JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 69, 341–352 (1979)

# Applications of Nonstandard Analysis to Spin Models*

L. L. HELMS AND P. A. LOEB

*University of Illinois at Urbana–Champaign, Champaign, Illinois 61820*

*Submitted by G.-C. Rota*

Methods of nonstandard analysis are used to construct a Markov semigroup representing the stochastic evolution of an infinite spin system with finite range interaction by means of a hyperfinite spin system. The hyperfinite spin system is then used to derive classical results about phase transitions of the stochastic Ising model without the use of thermodynamic limits.

## 1. INTRODUCTION

An infinite spin system is constructed as follows. Let $\Lambda$ be a $d$-dimensional integer lattice. For $x = (x_1, ..., x_d) \in \Lambda$ and $r > 0$, let $\| x \| = \max_{1 \leqslant i \leqslant d} | x_i |$ and $B_{x,r} = \{y \in \Lambda : \| y - x \| \leqslant r\}$. Equipped with the product topology, $S = \{-1, +1\}^{\Lambda}$ is a compact metric space. An element $\eta \in S$ is called a configuration and $\eta(x)$, $x \in \Lambda$, is called the spin at $x$. For each $x \in \Lambda$ and $\eta \in S$, let $\eta_x$ be the configuration obtained from $\eta$ by reversing the spin at $x$; that is, $\eta_x(y)$ is $-\eta(x)$ if $y = x$ and $\eta(y)$ if $y \neq x$. The set of real-valued continuous functions on $S$ will be denoted by $C(S)$ with the usual supremum norm. If $f$ is a function and $J$ is in the domain of $f$, $f|_J$ will denote the restriction of $f$ to $J$. The collection of finite subsets of $\Lambda$ will be denoted by $\mathscr{D}$ and, for $J \in \mathscr{D}$, $\mathscr{T}(J)$ will denote the set of cylindrical functions on $S$ with base $J$. Letting $\mathscr{T} = \bigcup_{J \in \mathscr{D}} \mathscr{T}(J)$, the set of cylindrical or tame functions on $S$, $\mathscr{T}$ is a dense subset $C(S)$ by the Stone-Weierstrass Theorem. We are given a function $c: \Lambda \times S \to R$ satisfying

(i) there is an $M \in R$ such that $0 < c(x, \eta) \leqslant M$ for all $x \in \Lambda$, $\eta \in S$, and

(ii) there is an $L \in R$ such that $c(x, \eta) = c(x, \xi)$ whenever $x \in \Lambda$ and $\eta|_{B(x,L)} = \xi|_{B(x,L)}$.

The functions $c(x, \cdot)$, $x \in \Lambda$, are called speed functions and control the stochastic evolution of a spin system as follows. If the system is in the configuration $\eta$ at a given time, the probability that the spin at a single site $x \in \Lambda$ will be reversed in a subsequent time interval of length $\Delta t$ is $c(x, \eta)\, \Delta t + o(\Delta t)$ while

* The work of both authors was supported by NSF Grant MCS 76-07471.



0022-247X/79/060341-12$02.00/0

the probability that spins will be reversed at two or more sites in the same time interval is $o(\Delta t)$. This heuristic description of the stochastic evolution of a spin system can be formulated more precisely by defining an operator $\Omega$ on $\mathscr{T}$ as follows:

$$\Omega f(\eta) = \sum_{x \in \Lambda} c(x, \eta) [f(\eta_x) - f(\eta)], \qquad f \in \mathscr{T}, \quad \eta \in S. \tag{1.1}$$

The problem then becomes one of determining whether or not there is a Markov semigroup of operators $\{T(t): t \geqslant 0\}$ on $C(S)$ having infinitesimal generator which is an extension of $\Omega$. There are also problems connected with the uniqueness of the semigroup, the existence of invariant measures for the semigroup, and uniqueness of invariant measures. This approach to spin systems, and interacting particles in general, was formulated by Spitzer [15] and extensively treated by him in the case in which $\Lambda$ is replaced by a finite set. Dobrushin [1], Holley [3], and Liggett [8] proved the first existence and uniqueness theorems for the Markov semigroups, the latter two making use of the Hille–Yosida Theorem.

Using methods of nonstandard analysis, we will treat some of the same problems by constructing a spin model on a "large" rectangle containing all the points of $\Lambda$ and having the formal properties of a spin system based on a finite number of sites. In the case of the finite range speed functions assumed above, this approach yields simple proofs of existence theorems and has the advantage of representing the time evolving infinite spin system in terms of a spin system having the formal properties of a finite spin system.

