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The amenability of affine algebras

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Abstract

We introduce the notion of amenability for affine algebras. We characterize amenability by Følner-sequences, paradoxicality and the existence of finitely invariant dimension-measures. Then we extend the results of Rowen on ranks, from affine algebras of subexponential growth to amenable affine algebras.

Keywords: Amenable groups; Affine algebras

1. Introduction

First, let us recall the classical notion of amenability. Let Γ be a discrete group. We call Γ paradoxical, if it can be written as a disjoint union $\Gamma = A_1 \cup A_2 \cup \cdots \cup A_m$ such that for some elements $g_1, h_1, g_2, h_2, \ldots, g_m, h_m \in \Gamma$, the sets $A_1 g_1, A_1 h_1, A_2 g_2, A_2 h_2, \ldots, A_m g_m, A_m h_m$ are disjoint as well. The group Γ is called amenable if it is not paradoxical. The theorem below is one the fundamental results on amenability.

Theorem 1. *The following conditions are equivalent*:

(1) *Γ is amenable.*
(2) *There exists a finitely additive measure on the subsets of Γ such that $\mu(\Gamma) = 1$ and $\mu(Ag) = \mu(A)$ for any $A \subseteq \Gamma$ and $g \in \Gamma$.*

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(3) *There exists a sequence of finite subsets* (*Følner-exhaustion*) $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$, $\bigcup_{n=1}^{\infty} \mathcal{F}_n = \Gamma$, *such that for any* $g \in \Gamma$,

$$\lim_{n\to\infty} \frac{|\mathcal{F}_n \cup \mathcal{F}_n g|}{|\mathcal{F}_n|} = 1.$$

The goal of this paper is to define and study the appropriate version of amenability for affine algebras. Throughout this article R denotes a unital affine algebra over a commutative field K.

Definition 1.1. The affine algebra R is (left) amenable if there exists a sequence of finite dimensional linear subspaces $W_1 \subseteq W_2 \subseteq \cdots$, $\bigcup_{n=1}^{\infty} W_n = R$, such that for any $r \in R$,

$$\lim_{n\to\infty} \frac{\dim_K(W_n r + W_n)}{\dim_K(W_n)} = 1. \tag{1}$$

We call such an exhaustion by subspaces a Følner-exhaustion. Now we define the analogues of paradoxicality and the invariant finitely additive measure for algebras without zero-divisors.

Definition 1.2. Let R be an affine algebra without zero divisors. We say that R is paradoxical, if any basis of R over K, $\{f_i\}_{i=1}^{\infty}$ can be written as the disjoint union $A_1 \cup A_2 \cup \cdots \cup A_m$ such that for some non-zero elements $g_1, h_1, g_2, h_2, \ldots, g_m, h_m \in R$, the sets $A_1 g_1, A_1 h_1, A_2 g_2, A_2 h_2, \ldots, A_m g_m, A_m h_m$ are mutually independent.

Now let $\{e_i\}_{i=1}^{\infty}$ be a basis of R, where again R has no zero-divisors. An independent subset $L \subset R$ is called regular with respect to $\{e_i\}_{i=1}^{\infty}$ if there exists subsets of $\{e_i\}_{i=1}^{\infty}$: $A_1, A_2, A_3, \ldots, A_n$ and $\{r_1, r_2, \ldots, r_n\} \subset R$ such that L can be written as the disjoint union of $A_1 r_1, A_2 r_2, \ldots, A_n r_n$.

Definition 1.3. An invariant finitely additive dimension-measure with respect to $\{e_i\}_{i=1}^{\infty}$ is a non-negative function μ on the set of regular subsets satisfying the following conditions:

(1) $\mu(\{e_i\}_{i=1}^{\infty}) = 1$ and $\mu(A) \leqslant 1$ for any independent regular subset A.
(2) If A and B are independent regular subsets then $\mu(A \cup B) = \mu(A) + \mu(B)$.
(3) For any non-zero $r \in R$ and regular set A, $\mu(A) = \mu(Ar)$.

The main result of the paper is that the following theorem.

Theorem 2. *The following conditions are equivalent for affine algebras R without zero-divisors.*

(1) *R is amenable.*
(2) *R is not paradoxical.*
(3) *There exists a finitely additive invariant dimension-measure on R with respect to some basis $\{e_i\}_{i=1}^{\infty}$.*

We shall also study the algebraic properties of amenable algebras, extending Rowen's work on algebras of subexponential growth, e.g., we prove that amenable affine algebras has the unique rank property.

