{n+1}$, $\forall n \geqslant 1$, *holds if and only if the following two properties hold*:

(i) *the zeros of* $f(z)$ *are all real, simple and negative, and*
(ii) *the zeros of any polynomial* $\sum_{k=m}^{n} a_k z^k$, *formed by taking any number of consecutive terms of* $f(z)$, *are all real and non-positive.*

Theorem C is closely connected with the above mention sufficient condition for a matrix to have positive minors. About the connection between the property of a polynomial to have only real non-positive zeros and the positivity of minors of certain matrix see, for example, [1].

2. Proof of Theorems 1 and 3

It is well known (and easy verified) that polynomials of degree 1 and 2 with positive leading coefficient are stable if and only if all their coefficients are positive.

Let $F(z)=a_0+a_1z+\cdots+a_nz^n$ be a polynomial with positive coefficients. Denote by

$$s_j=\frac{a_j a_{j+1}}{a_{j-1}a_{j+2}},\quad 1\leqslant j\leqslant n-2. \tag{1}$$

Let $F(z)=\sum_{j=0}^{3} a_j z^j$ be a polynomial with positive coefficients. Polynomial F is stable if and only if $s_1>1$ (see, for example, [12, p. 34]). Really, for $F(z)=\sum_{j=0}^{3} a_j z^j$ we have $f(z)=a_0-a_2z$ and $g(z)=z(a_1-a_3z)$. Both polynomials have simple real zeros, and these zeros interlace if and only if $0<\frac{a_0}{a_2}<\frac{a_1}{a_3}\Leftrightarrow s_1>1$.

Proof of Theorem 1. To prove Theorem 1 we will use the Hermite–Biehler Criterion of stability. Let us prove the statement 1 of Theorem 1.

For $F(z)=\sum_{j=0}^{4} a_j z^j$ we have $f(z)=a_0-a_2z+a_4z^2$ and $g(z)=z(a_1-a_3z)$. Using our notations we can express two zeros of the polynomial $f(z)$ in such a way:

$$t_{1,2}=\frac{a_0}{a_2}\frac{s_1s_2}{2}\left(1\mp\sqrt{1-\frac{4}{s_1s_2}}\right).$$

These zeros are real and distinct since $\min(s_1,s_2)>2$. Two zeros of the polynomial $g(z)$ are $t_0^*=0$, $t_1^*=\frac{a_0}{a_2}s_1$, they are real and distinct. The polynomial $F(z)$ is stable if and only if

$$0<\frac{a_0}{a_2}\frac{s_1s_2}{2}\left(1-\sqrt{1-\frac{4}{s_1s_2}}\right)<\frac{a_0}{a_2}s_1<\frac{a_0}{a_2}\frac{s_1s_2}{2}\left(1+\sqrt{1-\frac{4}{s_1s_2}}\right). \tag{2}$$

The first inequality in (2) obviously holds.

The second inequality in (2) is equivalent to $s_2(1-\sqrt{1-\frac{4}{s_1s_2}})<2$. The left-hand side of this inequality is strictly decreasing in s_1, so this inequality follows from $s_2(1-\sqrt{1-\frac{2}{s_2}})\leqslant 2$ (we paste $s_1=2$). The left-hand side of the last inequality is strictly decreasing in s_2, and for $s_2=2$ the left-hand side is equal to the right-hand side. By these reasons the second inequality in (2) holds.

The third inequality in (2) is equivalent to $s_2(1+\sqrt{1-\frac{4}{s_1s_2}})>2$. Since $\min(s_1,s_2)>2$ the last inequality is true.

Thus, polynomial $F(z)=\sum_{j=0}^{4} a_j z^j$ with positive coefficients is stable provided $\min(s_1,s_2)>2$.

Let us prove the statement 2 of Theorem 1. For $F(z)=\sum_{j=0}^{5} a_j z^j$ we have $f(z)=a_0-a_2z+a_4z^2$ and $g(z)=z(a_1-a_3z+a_5z^2)$. Using our notations we can express two zeros of the polynomial $f(z)$ in such a way:

$$t_{1,2}=\frac{a_0}{a_2}\frac{s_1s_2}{2}\left(1\mp\sqrt{1-\frac{4}{s_1s_2}}\right).$$

These zeros are real and distinct since $\min(s_1, s_2) > x_0 > 2$. The polynomial $g(z)$ has three distinct real zeros which can be written in such a way:

$$t_0^* = 0, \qquad t_1^* = \frac{a_0}{a_2}\frac{s_1s_2s_3}{2}\left(1 - \sqrt{1 - \frac{4}{s_2s_3}}\right), \qquad t_2^* = \frac{a_0}{a_2}\frac{s_1s_2s_3}{2}\left(1 + \sqrt{1 - \frac{4}{s_2s_3}}\right).$$