## 2. Hyperfinite Spin Systems

We will employ a denumerably comprehensive, i.e., $\aleph_1$-saturated, enlargement of a structure containing the real numbers $R$ (see [14] for terminology and notation). Consider $^*\Lambda = \{(x_1, ..., x_d): x_1, ..., x_d \in {}^*Z\}$ and $^*S = {}^*(\{-1, +1\}^\Lambda)$ If $\xi \in {}^*S$ and $A$ is an internal subset of $^*\Lambda$, we will let $[\eta, \xi]_A$ be the element of $^*S$ which is equal to $\eta$ on $A$ and $\xi$ on $^*\Lambda \sim A$. Let $^*c$ denote the nonstandard extension of the function $c$ of the preceding section, let $\gamma$ be a fixed element of $^*N \sim N$, let $\Gamma = B_{0,\gamma}$, let $S_\Gamma$ be the set of internal mappings from $\Gamma$ into $\{-1, +1\}$, and let $C(S_\Gamma)$ be the set of internal hyperreal-valued functions on $S_\Gamma$ with the maximum norm. If $f$ is any function on $^*S$ and $\phi \in {}^*S$, $f_\phi$ will denote the function on $S_\Gamma$ defined by $f_\phi(\eta) = f([\eta, \phi]_\Gamma)$. We will make use of the standard part map $st_\Gamma$ defined for $\eta \in S_\Gamma$ by $st_\Gamma \eta = \eta\,|_\Lambda$.

We can now define an operator analogous to the operator $\Omega$ given by (1.1). For each $\phi \in {}^*S$, an operator $\Omega_{\Gamma,\phi}$ on $C(S_\Gamma)$ is defined by the equation

$$\Omega_{\Gamma,\phi} f(\eta) = \sum_{x \in \Gamma} {}^*c(x, [\eta, \phi]_\Gamma) [f(\eta_x) - f(\eta)], \qquad \eta \in S_\Gamma, \quad f \in C(S_\Gamma). \tag{2.1}$$

Each $\Omega_{\Gamma,\phi}$ is an internal bounded operator on $C(S_\Gamma)$ with $\| \Omega_{\Gamma,\phi} \| \leqslant 2M \mid \Gamma \mid$, where $\mid \Gamma \mid$ is the internal cardinality of $\Gamma$, and $\Omega_{\Gamma,\phi} 1 = 0$. We can therefore define an internal Markov semigroup of operators $\{S_{\Gamma,\phi}(t) : t \in {}^*[0, \infty)\}$ on $C(S_\Gamma)$ subject to the boundary condition $\phi$ by putting $S_{\Gamma,\phi}(t) = \exp(t\Omega_{\Gamma,\phi})$. The following result is known in other contexts (cf. [6]).

LEMMA 1. *If $f \in \mathscr{T}(J)$, $J \in \mathscr{D}$, and $n \in {}^*N$, then* $\| \Omega^n_{\Gamma,\phi} {}^*f_\phi \| \leqslant \| f \| 2^n M^n n! \times \exp[\mid J \mid + n(2L + 1)^d]$.

*Proof.* Define operators $M_x$, $U_x$, $x \in \Gamma$, on $C(S_\Gamma)$ by putting $U_x f(\eta) = f(\eta_x) - f(\eta)$ and $M_x f(\eta) = {}^*c(x, [\eta, \phi]_\Gamma) f(\eta)$ for $\eta \in S_\Gamma$, $f \in C(S_\Gamma)$. Note that $U_x {}^*f_\phi = 0$ if $f \in \mathscr{T}(J)$ and $x \notin J$. If $f \in \mathscr{T}(J)$, then $\Omega_{\Gamma,\phi} {}^*f_\phi = \sum_{x \in \Gamma} M_x U_x {}^*f_\phi$ and more generally,

$$\Omega^n_{\Gamma,\phi} {}^*f_\phi = \sum_{x_n \in \Gamma} \cdots \sum_{x_1 \in \Gamma} M_{x_n} U_{x_n} \cdots M_{x_1} U_{x_1} {}^*f_\phi .$$