2. The proof of Theorem 2

2.1. The Doubling Lemma

Lemma 2.1. *Let R be a non-amenable affine algebra with no zero-divisor. Then there exists a finite dimensional linear subspace $Z \subset R$ containing the unit and $\varepsilon > 0$ such that for any finite dimensional linear subspace $V \subset R$,*

$$\frac{\dim_K(VZ)}{\dim_K(V)} > 1 + \varepsilon.$$

Proof. Let $Z_1 \subset Z_2 \subset \cdots$, $\bigcup_{n=1}^{\infty} Z_n = R$ be a sequence of finite dimensional subspaces containing the unit. Suppose that the statement of the lemma is not true, then there exist finite dimensional linear subspaces $V_1, V_2, \ldots, V_n, \ldots$ such that

$$\frac{\dim_K(V_n Z_n)}{\dim_K(V_n)} < 1 + \frac{1}{2^n}.$$

Obviously, $\dim_K(V_n) \to \infty$. Now we construct a Følner-exhaustion for R inductively. Let $W_1 = V_1$. If we have already constructed $W_1 \subset W_2 \subset \cdots \subset W_{n-1}$ then choose k such a way that

$$\dim_k(V_k) \geqslant \left(\dim_K(W_{n-1}) + \dim_K(Z_n)\right) \cdot 2^n.$$

Let $W_n = V_k + W_{n-1} + Z_n$. Then $\{W_n\}_{n=1}^{\infty}$ will satisfy (1), leading to a contradiction. □

As a corollary we have the following Doubling Lemma.

Lemma 2.2. *Let R be a non-amenable affine algebra with no zero-divisor. Then there exists a finite dimensional linear subspace $Z \subset R$ containing the unit such that for any finite dimensional linear subspace $V \subset R$,*

$$\frac{\dim_K(VZ)}{\dim_K(V)} > 2.$$

2.2. Amenability implies the existence of finitely additive invariant dimension-measure

Lemma 2.3. *Let R be an amenable affine algebra with no zero divisor. Then one can construct a sequence of finite dimensional vector spaces, $\overline{V}_1 \subset V_1 \subset \overline{V}_2 \subset V_2 \subset \cdots \subset R$ with the following properties.*

- $\{V_n\}_{n=1}^{\infty}$ *satisfy* (1).
- $\lim_{n\to\infty} \frac{\dim_K(\overline{V}_n)}{\dim_K(V_n)} = 1$.
- *For any* $0 \neq s \in R$ *there exists* $k > 0$, *such that if* $n > k$, *then* $\overline{V}_n s \subset V_n$.

Proof. First choose an exhaustion $W_1 \subset W_2 \subset \cdots$, $\bigcup_{n=1}^{\infty} W_n = R$ satisfying (1). Let $\overline{V}_1 = W_1$, $V_1 = W_1$. Suppose that we have already chosen

$$\overline{V}_1 \subset V_1 \subset \overline{V}_2 \subset V_2 \subset \cdots \subset \overline{V}_{n-1} \subset V_{n-1}.$$

Let us pick k so large that $V_{n-1} \subset W_k$. Then we choose $l > k$ so that

$$\frac{\dim_K(W_l W_k + W_l)}{\dim_K(W_l)} \leqslant 1 + \frac{1}{2^n}.$$

Then let $\overline{V}_n = W_l$ and $V_n = W_l W_k + W_l$. $\square$

Proposition 2.1. *Let R be an amenable affine algebra with no zero divisor. Then there exists a finitely additive invariant dimension-measure on R with respect to some basis* $\{e_i\}_{i=1}^{\infty}$.