The polynomial $F(z)$ is stable if and only if

$$0 < \frac{a_0}{a_2}\frac{s_1s_2}{2}\left(1 - \sqrt{1 - \frac{4}{s_1s_2}}\right) < \frac{a_0}{a_2}\frac{s_1s_2s_3}{2}\left(1 - \sqrt{1 - \frac{4}{s_2s_3}}\right) < \frac{a_0}{a_2}\frac{s_1s_2}{2}\left(1 + \sqrt{1 - \frac{4}{s_1s_2}}\right) < \frac{a_0}{a_2}\frac{s_1s_2s_3}{2}\left(1 + \sqrt{1 - \frac{4}{s_2s_3}}\right). \tag{3}$$

The first inequality in (3) obviously holds.

The second inequality in (3) is equivalent to $(1 - \sqrt{1 - \frac{4}{s_1s_2}}) < s_3(1 - \sqrt{1 - \frac{4}{s_2s_3}})$. The left-hand side of this inequality is strictly decreasing in s_1, and $s_1 > x_0 > 2$ so this inequality follows from $(1 - \sqrt{1 - \frac{2}{s_2}}) \leqslant s_3(1 - \sqrt{1 - \frac{4}{s_2s_3}})$ (we paste $s_1 = 2$). The right-hand side of the last inequality is strictly decreasing in s_3 and $\lim_{s_3\to\infty} s_3(1 - \sqrt{1 - \frac{4}{s_2s_3}}) = \frac{2}{s_2}$, so the second inequality in (3) follows from the obvious one $(1 - \sqrt{1 - \frac{2}{s_2}}) \leqslant \frac{2}{s_2}$. Thus, the second inequality in (3) is true.

Let us check that under our assumptions the third inequality in (3) holds, or, equivalently $s_3(1 - \sqrt{1 - \frac{4}{s_2s_3}}) < (1 + \sqrt{1 - \frac{4}{s_1s_2}})$. The left-hand side of this inequality is strictly decreasing in s_3, the right-hand side is strictly increasing in s_1, so this inequality follows from $x_0(1 - \sqrt{1 - \frac{4}{s_2x_0}}) \leqslant (1 + \sqrt{1 - \frac{4}{x_0s_2}})$, or, equivalently, $(1 + x_0)\sqrt{1 - \frac{4}{s_2x_0}} \geqslant x_0 - 1$. The left-hand side of the last inequality is strictly increasing in s_2, and the last inequality follows from $(1 + x_0)\sqrt{1 - \frac{4}{x_0^2}} \geqslant x_0 - 1$ or, equivalently, $x_0^3 - x_0^2 - 2x_0 - 1 \geqslant 0$. By the definition of x_0 this is true. Thus, the third inequality in (3) holds.

Let us check that under our assumptions the fourth inequality in (3) holds, or, equivalently $1 + \sqrt{1 - \frac{4}{s_1s_2}} < s_3(1 + \sqrt{1 - \frac{4}{s_2s_3}})$. The left-hand side is strictly increasing in s_1 and $\lim_{s_1\to\infty} 1 + \sqrt{1 - \frac{4}{s_1s_2}} = 2$, so the last inequality follows from $2 \leqslant s_3(1 + \sqrt{1 - \frac{4}{s_2s_3}})$. The right-hand side is strictly increasing in $s_2 > x_0 > 2$, so the last inequality follows from the obvious inequality $2 \leqslant s_3(1 + \sqrt{1 - \frac{2}{s_3}})$. So, the fourth inequality in (3) holds.

Thus, polynomial $F(z) = \sum_{j=0}^{5} a_j z^j$ with positive coefficients is stable provided $\min(s_1, s_2, s_3) > x_0$.

Remark. It follows from the proof of the statement 2 of Theorem 1 that if $\min(s_1, s_2, s_3) \geqslant x_0$ then $t_0^* < t_1 < t_1^* \leqslant t_2 < t_2^*$ (the notation is the same as in the proof of statement 2 of Theorem 1).