Since the order of summation can be interchanged

$$\Omega^n_{\Gamma,\phi} {}^*f_\phi = \sum_{x_1 \in \Gamma} \cdots \sum_{x_n \in \Gamma} M_{x_n} U_{x_n} M_{x_{n-1}} U_{x_{n-1}} \cdots M_{x_1} U_{x_1} {}^*f_\phi$$

Fix $x_1, \ldots, x_{n-1}$. Then $M_{x_{n-1}} U_{x_{n-1}} \cdots M_{x_1} U_{x_1} {}^*f_\phi$ involves only the functions ${}^*f_\phi$, ${}^*c(x_1, \cdot)$, ${}^*c(x_2, \cdot), \ldots, {}^*c(x_{n-1}, \cdot)$ and is cylindrical with base $J \cup B_{x_1,L} \cup \cdots \cup B_{x_{n-1},L}$. The summation over $x_n$ can therefore be reduced to $J \cup B_{x_1,L} \cup \cdots \cup B_{x_{n-1},L}$. Since $\| M_{x_n} \| \leqslant M$ and $\| U_{x_n} \| \leqslant 2$,

$$\| \Omega^n_{\Gamma,\phi} {}^*f_\phi \| \leqslant 2M(\mid J \mid + (n-1)(2L+1)^d) \sum_{x_1 \in \Gamma} \cdots \sum_{x_{n-1} \in \Gamma} \| M_{x_{n-1}} U_{x_{n-1}} \cdots M_{x_1} U_{x_1} {}^*f_\phi \| .$$

Repeating this argument,

$$\begin{aligned} \| \Omega^n_{\Gamma,\phi} {}^*f_\phi \| &\leqslant 2^n M^n \| f \| \prod_{k=1}^{n} (\mid J \mid + (k-1)(2L+1)^d) \\ &\leqslant 2^n M^n \| f \| (\mid J \mid + n(2L+1)^d)^n \\ &\leqslant 2^n M^n \| f \| n! \exp(\mid J \mid + n(2L+1)^d). \end{aligned}$$

Recalling that a hyperreal-valued function $g$ on $S_\Gamma$ is $S$-continuous if $g(\eta) \simeq g(\xi)$ whenever $\eta, \xi \in S_\Gamma$ with $st_\Gamma \eta = st_\Gamma \xi$, we now show that $S_{\Gamma,\phi}(t) f$ is $S$-continuous for suitable $f$ and $t \in \mathcal{O}^+$, the set of finite nonnegative hyperreal numbers.

LEMMA 2. *If $f \in C(S)$, $\phi$, $\psi \in {}^*S$, $\xi$, $\eta \in S_\Gamma$ with $st_\Gamma \xi = st_\Gamma \eta$, and $t \in \mathcal{O}^+$ satisfies* $2Mt \exp(2L+1)^d < 1$, *then* $S_{\Gamma,\phi}(t)\, {}^*f_\phi(\eta) \simeq S_{\Gamma,\psi}(t)\, {}^*f_\psi(\xi)$.

*Proof.* Suppose $f \in \mathcal{T}(J)$, $J \in \mathcal{D}$, $t$ satisfies the stated condition, and $n \in N$. Since $\Omega^n_{\Gamma,\phi}{}^*f_\phi$ involves only the tame functions $^*f$, $^*c(x_1, \cdot)$ for $x_1 \in J$, $^*c(x_2, \cdot)$ for $x_2 \in B_{x_1,L} \cup J$, etc., and these functions are unaffected by the boundary condition, $\Omega^n_{\Gamma,\phi}{}^*f_\phi(\eta) = \Omega^n_{\Gamma,\psi}{}^*f_\psi(\xi)$. Since this is true for all $n \in N$ and $N$ is external in $^*N$ there is an $\alpha \in {}^*N \sim N$ such that $\Omega^n_{\Gamma,\phi}{}^*f_\phi(\eta) = \Omega^n_{\Gamma,\psi}{}^*f_\psi(\xi)$ for $0 \leqslant n \leqslant \alpha$. Noting that

$$S_{\Gamma,\phi}(t)\,{}^*f_\phi(\eta) = \sum_{n=0}^{\alpha} \frac{t^n}{n!}\,\Omega^n_{\Gamma,\phi}{}^*f_\phi(\eta) + \sum_{n \geqslant \alpha+1} \frac{t^n}{n!}\,\Omega^n_{\Gamma\,\phi}{}^*f_\phi(\eta),$$

we have

$$S_{\Gamma,\phi}(t)\,{}^*f_\phi(\eta) - S_{\Gamma,\psi}(t)\,{}^*f_\psi(\xi) = \sum_{n \geqslant \alpha+1} \frac{t^n}{n!}\,[\Omega^n_{\Gamma,\phi}{}^*f_\phi(\eta) - \Omega^n_{\Gamma,\psi}{}^*f_\psi(\xi)].$$