Proof. Let us choose a basis $\{e_i\}_{i=1}^{\infty}$ of R inductively, such a way that if $\overline{V}_i$ is a k_i-dimensional space, then $\{e_1, e_2, \ldots, e_{k_i}\}$ form a basis of $\overline{V}_i$, similarly if V_i is a l_i-dimensional space, then $\{e_1, e_2, \ldots, e_{l_i}\}$ form a basis of V_i. $\square$

Lemma 2.4. *For* $0 \neq s \in R$, *using the notation of Lemma* 2.3 *let*

$$F_k(s) = \{e_j \in V_k\colon\ e_j s \notin V_k\},$$
$$B_k(s) = \{e_j \notin V_k\colon\ e_j s \in V_k\}.$$

Then

$$\lim_{k\to\infty} \frac{|F_k(s)|}{\dim_K(V_k)} = 0, \tag{2}$$

$$\lim_{k\to\infty} \frac{|B_k(s)|}{\dim_K(V_k)} = 0. \tag{3}$$

Proof. First note that if k is large, then $\overline{V}_k s \subset V_k$, consequently (2) holds. Then (3) follows from the fact that the right multiplication by s is an injective map. $\square$

Now we define the finitely additive invariant dimension-measure. For any regular independent subset L,

$$\mu(L) = \lim_{\omega} \frac{|L \cap V_k|}{\dim_K(V_k)}.$$

Then, of course, $\mu(L) \leqslant 1$, $\mu(\{e_i\}_{i=1}^{\infty}) = 1$ and $\mu(A) + \mu(B) = \mu(A \cup B)$ if A and B are independent. In order to finish the proof of Proposition 2.1 it is enough to see that for any $0 \neq r \in R$ and regular independent subset L,

$$\lim_{k\to\infty} \frac{|\dim_K(Lr \cap V_k) - \dim_K(L \cap V_k)|}{\dim_K(V_k)} = 0. \tag{4}$$

However, by additivity, we may suppose that L is constructed by using only one translation, that is for any $a_i \in L$ there exists e_{n_i} such that $a_i = e_{n_i} s$. Let $N_L \subset \{e_i\}_{i=1}^{\infty}$ be the set of all such e_{n_i}'s. Then $L = N_L s$. By Lemma 2.4, $\mu(N_L) = \mu(N_L s)$ and $\mu(N_L) = \mu(N_L sr)$ that implies the invariance of μ. □

2.3. Non-amenability implies paradoxicality

The goal of this subsection is to prove the following proposition.

Proposition 2.2. *If the affine algebra R with no zero divisor is not amenable then it is paradoxical.*

We apply the "algebraization" of the tools used in [2]. Our first lemma is just the linear algebraic analog of the classical Hall lemma of graph theory.

Lemma 2.5. *Let $e_1, e_2, \dots, e_m$ be a basis for the m-dimensional vector space K^m and let $T_1, T_2, \dots, T_k$ be a finite collection of linear transformations from K^m to K^n. Suppose that for any l-tuple $\{e_{i_1}, e_{i_2}, \dots, e_{i_l}\}$, the linear vector space spanned by the vectors $\{\bigcup_{t=1}^{l} \bigcup_{j=1}^{k} T_j(e_{i_t})\}$ is at least l-dimensional. Then, there exists a function $\phi : \{1, 2, \dots, m\} \to \{1, 2, \dots, k\}$ such that the vectors $\{T_{\phi(1)}(e_1), T_{\phi(2)}(e_2), \dots, T_{\phi(m)}(e_m)\}$, are independent.*

Proof. We proceed by induction. The lemma obviously holds for $m = 1$. Suppose that the lemma holds for any $1 \leqslant k < m$.

If for any l-tuple $l < m$, $\{e_{i_1}, e_{i_2}, \dots, e_{i_l}\}$, the linear vector space spanned by the vectors $\{\bigcup_{t=1}^{l} \bigcup_{j=1}^{k} T_j(e_{i_t})\}$ is at least $(l+1)$-dimensional, then first define $\phi(1)$ such a way that $T_{\phi(1)}(e_1)$ is non-zero. Then for the remaining basis vectors $\{e_2, e_3, \dots, e_m\}$ let us consider the quotient maps $T_j' : K^{m-1} \to K^n/\{T_{\phi(1)}(e_1)\}$. This new system of vector spaces and maps must satisfy the conditions of our lemma. Hence we can extend ϕ to the whole set $\{1, 2, \dots, m\}$.

Now, if for some l-tuple $\{i_1, i_2, \dots, i_l\}$, $l < m$, the linear vector space spanned by the vectors $\{\bigcup_{t=1}^{l} \bigcup_{j=1}^{k} T_j(e_{i_t})\}$ is exactly l-dimensional, then first define ϕ for $\{i_1, i_2, \dots, i_l\}$. Then for the remaining vectors, we can again consider the quotient maps $T_j' : K^{m-1} \to K^n/\{T_{\phi(i_1)}(e_{i_1}), T_{\phi(i_2)}(e_{i_2}), \dots, T_{\phi(i_l)}(e_{i_l})\}$. Again, the new system of vector spaces and maps must satisfy the conditions of our lemma, hence we can extend ϕ onto the whole set $\{1, 2, \dots, m\}$. □

Now we have the following corollary.