Let $F(z) = \sum_{k=0}^{n} a_k z^k$, $a_k > 0$, be a polynomial satisfying the condition $s_j \geqslant x_0$ for $j = 1, 2, \ldots, n-2$. Without loss of generality we can assume that $a_0 = 1$. Denote by

$$p_j = \frac{a_{j-1}}{a_j}, \quad j = 1, 2, \ldots, n. \tag{4}$$

Using this notation we can write

$$F(z) = 1 + a_1z + a_2z^2 + \cdots + a_nz^n = 1 + \frac{z}{p_1} + \frac{z^2}{p_1p_2} + \cdots + \frac{z^n}{p_1p_2\cdots p_n}.$$

To prove the statement 3 of Theorem 1 we need some lemmas. The statement below is the direct consequence of the statements 1 and 2 of Theorem 1 and Remark.

Lemma 1. *Let* $\tilde{f}(z) = 1 - \frac{z}{p_1p_2} + \frac{z^2}{p_1p_2p_3p_4}$, $\tilde{g}(z) = 1 - \frac{z}{p_2p_3} + \frac{z^2}{p_2p_3p_4p_5}$, *where* $p_j > 0$, $1 \leqslant j \leqslant 5$. *Suppose that* $p_{j+2}/p_j \geqslant x_0$, *where* x_0 *is equal to the unique positive root of the polynomial* $x^3 - x^2 - 2x - 1$. *Denote by* $0 < x_1 < x_2$ *the zeros of* $\tilde{f}$, $0 < x_1^* < x_2^*$ *the zeros of* $\tilde{g}$. *Then*

$$\tilde{g}(x_2) \leqslant 0, \qquad \tilde{f}(x_1^*) \leqslant 0. \tag{5}$$

If $\hat{f} = -1 + \frac{z}{p_3p_4}$, $\hat{g} = 1 - \frac{z}{p_2p_3}$, *then*

$$\hat{g}(x_2) < 0, \qquad \hat{f}(x_1^*) < 0. \tag{6}$$

We need the following lemma.

Lemma 2. *Let*

$$F(z) = 1 - \frac{z}{\rho_1} + \frac{z^2}{\rho_1\rho_2} - \cdots + (-1)^n \frac{z^n}{\rho_1\rho_2\cdots\rho_n}, \quad n \geqslant 3, \tag{7}$$

be a polynomial with $\rho_j > 0$ *and* $\min\{\frac{\rho_{j+1}}{\rho_j}, 1 \leqslant j \leqslant n-1\} > 1$. *Denote by*

$$R_j(z, F) = \begin{cases} 1 - \frac{z}{\rho_j} + \frac{z^2}{\rho_j\rho_{j+1}}, & \text{if } j = 1, 2, \ldots, n-1; \\ -1 + \frac{z}{\rho_1}, & \text{if } j = 0; \\ 1 - \frac{z}{\rho_n}, & \text{if } j = n. \end{cases} \tag{8}$$

Then

$$(-1)^j F(x) > -K_j(x) R_j(x, F), \quad \rho_{j-2} < x < \rho_{j+3}, \ j = 0, 1, \ldots, n, \tag{9}$$

where $K_j(x) > 0$, $\rho_j = 0$ *for* $j < 1$, *and* $\rho_j = \infty$ *for* $j > n$.

Proof of Lemma 2. Let us fix $j \in \{1, 2, \ldots, n-1\}$. We have

$$\begin{aligned}(-1)^j F(x) &= \left((-1)^j + \sum_{k=1}^{j-2}(-1)^{k+j}\frac{x^k}{\rho_1\rho_2\cdots\rho_k}\right) + \left[-\frac{x^{j-1}}{\rho_1\rho_2\cdots\rho_{j-1}} + \frac{x^j}{\rho_1\rho_2\cdots\rho_j} - \frac{x^{j+1}}{\rho_1\rho_2\cdots\rho_{j+1}}\right] \\ &\quad + \sum_{k=j+2}^{n}(-1)^{j+k}\frac{x^k}{\rho_1\rho_2\cdots\rho_k} \\ &=: \Sigma_1(x) + \left[-\frac{x^{j-1}}{\rho_1\rho_2\cdots\rho_{j-1}} + \frac{x^j}{\rho_1\rho_2\cdots\rho_j} - \frac{x^{j+1}}{\rho_1\rho_2\cdots\rho_{j+1}}\right] + \Sigma_2(x). \end{aligned} \tag{10}$$

