Existence and Iteration of \( n \) Symmetric Positive Solutions for a Singular Two-Point Boundary Value Problem

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Abstract—Let \( n \) be an arbitrary natural number. In this paper, we consider the existence of \( n \) symmetric positive solutions and establish a corresponding iterative scheme for the two-point boundary value problem

\[
\begin{align*}
  w''(t) + h(t)f(w(t)) &= 0, & 0 < t < 1, \\
  \alpha w(0) - \beta w'(0) &= 0, & \alpha w(1) + \beta w'(1) = 0.
\end{align*}
\]

The main tool is the monotone iterative technique. Here, the coefficient \( h(t) \) is symmetric on \((0, 1)\) and may be singular at both end points \( t = 0 \) and \( t = 1 \). © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Let \( n \) be an arbitrary natural number. The purpose of this paper is to consider the existence of \( n \) symmetric positive solutions and establish a corresponding iterative scheme for the two-point boundary value problem

\[
\begin{align*}
  w''(t) + h(t)f(w(t)) &= 0, & 0 < t < 1, \\
  \alpha w(0) - \beta w'(0) &= 0, & \alpha w(1) + \beta w'(1) = 0,
\end{align*}
\]

\((P)\)

where \( \alpha, \beta \geq 0, \rho = \alpha^2 + 2\alpha \beta > 0 \), and the coefficient \( h(t) \) may be singular at both end points \( t = 0 \) and \( t = 1 \). In this paper, a symmetric positive solution \( w^* \) of \((P)\) means a solution \( w^* \) of \((P)\) satisfying \( w^*(t) > 0, 0 < t < 1 \) and \( w^*(t) = w^*(1 - t), 0 \leq t \leq 1 \).

Problem \((P)\) describes many phenomena in applied mathematics, which can be found in the theory of nonlinear diffusion generated by nonlinear sources, and in the thermal ignition of gases, see, for example, [1-4]. Here, only positive solutions are meaningful. Recently, by making use of the Leray-Schauder degree theory on cones, the existence and multiplicity of positive solutions of
problem (P) have been studied by many authors, see, for example, [5–8]. We notice that Fink [4], Wong [5] and Yao [8] established the existence of one and two positive solutions for problem (P) with singular coefficient \( h(t) \). Very recently, by applying the Leggett-Williams theorem, Henderson and Thompson [9] obtained the existence of three symmetric nonnegative solutions of problem (P) with \( h(t) \equiv 1 \). In this paper, by improving the classical monotone iterative technique of Amann [10], we obtain not only the existence of \( n \) symmetric positive solutions, but also give an iterative scheme for approximating the solutions. It is worth stating that the first term of our iterative scheme is a constant function or a simple function. Therefore, the iterative scheme is significant and feasible. The idea of this paper comes from our papers [11,12].

We consider the Banach space \( C[0, 1] \) equipped with norm \( \|w\| = \max_{0 \leq t \leq 1} |w(t)| \). Let

\[
G(t, s) = \begin{cases} 
\frac{1}{\rho} (\beta + \alpha t)(\alpha + \beta - \alpha s), & 0 \leq t \leq s \leq 1, \\
\frac{1}{\rho} (\alpha + \beta - \alpha t)(\beta + \alpha t), & 0 \leq s \leq t \leq 1.
\end{cases}
\]

It is well known that \( G(t, s) \) means the Green function of problem (P) with \( h(t) \equiv 0 \). It is easy to see that

\[
G(t, s) > 0, \quad (t, s) \in (0, 1) \times (0, 1), \\
G(t, s) \leq G(s, s), \quad (t, s) \in [0, 1] \times [0, 1].
\]

The following conditions will be applied in this paper.

(H1) \( f: [0, +\infty) \to [0, +\infty) \) is continuous.

(H2) \( h: (0, 1) \to [0, +\infty) \) is continuous, \( 0 < \int_0^1 G(s, s)h(s)\,ds < +\infty \), and \( h(t) = h(1 - t) \), \( t \in [0, 1] \).