Lemma 2.6. *Let $e_1, e_2, \ldots, e_m$ be a basis for the m-dimensional vector space K^m and let $T_1, T_2, \ldots, T_k$ be a finite collection of linear transformations from K^m to K^n. Suppose that for any l-tuple $\{e_{i_1}, e_{i_2}, \ldots, e_{i_l}\}$, the linear vector space spanned by the vectors $\{\bigcup_{t=1}^{l} \bigcup_{j=1}^{k} T_j(e_{i_t})\}$ is at least $2l$-dimensional. Then, there exist two functions $\phi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$ and $\psi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$ such that the vectors $\{T_{\phi(1)}(e_1), T_{\phi(2)}(e_2), \ldots, T_{\phi(m)}(e_m), T_{\psi(1)}(e_1), T_{\psi(2)}(e_2), \ldots, T_{\psi(m)}(e_m)\}$ are independent.*

Proof. First define ϕ by our previous lemma then apply the same lemma for the quotient map $T_j' : K^m \to K^n / [\{T_{\phi(i_1)}(e_1), T_{\phi(i_2)}(e_2), \ldots, T_{\phi(i_n)}(e_n)\}]$. The next proposition is a simple corollary of the previous lemma and the classical König-lemma (or compactness) argument (see also [2]). □

Proposition 2.3. *Let $\{e_i\}_{i=1}^{\infty}$ be a basis for the infinite dimensional affine algebra R. Let $S = \{r_1, r_2, \ldots, r_s\}$ be a set of elements in R. Suppose that for any l-tuple $\{e_{i_1}, e_{i_2}, \ldots, e_{i_l}\}$ the linear vector space spanned by the vectors $\{\bigcup_{t=1}^{l} \bigcup_{j=1}^{s} e_{i_t} \cdot r_j\}$ is at least $2l$-dimensional. Then one has a partition of $\{e_i\}_{i=1}^{\infty} = A_1 \cup A_2 \cup \cdots \cup A_m$ and elements $g_1, h_1, g_2, h_2, \ldots, g_m, h_m \in S$ such that the sets $A_1 g_1, A_1 h_1, A_2 g_1, A_2 h_1, \ldots, A_m g_m, A_m h_m$ are mutually independent.*

Now we prove Proposition 2.2. If R is non-amenable, then by Lemma 2.2, for any basis $\{e_i\}_{i=1}^{\infty}$, there exist a subset $\{r_1, r_2, \ldots, r_s\} \subset R$ satisfying the conditions of Proposition 2.3. Consequently, R is paradoxical.

2.4. Paradoxicality implies the non-existence of finitely additive invariant dimension-measure

Proposition 2.4. *If R is a paradoxical amenable algebra with no zero-divisor, then there is no finitely additive dimension-measure on R.*

Proof. Suppose that μ is a finitely additive invariant dimension-measure with respect to the basis $\{e_i\}_{i=1}^{\infty}$. Then consider the paradoxical decomposition $\{e_i\}_{i=1}^{\infty} = A_1 \cup A_2 \cup \cdots \cup A_m$ as in the definition of paradoxicality. Then $B = A_1 g_1 \cup A_1 h_1 \cup \cdots \cup A_m g_m \cup A_m h_m$ is a regular independent subset of dimension-measure 2. This is a contradiction. □

Now Theorem 2 follows from Propositions 2.1, 2.2, and 2.4.

3. The algebraic properties of amenable algebras

3.1. The basic properties

In this section we prove some of the basic algebraic properties of the amenable algebras.