Using Lemma 1, the right side is dominated by the infinitesimal

$$2\,\|f\|\,e^{|J|} \sum_{n \geqslant \alpha+1} t^n 2^n M^n \exp n(2L+1)^d.$$

This proves the assertion for $f \in \mathcal{T}$. The extension to $f \in C(S)$ is immediate since $\mathcal{T}$ is dense in $C(S)$.

According to Lemma 2, the boundary condition $\phi$ has only an infinitesimal effect on $S_{\Gamma,\phi}(t)\,{}^*f_\phi$; we note in passing that the hyperfinite rectangle $\Gamma$ has the same effect as can be seen from examining the above proof. In view of this fact, we will drop the "$\Gamma$" from the notation so that $\Omega_{\Gamma,\phi}$ and $S_{\Gamma,\phi}(t)$ become $\Omega_\phi$ and $S_\phi(t)$, respectively. Until noted otherwise, $\phi$ will be a fixed element of $^*S$.

We will make repeated use of the following facts. Let $f$ be a finite-valued $S$-continuous internal function on $S_\Gamma$. The standard part $^\circ f$ of $f$ is then defined on $S$ by $^\circ f(\eta) = {}^\circ[f({}^*\eta\,|_\Gamma)]$. The standard function $^\circ f$ is continuous on $S$ and $\|\,{}^*({}^\circ f)_\phi - f\,\| \simeq 0$. (See [12], p. 116).

Lemma 3. *If $t \in \mathcal{O}^+$ and $f \in C(S_\Gamma)$ is a finite-valued $S$-continuous function on $S_\Gamma$, then $S_\phi(t) f$ is finite-valued and $S$-continuous on $S_\Gamma$.*

*Proof.* If we can establish the assertion for $t \in \mathcal{O}^+$ satisfying $2Mt\exp(2L+1)^d < 1$, then the assertion would hold for all $t \in \mathcal{O}^+$ by the semigroup property. Consider such $t$ only. We know from Lemma 2 that the assertion is true for $S_\phi(t)\,{}^*f_\phi$ whenever $f \in C(S)$. Suppose now that $f \in C(S_\Gamma)$ is finite-valued and $S$-continuous. Since the set $\{n \in {}^*N: |f(\xi)| \leqslant n$ for all $\xi \in S_\Gamma\}$ has a finite first element, $\|f\|$ is finite and $\|S_\phi(t) f\| \leqslant \|f\| < \infty$ for all $t \in \mathcal{O}^+$. Since $\|f - {}^*({}^\circ f)_\phi\| \simeq 0$ and $S_\phi(t)\,{}^*({}^\circ f)_\phi$ is $S$-continuous, $S_\phi(t) f$ is $S$-continuous.

For $t \in {}^*[0, +\infty)$ and $\eta \in S_\Gamma$ define a transition function $U_t^\phi(\eta, \cdot)$ by means of the equation

$$S_\phi(t) f(\eta) = \int_{S_\Gamma} f(\xi)\, U_t^\phi(\eta, d\xi), \qquad f \in C(S_\Gamma).$$

Although $S_\phi(t)$ is really an internal matrix operator, we will employ the usual notation of Markov processes. Then for each $\eta \in S_\Gamma$, $U_t^\phi(\eta, \cdot)$ is a probability measure on the algebra $\mathcal{O}\!\mathcal{l}$ of internal subsets of $S_\Gamma$ and for each $A \in \mathcal{O}\!\mathcal{l}$, $U_t^\phi(\cdot, A)$ is measurable relative to the algebra $\mathcal{O}\!\mathcal{l}$. By the semigroup property, for $s, t \in {}^*[0, \infty)$, $\eta \in S_\Gamma$, and $A \in \mathcal{O}\!\mathcal{l}$

$$U_{t+s}^\phi(\eta, A) = \int_{S_\Gamma} U_t^\phi(\xi, A)\, U_s^\phi(\eta, d\xi).$$

Having constructed the semigroup $S_\phi(t)$, we will now show how the semigroup induces a Markov semigroup in the standard model for the reals.