We recall that the function \( w \) is said to be concave on \([0, 1] \), if

\[
w(\tau t_2 + (1 - \tau)t_1) \geq \tau w(t_2) + (1 - \tau)w(t_1), \quad t_1, t_2, \tau \in [0, 1],
\]

and the function \( w \) is said to be symmetric on \([0, 1] \), if \( w(t) = w(1 - t) \), \( t \in [0, 1] \).

We denote

\[
C^+[0, 1] = \{w \in C[0, 1] : w(t) \geq 0, \ t \in [0, 1]\},
\]

\[
K = \{w \in C^+[0, 1] : w \text{ is symmetric and concave on } [0, 1]\},
\]

\[
(Tw)(t) = \int_0^1 G(t, s)h(s)f(w(s))\,ds, \quad t \in [0, 1].
\]

It is easy to see that \( K \) is a cone of nonnegative functions in \( C[0, 1] \). For \( w \in K \), we have \( \|w\| = w(1/2) \) and

\[
2\|w\| \min\{t, 1 - t\} \leq w(t) \leq \|w\|, \quad t \in [0, 1].
\]

2. BASIC LEMMA

In order to prove our results, we need the following lemma.

**LEMMA 2.1.** Assume (H1) and (H2) hold. Then, \( T : K \to K \) is completely continuous.

**PROOF.** Let \( w \in K \), then

\[
(Tw)(0) = \frac{\beta}{\rho} \int_0^1 (\alpha + \beta - \alpha s)h(s)f(w(s))\,ds,
\]

\[
(Tw)(1) = \frac{\beta}{\rho} \int_0^1 (\beta + \alpha s)h(s)f(w(s))\,ds.
\]
By differentiation,

\[(Tw)'(t) = -\frac{\alpha}{\rho} \int_0^t (\beta + \alpha s)h(s)f(w(s)) \, ds + \frac{\alpha}{\rho} \int_t^1 (\alpha + \beta - \alpha s)h(s)f(w(s)) \, ds.\]

Thus,

\[(Tw)'(0) = \frac{\alpha}{\rho} \int_0^1 (\alpha + \beta - \alpha s)h(s)f(w(s)) \, ds,
(Tw)'(1) = -\frac{\alpha}{\rho} \int_0^1 (\beta + \alpha s)h(s)f(w(s)) \, ds.
\]

It follows that

\[\alpha(Tw)(0) - \beta(Tw)'(0) = 0, \quad \alpha(Tw)(1) + \beta(Tw)'(1) = 0.\]

Since

\[(Tw)''(t) = -h(t)f(w(t)) \leq 0, \quad t \in [0, 1],\]

we see that \(Tw\) is a nonnegative concave function on \([0, 1]\). Noticing that \(h\) and \(w\) are symmetric, let \(v = 1 - s\), then

\[TW(t) = \frac{1}{\rho} \int_0^t (\alpha + \beta - \alpha t)(\beta + \alpha s)h(s)f(w(s)) \, ds
+ \frac{1}{\rho} \int_t^1 (\beta + \alpha t)(\alpha + \beta - \alpha v)h(1 - v)f(w(1 - v)) \, dv\]
\[= -\frac{1}{\rho} \int_1^{1-t} (\alpha + \beta - \alpha t)(\alpha + \beta - \alpha v)h(1 - v)f(w(1 - v)) \, dv
- \frac{1}{\rho} \int_0^t (\beta + \alpha t)(\beta + \alpha v)h(v)f(w(v)) \, dv
+ \frac{1}{\rho} \int_0^{1-t} (\alpha + \beta - \alpha t)(\alpha + \beta - \alpha v)h(v)f(w(v)) \, dv\]
\[= \int_0^1 G(1-t,s)h(s)f(w(s)) \, ds = (Tw)(1-t).\]

Thus, \(T : K \to K\).