Proposition 3.1. *Any affine algebra of subexponential growth is amenable.*

Proof. Suppose that $S = \{r_1, r_2, \ldots, r_k\} \subseteq R$ is a generator system for R that is $R = K(r_1, r_2, \ldots, r_k)$. We denote by R_m the m-ball with respect to S that is $R_m = \sum_{j=1}^{m} KS^j$. Let $d_m = \dim_K(R_m)$. Since R has subexponential growth, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $d_m \leqslant C_\varepsilon(1+\varepsilon)^m$ for all $m \geqslant 1$. Therefore there exists a subsequence $\{d_{m_n}\}_{n=1}^{\infty}$ such that

$$d_{m_n+n} \leqslant d_{m_n}\left(1 + \frac{1}{2^n}\right).$$

Consequently, if $W_n = R_{m_n}$, then

$$\frac{\dim_K(W_n r + W_n)}{\dim_K(W_n)} \leqslant 1 + \frac{1}{2^n},$$

provided that $r \in \sum_{j=1}^{n} KS^j$. □

On the other hand, there are amenable algebras of exponential growth. It is easy to check that if Γ is a finitely generated amenable group, then the group algebra $K\Gamma$ is amenable. Indeed, W_n can be chosen as the linear subspace spanned by the elements of $\mathcal{F}_n$, where $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ is a Følner-exhaustion. If $r = k_1 g_1 + k_2 g_2 + \cdots + k_s g_s \in K\Gamma$ and $\varepsilon > 0$, then for sufficiently large n,

$$\frac{\dim_K(W_n r + W_n)}{\dim_K(W_n)} \leqslant \frac{\dim_K(W_n + W_n g_1 + W_n g_2 + \cdots + W_n g_s)}{\dim_K(W_n)} \leqslant 1 + \varepsilon.$$

As it is well-known, there are amenable groups of exponential growth. In this case $K\Gamma$ has exponential growth.

Proposition 3.2. *If R is an amenable affine algebra and R has no zero-divisors, then R has Goldie dimension* 1*, that is R does not contain two independent left ideals.*

Proof. Let $I, J \lhd R$ be left ideals, $0 \neq a \in I$, $0 \neq b \in J$. If n is large enough, then

$$\dim_K(W_n a \cap W_n) > \frac{1}{2}\dim_K(W_n)$$

and

$$\dim_K(W_n b \cap W_n) > \frac{1}{2}\dim_K(W_n),$$

hence

$$\dim_K(W_n a \cap W_n b) > 0. \quad \square$$

The previous proposition shows that the group algebra of the free group of two generators is *not* amenable.

3.2. The ranks of finitely generated modules

Slightly modifying the arguments of Rowen [4] we define a real-valued rank function on finitely generated (left) modules over amenable affine algebras. Let ω be an ultrafilter and $\lim_\omega : l^\infty(\mathbb{N}) \to \mathbb{R}$ be the corresponding ultralimit that is a linear functional on the space of bounded sequences such that

$$\liminf_{n\to\infty}\{a_n\} \leqslant \lim_\omega\{a_n\} \leqslant \limsup_{n\to\infty}\{a_n\}$$

and $\lim_\omega\{a_n\} = \lim_{n\to\infty}\{a_n\}$ if $\{a_n\}_{n=1}^\infty$ is a convergent sequence of real numbers. Note that for any finite dimensional linear subspace $Z \subseteq R$ containing the unit,

$$\lim_{n\to\infty} \frac{\dim_K(W_n Z)}{\dim_K(W_n)} = 1.$$

Let R be an amenable affine algebra with a given sequence of subspaces $\{W_n\}_{n=1}^\infty$ satisfying (1). Suppose that M is a finitely generated R-module such that $M = \sum_{j=1}^r Rx_i$, where $\{x_1, x_2, \dots, x_r\} \subseteq M$. Then the rank of M is defined as follows:

$$\operatorname{rank}(M) = \lim_\omega \frac{\dim_K(W_n x_1 + W_n x_2 + \cdots + W_n x_r)}{\dim_K(W_n)}.$$

We shall see that the rank function might depend on the choice of the exhaustion $\{W_n\}_{n=1}^\infty$.

Proposition 3.3. *The rank defined above does not depend on the particular choice of the generator system $\{x_1, x_2, \dots, x_r\}$. Also, the rank is bounded above by the minimal number of elements spanning M.*

Proof. It is enough to prove that if $Z \subseteq R$ is a finite dimensional linear subspace containing the unit then,

$$\lim_{n\to\infty} \frac{\dim_K(\sum_{i=1}^r W_n Z x_i) - \dim_K(\sum_{i=1}^r W_n x_i)}{\dim_K(W_n)} = 0.$$