For every $x \in (\rho_{j-2}, \rho_{j+3})$, $\frac{x^k}{\rho_1\rho_2\cdots\rho_k} < \frac{x^{k+1}}{\rho_1\rho_2\cdots\rho_{k+1}}$ holds for $0 \leqslant k \leqslant j-3$. So for all $x \in (\rho_{j-2}, \rho_{j+3})$ summands in $\Sigma_1(x)$ are alternating in sign and their moduli are increasing. Analogously for all $x \in (\rho_{j-2}, \rho_{j+3})$ summands in $\Sigma_2(x)$ are alternating in sign and their moduli are decreasing. So $\Sigma_1(x) \geqslant 0$, $\Sigma_2(x) \geqslant 0$ for all $x \in (\rho_{j-2}, \rho_{j+3})$, so for $j = 1, 2, \ldots, n-1$

$$(-1)^j F(x) \geqslant -\frac{x^{j-1}}{\rho_1\rho_2\cdots\rho_{j-1}} + \frac{x^j}{\rho_1\rho_2\cdots\rho_j} - \frac{x^{j+1}}{\rho_1\rho_2\cdots\rho_{j+1}}, \quad x \in (\rho_{j-2}, \rho_{j+3}). \tag{11}$$

Thus (9) is proved for $j = 1, 2, \ldots, n-1$.

For $j = 0$ (9) follows from the inequality

$$F(x) = -\left(-1 + \frac{x}{\rho_1}\right) + \sum_{k=2}^{n}(-1)^k\frac{x^k}{\rho_1\rho_2\cdots\rho_k} > -R_0(x, F), \quad 0 < x < \rho_3.$$

We use the fact that for $0 < x < \rho_3$ the summands in $\sum_{k=2}^{n}(-1)^k\frac{x^k}{\rho_1\rho_2\cdots\rho_k}$ are alternating in sign and their moduli are decreasing. So the sign of this sum coincides with the sign of the first summand (for $k = 2$) and this sign is positive.

For $j = n$ (9) follows from the inequality

$$(-1)^n F(x) = -\frac{x^{n-1}}{\rho_1\rho_2\cdots\rho_{n-1}}\left(1 - \frac{x}{\rho_n}\right) + (-1)^n + \sum_{k=1}^{n-2}(-1)^{k+n}\frac{x^k}{\rho_1\rho_2\cdots\rho_k} > -\frac{x^{n-1}}{\rho_1\rho_2\cdots\rho_{n-1}} R_n(x, F),$$
$$\rho_{n-2} < x < \infty.$$

We use the fact that for $\rho_{n-2} < x < \infty$ the summands in the expression $((-1)^n + \sum_{k=1}^{n-2}(-1)^{k+n}\frac{x^k}{\rho_1\rho_2\cdots\rho_k})$ are alternating in sign and their moduli are increasing. So the sign of this expression coincides with the sign of the last summand (for $k = n-2$) and this sign is positive. □

Lemma 3. *Let* $F(z) = 1 - \frac{z}{\rho_1} + \frac{z^2}{\rho_1\rho_2} - \cdots + (-1)^n \frac{z^n}{\rho_1\rho_2\ldots\rho_n}$, $n \geqslant 3$, *be a polynomial with* $\rho_j > 0$ *and* $\min\{\frac{\rho_{j+1}}{\rho_j},\ 1 \leqslant j \leqslant n-1\} > 4$. *Let polynomials* $R_j(z, F)$, $j = 0, 1, \ldots, n$, *be defined by* (8).

1. *Polynomial* $R_j(z)$, $j = 1, 2, \ldots, n-1$, *has simple real zeros, which we denote by* $0 < \omega_1(j) < \omega_2(j)$.
2. *Polynomial* $F(z)$ *has simple real zeros* $0 < x_1 < x_2 < \cdots < x_n$.
3. $\omega_1(j) > \rho_j$, $\omega_2(j) < \rho_{j+1}$.

4. $\sqrt{\rho_{j-1}\rho_j} < x_j < \sqrt{\rho_j\rho_{j+1}}$, $j = 1, 2, \ldots, n$ (*where we put* $\rho_0 = 1$, $\rho_{n+1} = +\infty$).
5. $x_j < \omega_1(j)$, $x_{j+1} > \omega_2(j)$, $j = 1, 2, \ldots, n-1$.