Now, let \(n \geq 3\) and

\[(T_n w)(t) = \int_0^1 G(t,s)h_n(s)f(w(s)) \, ds, \quad t \in [0, 1],\]

where

\[h_n(t) = \begin{cases} \inf_{0 \leq s \leq 1/n} h(s), & 0 \leq t \leq \frac{1}{n}, \\ h(t), & \frac{1}{n} \leq t \leq \frac{(n-1)}{n}, \\ \inf_{(n-1)/n \leq s \leq 1} h(s), & \frac{(n-1)}{n} \leq t \leq 1. \end{cases}\]

As proven above, \(T_n : K \to K\). Since \(h_n : [0, 1] \to [0, +\infty)\) is a piecewise continuous function, we see that \(T_n : K \to K\) is completely continuous (see [6]).
Let \( d > 0 \) and \( M_d = \max\{f(l) : 0 \leq l \leq d\} < +\infty. \) Since \( \int_0^1 G(s,s)h(s)\,ds < +\infty, \) by the absolute continuity of integral, we have
\[
\lim_{n \to \infty} \int_{e(n)} G(s,s)h(s)\,ds = 0,
\]
where \( e(n) = [0,1/n] \cup [(n-1)/n,1]. \) So,
\[
\sup\{\|Tw - T_nw\| : w \in K, \|w\| \leq d\} = \sup \left\{ \max_{0 \leq t \leq 1} \int_0^1 G(t,s)[h(s) - h_n(s)]f(w(s))\,ds : w \in K, \|w\| \leq d \right\}
\leq M_d \max_{0 \leq t \leq 1} \int_0^1 G(t,s)[h(s) - h_n(s)]\,ds \leq M_d \int_0^1 G(s,s)[h(s) - h_n(s)]\,ds
\leq M_d \int_{e(n)} G(s,s)h(s)\,ds \to 0, \quad n \to +\infty.
\]
It implies that the completely continuous operators \( T_n \) uniformly approximate \( T \) on any bounded subset of \( K. \) Therefore, \( T : K \to K \) is completely continuous. 

3. MAIN RESULTS

In this section, we denote
\[
A = \left[ \max_{0 \leq t \leq 1} \int_0^1 G(t,s)h(s)\,ds \right]^{-1}, \quad B = \left[ \max_{0 \leq t \leq 1} \int_{1/4}^{3/4} G(t,s)h(s)\,ds \right]^{-1}.
\]
Constants \( A, B \) are not easy to compute explicitly. If \( \min_{1/4 \leq t \leq 3/4} h(t) > 0, \) for convenience, we can replace \( A \) by \( A', B \) by \( B', \) where
\[
A' = \left[ \max_{0 \leq t \leq 1} h(t) \int_0^1 G(s,s)\,ds \right]^{-1} = \frac{6(\alpha^2 + 2\alpha\beta)}{(\alpha^2 + 6\alpha\beta + 6\beta^2) \max_{0 \leq t \leq 1} h(t)},
\]
\[
B' = \left[ \min_{1/4 \leq t \leq 3/4} h(t) \int_{1/4}^{3/4} G \left( \frac{1}{2}, s \right)\,ds \right]^{-1} = \frac{32(\alpha^2 + 2\alpha\beta)}{(\alpha + 2\beta)(3\alpha + 8\beta) \min_{1/4 \leq t \leq 3/4} h(t)}.
\]
Obviously, \( 0 < A' < A < B < B'. \)

3.1. Existence of One Symmetric Positive Solution

THEOREM 3.1. Assume that \((H_1)\) and \((H_2)\) hold. If there exist two positive numbers \( b < a \) such that
(1) \( f : [0, a] \to [0, +\infty) \) is nondecreasing,
(2) \( f(b/2) \geq bB, f(a) < aA, \)
then, problem \((P)\) has one symmetric positive solution \( w^* \in K \) such that \( b \leq \|w^*\| \leq a \) and \( \lim_{n \to \infty} T_n\bar{w} = w^*, \) i.e.,
\[
\lim_{n \to \infty} \max_{0 \leq t \leq 1} \|\left( T_n\bar{w}\right)(t) - w^*(t)\| = 0,
\]
where \( \bar{w}(t) = 2b \min\{t, 1-t\} \) or \( \bar{w}(t) \equiv a, t \in [0,1]. \)