We have the following inequalities:

$$\begin{aligned} 0 &\leqslant \frac{\dim_K(\sum_{i=1}^r W_n Z x_i) - \dim_K(\sum_{i=1}^r W_n x_i)}{\dim_K(W_n)} \\ &\leqslant \sum_{i=1}^r \frac{\dim_K(W_n Z x_i) - \dim_K(W_n x_i)}{\dim_K(W_n)} \\ &\leqslant r \cdot \frac{\dim_K(W_n Z) - \dim_K(W_n)}{\dim_K(W_n)}. \end{aligned}$$

However, by amenability,

$$\lim_{n\to\infty} \frac{\dim_K(W_n Z) - \dim_K(W_n)}{\dim_K(W_n)} = 0. \quad \square$$

Corollary 3.1. *If R is an amenable affine algebra, then*

(1) $\operatorname{rank}(R^n) = n$, *that is an amenable affine algebra always satisfies the unique rank property* [1,4].
(2) *If M and N are finitely generated R-modules and M is either a submodule or a homomorphic image of N, then* $\operatorname{rank}(M) \leqslant \operatorname{rank}(N)$.
(3) *If N and M are finitely generated R-modules, then* $\operatorname{rank}(M \oplus N) = \operatorname{rank}(M) + \operatorname{rank}(N)$.

3.3. Exact sequences

Definition 3.1. Let $0 \to M \to N \to N/M \to 0$ be an exact sequence of finitely generated R-modules and let $X = \{x_1, x_2, \dots, x_n\}$ be a system of generators for N, containing a system of generators for M. Then the relative rank is defined as follows:

$$\operatorname{rank}_X(M) = \lim_{\omega} \frac{\dim_K(M \cap \sum_{i=1}^r W_n x_i)}{\dim_K W_n}.$$

Obviously, $\operatorname{rank}_X(M) \geqslant \operatorname{rank}(M)$.

Proposition 3.4. $\operatorname{rank}(N) = \operatorname{rank}(M/N) + \operatorname{rank}_X(M)$.

Proof. Denote by $[x_i]$ the image of the quotient map $N \to N/M$. Then

$$\dim_K\left(\sum_{j=1}^r W_n x_i\right) = \dim_K\left(\sum_{j=1}^r W_n[x_i]\right) + \dim_K\left(M \cap \sum_{i=1}^r W_n x_i\right).$$

Hence the statement follows. $\square$

Corollary 3.2. $\operatorname{rank}(N) \geqslant \operatorname{rank}(M/N) + \operatorname{rank}(M)$.

Example. Let R be the algebra generated by $1, x, y$, where $x^2 = 0$, $xy = 0$. Let W_n be the linear subspace with basis $\{1, y, y^2, \dots, y^n, x, yx, y^2x, \dots, y^{n^2}x\}$ and let $M = Rx + Ry$, $N = R$. Then it is easy to see that $\operatorname{rank}(N) = 1$, $\operatorname{rank}(M) = 0$, $\operatorname{rank}(N/M) = 0$. Note, however, that if the linear subspaces W_n are defined as $\{1, y, y^2, \dots, y^n, x, yx, y^2x, \dots, y^n x\}$, then $\operatorname{rank}(M) = 1$, $\operatorname{rank}(N/M) = 0$, that is the additivity holds. In [4] the author claims that for his rank function

$$\operatorname{rank}_S(N) \leqslant \operatorname{rank}_S(M) + \operatorname{rank}_S(N/M). \tag{5}$$

It seems to me that there might be a gap in his argument. The previous example suggests that the space of the exhaustion must play a greater role, and if (5) is true then an ultralimit construction would result in an actual additive real valued rank function on the set of finitely generated modules over affine algebras of subexponential growth. That is

$$\operatorname{rank}(N) = \operatorname{rank}(M/N) + \operatorname{rank}(M).$$

It would immediately imply that $[R^n] = [R^m]$ in the Grothendieck group $G_0(R)$. This would be much stronger than the unique rank property (see [3] for a discussion).

References

[1] E. Aljadeff, S. Rosset, Growth and uniqueness of rank, Israel J. Math. 64 (1988) 251–264.

[2] W.A. Deuber, M. Simonovits, V.T. Sós, A note on paradoxical metric spaces, Studia Sci. Hung. Math. 30 (1995) 17–23.

[3] W. Lück, Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers, II: Applications to Grothendieck groups, L^2-Euler characteristics and Burnside groups, J. Reine Angew. Math. 496 (1998) 213–236.

[4] L. Rowen, Modules over affine algebras having subexponential growth, J. Algebra 133 (1990) 527–532.