Proof of Lemma 3. Since $\min\{\frac{\rho_{j+1}}{\rho_j},\ 1 \leqslant j \leqslant n-1\} > 4$ then

$$R_j(\sqrt{\rho_j\rho_{j+1}}, F) = 2 - \sqrt{\frac{\rho_{j+1}}{\rho_j}} < 0, \quad 1 \leqslant j \leqslant n-1.$$

The statements 1 and 3 of lemma immediately follow from this. The fact that $\min\{\frac{\rho_{j+1}}{\rho_j}, 1 \leqslant j \leqslant n-1\} > 4$ implies that polynomial $F(z)$ has simple real zeros is the well-known (see, for example, [9]). This fact and statement 4 of lemma follow from the statements:

$$F(\sqrt{\rho_0\rho_1}) > 0, \qquad -F(\sqrt{\rho_1\rho_2}) > 0, \qquad F(\sqrt{\rho_2\rho_3}) > 0, \qquad \ldots,$$
$$(-1)^{n-1}F(\sqrt{\rho_{n-1}\rho_n}) > 0, \qquad \lim_{x\to\infty}\big[(-1)^n F(x)\big] = +\infty. \tag{12}$$

The last statement is obvious, the rest of inequalities is the direct consequence of (9). Since for all $z \in (\omega_1(j), \omega_2(j))$ we have $R_j(z, F) < 0$, $j = 1, 2, \ldots, n-1$, than the statement 5 is the direct consequence of (9). □

Let us prove now the statement 3 of Theorem 1.

By the Hermite–Biehler Criterion the polynomial F is stable if and only if the following two polynomials

$$f(z) = \sum_{m=0}^{\lfloor n/2\rfloor} (-1)^m a_{2m} z^m$$

and

$$g(z) = z \sum_{m=0}^{\lfloor (n-1)/2\rfloor} (-1)^m a_{2m+1} z^m$$

have simple real interlacing zeros.

Since

$$\frac{a_{2m}^2}{a_{2m-2}a_{2m+2}} = \frac{a_{2m-1}a_{2m}}{a_{2m-2}a_{2m+1}}\,\frac{a_{2m}a_{2m+1}}{a_{2m-1}a_{2m+2}} \geqslant x_0^2 > 4, \quad m = 1, 2, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor,$$

and

$$\frac{a_{2m+1}^2}{a_{2m-1}a_{2m+3}} = \frac{a_{2m}a_{2m+1}}{a_{2m-1}a_{2m+2}}\,\frac{a_{2m+1}a_{2m+2}}{a_{2m}a_{2m+3}} \geqslant x_0^2 > 4, \quad m = 1, 2, \ldots, \left\lfloor \frac{n-3}{2} \right\rfloor,$$

polynomials $f(z)$ and $g(z)$ have simple real zeros. It remains to prove that under our assumptions zeros of polynomials $f(z)$ and $g(z)$ are interlacing.

We need the following notations. Let P be a real polynomial. Denote by $N_{(a,b)}(P)$ the number of zeros of P in the interval (a, b). Denote by $0 < t_1 < t_2 < \cdots < t_{\lfloor \frac{n}{2}\rfloor}$ zeros of $f(z)$ and by $0 = t_0^* < t_1^* < t_2^* < \cdots < t^*_{\lfloor \frac{n-1}{2}\rfloor}$ zeros of $g(z)$. We obtain the fact that zeros of polynomials $f(z)$ and $g(z)$ are interlacing as a consequence of the following lemma.

Lemma 4.

$$N_{(t_j,t_{j+1})}(g) \geqslant 1, \quad j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1, \tag{13}$$

$$N_{(t_j^*,t_{j+1}^*)}(f) \geqslant 1, \quad j = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor - 1. \tag{14}$$

Proof of Lemma 4. For $F(z) = 1 + \frac{z}{p_1} + \frac{z^2}{p_1p_2} + \cdots + \frac{z^n}{p_1p_2\cdots p_n}$ we have $f(z) = 1 - \frac{z}{p_1p_2} + \frac{z^2}{p_1p_2p_3p_4} + \cdots$ and $g(z) = \frac{z}{p_1}(1 - \frac{z}{p_2p_3} + \frac{z^2}{p_2p_3p_4p_5} + \cdots)$. Put $g_1(z) = g(z)p_1/z$. Note that the polynomial f has the form (7) with $\rho_j = p_{2j-1}p_{2j}$ and the polynomial g_1 has the form (7) with $\rho_j = p_{2j}p_{2j+1}$. We will consider the polynomials $R_j(z, f)$ for $j = 0, 1, 2, \ldots, \lfloor \frac{n}{2}\rfloor$ and $R_j(z, g_1)$ for $j = 0, 1, 2, \ldots, \lfloor \frac{n-1}{2}\rfloor$, these polynomials are defined by (8).