PROOF. We denote \( K[b,a] = \{w \in K : b \leq \|w\| \leq a\}. \) Let \( w \in K[b,a], \) then \( \max_{0 \leq t \leq 1} w(t) \leq a \) and
\[
\min_{1/4 \leq t \leq 3/4} w(t) \geq 2b \min_{1/4 \leq t \leq 3/4} \min\{t, 1-t\} = \frac{b}{2}.
\]
Existence and Iteration of n Symmetric Positive Solutions

So, by Assumptions (1) and (2), we have

\[ f(w(t)) \leq aA, \quad t \in [0,1], \]
\[ f(w(t)) \geq bB, \quad t \in \left[ \frac{1}{4}, \frac{3}{4} \right]. \]

It follows that

\[ \|Tw\| = \max_{0 \leq t \leq 1} \int_0^1 G(t,s)h(s)f(w(s)) \, ds \leq aA \max_{0 \leq t \leq 1} \int_0^1 G(t,s)h(s) \, ds = a, \]
\[ \|Tw\| \geq \max_{0 \leq t \leq 1} \int_{1/4}^{3/4} G(t,s)h(s)f(w(s)) \, ds \geq bB \max_{0 \leq t \leq 1} \int_{1/4}^{3/4} G(t,s)h(s) \, ds = b. \]

Thus, we assert that \( T : K[b,a] \to K[b,a] \) by Lemma 2.1.

Let \( \bar{w}(t) = a, \, t \in [0,1] \), then \( \bar{w} \in K[b,a] \). Let \( \bar{w}_1 = T\bar{w} \), then \( \bar{w}_1 \in K[b,a] \). We denote

\[ \bar{w}_{n+1} = T\bar{w}_n, \quad n = 1, 2, \ldots. \]

Since \( T(K[b,a]) \subset K[b,a] \), we have \( \bar{w}_n \in K[b,a], \quad n = 1, 2, \ldots \). From Lemma 2.1, \( T \) is completely continuous. We assert that \( \{\bar{w}_n\}_{n=1}^{\infty} \) has a convergent subsequence \( \{\bar{w}_{n_k}\}_{k=1}^{\infty} \) and there exists \( \bar{w}^* \in K[b,a] \) such that \( \bar{w}_{n_k} \to \bar{w}^* \).

Now, since \( \bar{w}_1 \in K[b,a] \), we have

\[ \bar{w}_1(t) \leq \|\bar{w}_1\| \leq a = \bar{w}(t), \quad t \in [0,1]. \]

Since \( f : [0,a] \to [0, +\infty) \) is nondecreasing, we have

\[ \bar{w}_2(t) = (T\bar{w}_1)(t) = \int_0^1 G(t,s)h(s)f(\bar{w}_1(s)) \, ds \leq \int_0^1 G(t,s)h(s)f(\bar{w}(s)) \, ds = (T\bar{w})(t) = \bar{w}_1(t). \]

By induction, then

\[ \bar{w}_{n+1}(t) \leq \bar{w}_n(t), \quad t \in [0,1], \quad n = 1, 2, \ldots. \]

Hence, we assert that \( \bar{w}_n \to \bar{w}^* \) and \( T\bar{w}^* = \bar{w}^* \). Since \( \|\bar{w}^*\| \geq b > 0 \) and \( \bar{w}^* \) is a nonnegative concave function on \([0,1]\), we conclude that

\[ \bar{w}^*(t) > 0, \quad t \in (0,1). \]

It is well known that the fixed point of operator \( T \) is the solution of problem (P). Therefore, \( \bar{w}^* \) is a symmetric positive solution of problem (P).

Let \( \bar{w}(t) = 2b \min\{t, 1-t\}, \quad t \in [0,1] \). It is easy to see that, for any \( w \in K[b,a] \), we have \( w(t) \geq \bar{w}(t), \quad t \in [0,1] \). The remainder of the proof is similar to the case of \( \bar{w}(t) \equiv a \).