To do this, we will make use of the following result. Let $\nu$ be an internal measure on $\mathcal{O}\!\mathcal{l}$ and let $\nu_0$ be the measure on the external $\sigma$-algebra $\sigma(\mathcal{O}\!\mathcal{l})$ obtained by first taking the standard part of $\nu$ and then taking the Carathéodory extension. We then let $\nu_S$ denote the image of $\nu_0$ on $\mathscr{B}(S)$, the Borel subsets of $S$; that is, $\nu_S(A) = \nu_0(st_\Gamma^{-1}A)$ for $A \in \mathscr{B}(S)$. If $f \in C(S)$, then

$$\int_S f\, d\nu_S = \int_{S_\Gamma} {}^\circ({}^*f_\phi)\, d\nu_0 \simeq \int_{S_\Gamma} {}^*f_\phi\, d\nu.$$

Further details can be found in [9] and [10].

For $t \in {}^*[0, \infty)$ and $\eta \in S_\Gamma$, let ${}^\circ U_t^\phi(\eta, \cdot)$ be the probability measure on the external $\sigma$-algebra $\sigma(\mathcal{O}\!\mathcal{l})$ constructed as above. If $\eta \in S$, $t \in [0, \infty)$, and $A \in \mathscr{B}(S)$, put

$$T_t^\phi(\eta, A) = {}^\circ U_t^\phi({}^*\eta\,|_\Gamma, st_\Gamma^{-1}A)$$

and following the usual notational custom, $T_t^\phi f = \int_S f(\xi)\, T_t(\cdot, d\xi)$ for bounded Borel measurable functions $f$ on $S$. We now show that $\{T_t^\phi : t \in [0, \infty)\}$ is a Feller semigroup.

LEMMA 4. *If $f \in C(S)$ and $t \in [0, \infty)$, then $T_t^\phi f \in C(S)$; moreover, if $\eta \in S$, $A \in \mathscr{B}(S)$, and $s, t \in [0, \infty)$, then*

$$T_{t+s}^\phi(\eta, A) = \int T_t^\phi(\xi, A)\, T_s^\phi(\eta, d\xi).$$

*Proof.* Since $f$ is uniformly continuous on $S$, ${}^*f_\phi$ is $S$-continuous and finite-valued on $S_\Gamma$. By Lemma 3, $S_\phi(t)^* f_\phi$ is $S$-continuous and finite-valued on $S_\Gamma$. Since

$$S_\phi(t)\, {}^*f_\phi({}^*\eta\,|_\Gamma) = \int_{S_\Gamma} {}^*f_\phi(\xi)\, U_t^\phi({}^*\eta\,|_\Gamma, d\xi) \simeq \int_{S_\Gamma} {}^\circ[{}^*f_\phi(\xi)]\, {}^\circ U_t({}^*\eta\,|_\Gamma, d\xi)$$

$$= \int_S f(\xi)\, T_t^\phi(\eta, d\xi)$$

the last integral is just the standard part of $S_\phi(t)\ {}^*f_\phi$ and is continuous on $S$. This shows that $T_t^\phi f \in C(S)$. Suppose now that $A$ is a standard cylinder set in $S$ (that is, the indicator function of $A$ is tame). Then

$$\int_S T_t^\phi(\xi, A)\ T_s^\phi(\eta, d\xi) = \int_S {}^\circ U_t^\phi({}^*\xi\ |_\Gamma, st_\Gamma^{-1}A)\ T_s^\phi(\eta, d\xi).$$

Since $U_t^\phi(\cdot, st^{-1}A)$ is $S$-continuous and finite-valued on $S_\Gamma$, the non-standard extension of ${}^\circ U_t^\phi(\cdot, st_\Gamma^{-1}A)$ is infinitesimally close to $U_t^\phi(\cdot, st_\Gamma^{-1}A)$ and therefore

$$\int_S T_t^\phi(\xi, A)\ T_s^\phi(\eta, d\xi) \simeq \int_{S_\Gamma} U_t^\phi(\xi, st_\Gamma^{-1}A)\ U_s^\phi({}^*\eta\ |_\Gamma, d\xi)$$

$$= U_{t+s}^\phi({}^*\eta\ |_\Gamma, st_\Gamma^{-1}A) \simeq T_{t+s}^\phi(\eta, A).$$

Since the first and last terms are real, they are equal. The assertion can be extended to $A \in \mathscr{B}(S)$ by a monotone class argument.