At first we will prove (13) with $n \geqslant 7$. By Lemma 3 we have $t_j < \omega_1(j)$, $t_{j+1} > \omega_2(j)$, where $\omega_1(j)$, $\omega_2(j)$ are zeros of $R_j(z, f)$. To prove (13) it is sufficient to prove that

$$(-1)^{j-1} g_1\big(\omega_1(j)\big) > 0, \quad j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1, \tag{15}$$

$$(-1)^{j-1} g_1\big(\omega_2(j)\big) < 0, \quad j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1. \tag{16}$$

By (9) we have

$$(-1)^{j-1} g_1(t) > -K_{j-1}(t) R_{j-1}(t, g_1), \tag{17}$$

where $p_{2j-6}p_{2j-5} < t < p_{2j+4}p_{2j+5}$, $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$;

$$(-1)^{j} g_1(t) > -K_{j}(t) R_{j}(t, g_1), \tag{18}$$

where $p_{2j-4}p_{2j-3} < t < p_{2j+6}p_{2j+7}$, $j = 0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$. In (17) and (18) the functions $K_j(t)$ are positive. By the statement 3 of Lemma 3 we have $p_{2j-1}p_{2j} < \omega_1(j) < \omega_2(j) < p_{2j+1}p_{2j+2}$, so inequalities (17) and (18) are valid for $t \in (\omega_1(j), \omega_2(j))$, $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$. In (17) we apply Lemma 1 with $\tilde{f}(t) = R_{j-1}(t, g_1)$, $j = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, $\hat{f}(t) = R_0(t, g_1)$ and $\tilde{g}(t) = R_j(t, f)$, $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$. We have

$$R_j\big(\omega_1(j), g_1\big) < 0, \quad j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1. \tag{19}$$

From (17), (19) we obtain (15).

In (18) we apply Lemma 1 with $\tilde{f}(t) = R_j(t, f)$ and $\tilde{g}(t) = R_j(t, g_1)$, $j = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$. If n is an odd number, then $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. If n is an even number we also apply Lemma 1 with $\hat{g}(t) = R_{\lfloor \frac{n}{2} \rfloor - 1}(t, g_1)$ and $\tilde{f}(t) = R_{\lfloor \frac{n}{2} \rfloor - 1}(t, f)$. We have

$$R_j\big(\omega_2(j), g_1\big) < 0, \quad j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1. \tag{20}$$

From (18), (20) we obtain (16).

The statement (14) with $n \geqslant 6$ can be proved analogously. By Lemma 3 we have $t_j^* < \omega_1^*(j)$, $t_{j+1}^* > \omega_2^*(j)$, where $\omega_1^*(j)$, $\omega_2^*(j)$ are zeros of $R_j(z, g_1)$. To prove (14) it is sufficient to prove that

$$(-1)^{j-1} f\big(\omega_1^*(j)\big) < 0, \quad j = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor - 1, \tag{21}$$

$$(-1)^{j-1} f\big(\omega_2^*(j)\big) > 0, \quad j = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor - 1. \tag{22}$$

By (9) we have

$$(-1)^{j} f(t) > -K_{j}(t) R_{j}(t, f), \tag{23}$$

where $p_{2j-5}p_{2j-4} < t < p_{2j+5}p_{2j+6}$, $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$ and

$$(-1)^{j+1} f(t) > -K_{j+1}(t) R_{j+1}(t, f), \tag{24}$$

where $p_{2j-3}p_{2j-2} < t < p_{2j+7}p_{2j+8}$, $j = 0, 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1$. In (23) and (24) the functions $K_j(t)$ are positive. By statement 3 of Lemma 3 we have $p_{2j}p_{2j+1} < \omega_1^*(j) < \omega_2^*(j) < p_{2j+2}p_{2j+3}$, so inequalities (23) and (24) are valid for $t \in (\omega_1^*(j), \omega_2^*(j))$, $j = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1$. Using (23), (24) and Lemma 1 (analogously to the proof of (15) and (16)) we can obtain (21) and (22). Thus, (14) is proved.

Applying (21) and (22) for $n = 6$ we have

$$f\big(\omega_1^*\big) < 0, \qquad f\big(\omega_2^*\big) > 0,$$

where ω_1^*, ω_2^* are zeros of the polynomial $R_1(z, g_1) = g_1(z)$. Besides this

$$f(0) = 1 > 0, \qquad \lim_{t \to \infty} f(t) = -\infty.$$

Thus, (13) is proved for $n = 6$. □

So, zeros of polynomials $f(z)$ and $g(z)$ are interlacing. Applying (13) for $j = 1$ we have

$$g_1\big(\omega_1(1)\big) > 0,$$

where $\omega_1(1)$ is the smallest zero of the polynomial $R_1(z, f)$. Besides this $g_1(0) = 1 > 0$. Thus, $t_1 < \omega_1(1) < t_1^*$.