**COROLLARY 3.2.** Assume that \((H_1)\) and \((H_2)\) hold. If

1. \( f : [0, +\infty) \to [0, +\infty) \) is nondecreasing,
2. \( \lim_{t \to 0} f(l)/l > 2B \) and \( \lim_{t \to +\infty} f(l)/l < A \) (particularly, \( \lim_{t \to 0} f(l)/l = +\infty \) and \( \lim_{t \to +\infty} f(l)/l = 0 \)),

then, problem (P) has one symmetric positive solution \( w^* \in K \) and there exist two positive numbers \( b < a \) such that \( \lim_{n \to \infty} T^n\bar{w} = w^* \), i.e.,

\[ \lim_{n \to \infty} \max_{0 \leq t \leq 1} |(T^n\bar{w})(t) - w^*(t)| = 0, \]

where \( \bar{w}(t) = 2b \min\{t, 1-t\} \) or \( \bar{w}(t) \equiv a, \quad t \in [0,1] \).
3.2. Existence of \( n \) Symmetric Positive Solutions

**Theorem 3.3.** Assume that \((H_1)\) and \((H_2)\) hold. If there exist \( 2n \) positive numbers \( b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n \) such that

1. \( f: [0, a_n] \to [0, +\infty) \) is nondecreasing,
2. \( f(b_i/2) \geq b_i B \) and \( f(a_i) \leq a_i A \), \( i = 1, 2, \ldots, n \),

then, problem \((P)\) has \( n \) symmetric positive solutions \( w_i^* \in K \), \( i = 1, 2, \ldots, n \), such that \( b_i \leq \|w_i^*\| \leq a_i \) and \( \lim_{n \to \infty} T^n \tilde{w}_i = w_i^* \), i.e.,

\[
\lim_{n \to \infty} \max_{0 \leq t \leq 1} |(T^n \tilde{w}_i)(t) - w_i^*(t)| = 0,
\]

where \( \tilde{w}_i(t) = 2b_i \min\{t, 1-t\} \) or \( \tilde{w}_i(t) \equiv a_i, t \in [0, 1] \), \( i = 1, 2, \ldots, n \).

**Corollary 3.4.** Assume that \((H_1)\) and \((H_2)\) hold. If

1. \( f: [0, +\infty) \to [0, +\infty) \) is nondecreasing,
2. \( \lim_{l \to 0} f(l)/l > 2B \) and \( \lim_{l \to +\infty} f(l)/l < A \) (particularly, \( \lim_{l \to 0} f(l)/l = +\infty \) and \( \lim_{l \to +\infty} f(l)/l = 0 \)),
3. there exist \( 2(n-1) \) positive numbers \( a_1 < b_2 < a_2 < \cdots < b_{n-1} < a_{n-1} < b_n \) such that

\[
f(a_i) < a_i A, \quad i = 1, \ldots, n-1,
\]

\[
f\left(\frac{b_i}{2}\right) > b_i B, \quad i = 2, \ldots, n,
\]

then, problem \((P)\) has \( n \) symmetric positive solutions \( w_i^* \in K \), \( i = 1, 2, \ldots, n \), and there exist two positive numbers \( b_1 < a_1, a_n > b_n \) such that \( \lim_{n \to \infty} T^n \tilde{w}_i = w_i^* \), i.e.,

\[
\lim_{n \to \infty} \max_{0 \leq t \leq 1} |(T^n \tilde{w}_i)(t) - w_i^*(t)| = 0,
\]

where \( \tilde{w}_i(t) = 2b_i \min\{t, 1-t\} \) or \( \tilde{w}_i(t) \equiv a_i, t \in [0, 1] \), \( i = 1, 2, \ldots, n \).

**Remark.** If \( f(0) > 0 \), then \( \lim_{l \to 0} f(l)/l = +\infty > 2B \).

**References**