Consider now the effect of the boundary condition $\phi$ on the $T_t^\phi$ semigroup. By Lemma 2, if $\phi, \psi \in {}^*S$ and $f \in C(S)$, then $T_t^\phi f = T_t^\psi f$ for small $t$ and therefore for all $t$ by the semigroup property. Thus, $T_t^\phi$ is independent of $\phi$ (and the hyperfinite rectangle $\Gamma$ also) and the "$\phi$" will be dropped from the notation so that $T_t^\phi$ becomes simply $T_t$.

THEOREM 5. *The family of operators* $\{T_t: t \in [0, \infty)\}$ *is the unique Feller semigroup whose generator is an extension of the operator* $\Omega$ *defined by* (1.1). *Moreover, for each* $f \in C(S)$, $\eta \in S$, *and* $t \in [0, \infty)$, $T_t f(\eta) = {}^\circ[S_\phi(t)\ {}^*f_\phi({}^*\eta\ |_\Gamma)]$ *for any choice of* $\phi \in {}^*S$.

*Proof.* To show that $T_t$ is a strongly continuous semigroup on $C(S)$, we need only prove strong continuity at $t = 0$. If $f \in \mathscr{T}(J)$, $J \in \mathscr{D}$, and $\eta \in S$, then

$$|\ \Omega_\phi^n\ {}^*f_\phi({}^*\eta\ |_\Gamma)| \leqslant \|f\|\ 2^n M^n n! \exp[|\ J\ | + n(2L+1)^d]$$

and

$$|\ S_\phi(t)\ {}^*f_\phi({}^*\eta\ |_\Gamma) - {}^*f_\phi({}^*\eta\ |_\Gamma)| \leqslant \|f\|\ e^{|J|} \sum_{n \in {}^*N \sim \{0\}} [2Mt \exp(2L+1)^d]^n$$

Given standard $\epsilon > 0$, there is a standard $\delta > 0$ such that right side is less than $\epsilon$ for $t \in (0, \delta)$. Since $T_t f(\eta) \simeq S_\phi(t)\ {}^*f_\phi({}^*\eta\ |_\Gamma)$ and $f(\eta) = {}^*f_\phi({}^*\eta\ |_\Gamma)$, $|\ T_t f(\eta) - f(\eta)| < 2\epsilon$ for $t \in (0, \delta)$ and $\eta \in S$. This shows that $T_t f$ is continuous at $t = 0$ whenever $f \in \mathscr{T}$. The extension to $f \in C(S)$ is immediate since $\mathscr{T}$ is dense in $C(S)$. It remains only to show that

$$\lim_{t \to 0+} \frac{T_t f - f}{t} = \Omega f$$

for each $f \in \mathscr{T}$. Suppose $f \in \mathscr{T}(J)$, $J \in \mathscr{D}$. First note that for $\eta \in S_\Gamma$ and small $t$

$$\left| \frac{S_\phi(t)\ {}^*f_\phi(\eta) - {}^*f_\phi(\eta)}{t} - \Omega_\phi {}^*f_\phi(\eta) \right| \leqslant \|f\|\, e^{|J|} t^{-1} \sum_{\substack{n \in {}^*N \\ n \geqslant 2}} [2Mt \exp(2L+1)^d]^n.$$

The right side can be made less than a standard positive number by taking $t$ to be less than some standard positive number (uniformly in $\eta$). Since $S_\phi(t)\ {}^*f_\phi({}^*\eta\,|_\Gamma) \simeq T_t f(\eta)$, ${}^*f_\phi({}^*\eta\,|_\Gamma) \simeq f(\eta)$, and $\Omega_\phi {}^*f_\phi({}^*\eta\,|_\Gamma) \simeq \Omega f(\eta)$ for $\eta \in S$, the assertion follows. Uniqueness follows as in [6].

The second statement in the above theorem is the basis for the claim that the Markov semigroup defining an infinite spin system can be represented in terms of a spin system having the formal properties of a finite spin system.

We will now look into the existence of invariant measures for the $T_t$ semigroup. Recall that the probability measure $\mu$ is invariant for the $T_t$ semigroup if $\int T_t f\, d\mu = \int f\, d\mu$ for all $f \in C(S)$. Since the functions ${}^*c(x, [\eta, \phi]_\Gamma)$ are strictly positive on $S_\Gamma$, the semigroup $S_\phi(t)$ is internally irreducible and therefore has a unique invariant internal measure $\nu^\phi$ defined on $\mathcal{Cl}$. We will let $\nu_S{}^\phi$ denote the measure on $\mathscr{B}(S)$ induced by $\nu^\phi$.