Theorem 1 is proved. □

Proof of Theorem 3. To prove Theorem 3 we use just the same reasonings as in the proof of Theorem 1. Instead of the Hermite–Biehler Criterion of stability for polynomials we will apply the following generalization of the Hermite–Biehler theorem on entire functions of order not greater then 1 and minimal type of growth.

Theorem HB. *(See [10, Chapter 7].) Let $G(z)=\sum_{k=0}^{\infty}a_kz^k$, $a_k>0$, be an entire function of order not greater then* 1 *and minimal type of growth. All zeros of G have negative real parts if and only if the following two entire functions $f(z)=\sum_0^{\infty}(-1)^m a_{2m}z^m$ and $g(z)=z\sum_0^{\infty}(-1)^m a_{2m+1}z^m$ have simple real interlacing zeros.*

Let us consider a function $G(z)=\sum_{k=0}^{\infty}a_kz^k$ with positive coefficients under condition $a_ka_{k+1}\geqslant x_0a_{k-1}a_{k+2}$ for $k\in\mathbb{N}$. Note that

$$\frac{a_{2k+1}}{a_{2k+2}}\cdot\frac{a_2}{a_1}=\frac{a_{2k+1}a_{2k}}{a_{2k+2}a_{2k-1}}\cdot\frac{a_{2k-1}a_{2k-2}}{a_{2k}a_{2k-3}}\cdots\frac{a_3a_2}{a_4a_1}\geqslant x_0^k,\quad k\geqslant 1,$$

and

$$\frac{a_{2k}}{a_{2k+1}}\cdot\frac{a_1}{a_0}=\frac{a_{2k}a_{2k-1}}{a_{2k+1}a_{2k-2}}\cdot\frac{a_{2k-2}a_{2k-3}}{a_{2k-1}a_{2k-4}}\cdots\frac{a_2a_1}{a_3a_0}\geqslant x_0^k,\quad k\geqslant 1.$$

So we have

$$a_{2k+2}\leqslant\frac{1}{x_0^k}\cdot\frac{a_2}{a_1}\cdot a_{2k+1},\qquad a_{2k+1}\leqslant\frac{1}{x_0^k}\cdot\frac{a_1}{a_0}\cdot a_{2k},\quad k\geqslant 1.$$

Whence

$$a_{2k+2}\leqslant\frac{1}{x_0^k}\cdot\frac{a_2}{a_1}\cdot a_{2k+1}\leqslant\frac{1}{x_0^{2k}}\cdot\frac{a_2}{a_1}\cdot\frac{a_1}{a_0}\cdot a_{2k}\leqslant\frac{1}{x_0^{2k+2(k-1)}}\cdot\left(\frac{a_2}{a_0}\right)^2a_{2k-2}\leqslant\cdots\leqslant\frac{1}{x_0^{k(k+1)}}\cdot\left(\frac{a_2}{a_0}\right)^ka_2,\quad k\geqslant 1.$$

Analogously

$$a_{2k+1}\leqslant\frac{1}{x_0^k}\cdot\frac{a_1}{a_0}\cdot a_{2k}\leqslant\frac{1}{x_0^{2k-1}}\cdot\frac{a_1}{a_0}\cdot\frac{a_2}{a_1}\cdot a_{2k-1}\leqslant\frac{1}{x_0^{2k-1+2k-3}}\cdot\left(\frac{a_2}{a_0}\right)^2a_{2k-3}\leqslant\cdots\leqslant\frac{1}{x_0^{k^2}}\cdot\left(\frac{a_2}{a_0}\right)^ka_1,\quad k\geqslant 1.$$

It is well known that the order of an entire function with coefficients a_k is given by the expression $\limsup_{n\to\infty}\frac{n\log n}{\log|a_n|^{-1}}$ (see, for example [10, Chapter 1]). So, G is an entire function of order 0.

It is easy to prove that statements of Lemmas 2, 3, 4 remain true for the entire function G. It follows from this that Theorem 3 is true.

Theorem 3 is proved. □

3. Proof of Theorem 2

Let us prove the statement 1 of Theorem 2. We consider a polynomial $T_\beta(z)=\sum_{j=1}^{4}a_j(\beta)z^j=1+\beta^3z+\beta^4z^2+\beta^3z^3+z^4$, $\beta>0$, and note that for this polynomial $s_j(\beta)=\frac{a_j(\beta)a_{j+1}(\beta)}{a_{j-1}(\beta)a_{j+2}(\beta)}=\beta^4$, $j=1,2$. We have $T_\beta(z)=z^2((z^2+z^{-2})+\beta^3(z+z^{-1})+\beta^4)$. Denote by $w=z+z^{-1}$ and note that $\operatorname{Re}z<0\Leftrightarrow\operatorname{Re}w<0$. We have $T_\beta(z)=z^2(w^2+\beta^3w+\beta^4-2)$. It is obvious that quadratic real polynomial is stable if and only if all its coefficients have the same sign. So $T_\beta(z)$ is stable if and only if $\beta^4-2>0$. In other words, if $s_1(\beta)=s_2(\beta)=\beta^4\leqslant 2$ then polynomial $T_\beta(z)$ is not stable.

To prove the statement 2 of Theorem 2 we consider a polynomial $M_\beta(z)=\sum_{j=1}^{5}b_j(\beta)z^j=1+\beta^4z+\beta^6z^2+\beta^6z^3+\beta^4z^4+z^5$, $\beta>0$, and note that for this polynomial $s_j(\beta)=\frac{b_j(\beta)b_{j+1}(\beta)}{b_{j-1}(\beta)b_{j+2}(\beta)}=\beta^4$, $j=1,2,3$. We have

$$\begin{aligned}M_\beta(z)&=(z+1)\big(z^4+\big(\beta^4-1\big)z^3+\big(\beta^6-\beta^4+1\big)z^2+\big(\beta^4-1\big)z+1\big)\\&=(z+1)z^2\big(\big(z^2+z^{-2}\big)+\big(\beta^4-1\big)\big(z+z^{-1}\big)+\big(\beta^6-\beta^4+1\big)\big).\end{aligned}$$

Using the notation $w=z+z^{-1}$ we can write

$$M_\beta(z)=(z+1)z^2\big(w^2+\big(\beta^4-1\big)w+\big(\beta^6-\beta^4-1\big)\big).$$

Since $\operatorname{sign}(\operatorname{Re}z)=\operatorname{sign}(\operatorname{Re}w)$, $M_\beta(z)$ is stable if and only if $\beta^6-\beta^4-1>0$. Note that $(\beta^6-\beta^4-1)(\beta^6+\beta^4+1)=\beta^{12}-\beta^8-2\beta^4-1$ and polynomial $\beta^6+\beta^4+1$ has no positive zeros. Thus $M_\beta(z)$ is stable if and only if $\beta^4>x_0$, where x_0 is the unique positive root of the polynomial x^3-x^2-2x-1, and for $\beta^4<x_0$ the polynomial $M_\beta(z)$ has zeros with positive real parts. So if $s_1(\beta)=s_2(\beta)=s_3(\beta)=\beta^4<x_0$, then polynomial $M_\beta(z)$ has zeros with positive real parts thus it is not stable.

Let us prove the statement 3 of Theorem 2. Obviously, for every $\varepsilon>0$ we can choose β in such a way that $x_0-\varepsilon<\beta^4<x_0$. So, the polynomial $M_\beta(z)$ has zeros with positive real parts. For every $\varepsilon\in(0,x_0)$ we denote by $\delta=(x_0-\varepsilon/2)^{\frac{1}{4}}$, so $\delta>0$, $\delta^4>x_0-\varepsilon$. For $n=6$ we put $Q_{\gamma_1,6}(z)=M_\delta(z)+\gamma_1z^6$, $\gamma_1>0$. Since $M_\delta(z)$ has zeros with positive real parts, $Q_{\gamma_1,6}(z)$ has zeros with positive real parts for γ_1 being small enough. For the polynomial $Q_{\gamma_1,6}(z)$ we have $s_1=s_2=s_3=\delta^4>x_0-\varepsilon$, and $s_4=\frac{1}{\delta^2\gamma_1}>x_0-\varepsilon$ for γ_1 being small enough. For γ_1 chosen below and $n=7$ we put $Q_{\gamma_2,7}(z)=Q_{\gamma_1,6}(z)+\gamma_2z^7$, $\gamma_2>0$.

Since $Q_{\gamma_1,6}(z)$ has zeros with positive real parts, $Q_{\gamma_2,7}(z)$ has zeros with positive real parts for γ_2 being small enough. For the polynomial $Q_{\gamma_2,7}(z)$ we have $s_1 = s_2 = s_3 = \delta^4 > x_0 - \varepsilon$, $s_4 = \frac{1}{\delta^2\gamma_1} > x_0 - \varepsilon$ and $s_5 = \frac{\gamma_1}{\delta^4\gamma_2} > x_0 - \varepsilon$ for γ_2 being small enough. Reasoning analogously we can construct the example needed for every $n \geqslant 5$ and example of an entire function.

Theorem 2 is proved. □

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