THEOREM 6. *For each $\phi \in {}^*S$, $\nu_S{}^\phi$ is invariant for the $T_t$ semigroup.*

*Proof.* Suppose $f \in C(S)$ and $t \in [0, \infty)$. Then

$$\int_S T_t f\, d\nu_S{}^\phi \simeq \int_{S_\Gamma} {}^*(T_t f)_\phi\, d\nu^\phi.$$

Since $T_t f$ is the standard part of the $S$-continuous finite-valued function $S_\phi(t)\ {}^*f_\phi$, ${}^*(T_t f)_\phi(\eta) \simeq S_\phi(t) {}^*f_\phi(\eta)$ for all $\eta \in S_\Gamma$ and

$$\int_S T_t f\, d\nu_S{}^\phi \simeq \int_{S_\Gamma} S_\phi(t)\ {}^*f_\phi\, d\nu^\phi = \int_{S_\Gamma} {}^*f_\phi\, d\nu^\phi \simeq \int_S f\, d\nu_S{}^\phi.$$

As has been pointed out, the $T_t$ semigroup is independent of the hyperfinite rectangle $\Gamma$ and the boundary condition $\phi$. The measures $\nu_S{}^\phi$, however, may very well depend upon the boundary condition and conceivably upon the rectangle $\Gamma$. It is not known if each invariant measure for the $T_t$ semigroup is one of the $\nu_S{}^\phi$ or in the weak*-closure of the convex hull of the $\nu_S{}^\phi$.

## 3. STOCHASTIC ISING MODEL

In this section we specialize by looking at a particular choice of the speed functions which define the stochastic Ising model. For this purpose, let $\beta$ be a positive real parameter (inverse temperature) and let $h$ be a real parameter (strength of external field). We define speed functions $c(x, \eta)$ on $\Lambda \times S$ by

$$c(x, \eta) = \exp\left[-\frac{\beta}{2} \sum_{\|y-x\|=1} \eta(y)\,\eta(x) - h\eta(x)\right].$$

The speed functions depend upon the parameters $\beta$ and $h$ but the notational dependence will be suppressed. These functions satisfy the conditions imposed in Section 1 and define an operator $\Omega$ on $\mathscr{T}$ by (1.1). The nonstandard extension $^*c(x, \eta)$ (as a function of $x, \eta, \beta, h$) is defined by the above equation provided the objects therein are interpreted properly; in particular, $\beta$ and $h$ will be regarded as hyperreal parameters. Consider now the hyperfinite rectangle $\Gamma$ of the preceding section. The internal operator defined by (2.1) depends upon the parameter $h$ and will be denoted by $\Omega_{h,\phi}$ (the dependence upon $\beta$ will be suppressed for the time being) and the internal semigroup generated by $\Omega_{h,\phi}$ will be denoted by $S_{h,\phi}(t)$. The semigroup of operators on $C(S)$ induced by the $S_{h,\phi}(t)$ will be denoted by $T_h(t)$. Applying the transfer principle to a well known result, the unique internal invariant probability measure for the $S_{h,\phi}(t)$ semigroup is given by the density

$$\mu_{\Gamma,h,\phi}(\eta) = Z(\Gamma, h, \phi)^{-1} \exp\left[\frac{\beta}{2} \sum_{x\in\Gamma} \sum_{\|y-x\|=1} [\eta, \phi]_\Gamma(y)\,[\eta, \phi]_\Gamma(x) + h \sum_{x\in\Gamma} \eta(x)\right] \tag{3.1}$$

for $\eta \in S_\Gamma$ where $Z(\Gamma, h, \phi)$ is the normalizing constant

$$Z(\Gamma, h, \phi) = \sum_{\eta\in S_\Gamma} \exp\left[\frac{\beta}{2} \sum_{x\in\Gamma} \sum_{\|y-x\|=1} [\eta, \phi]_\Gamma(y)\,[\eta, \phi]_\Gamma(x) + h \sum_{x\in\Gamma} \eta(x)\right].$$

This result is true no matter what rectangle is used, standard or nonstandard. Expectations relative to the probability density $\mu_{\Gamma,h,\phi}$ will be denoted by $E^{\phi}_{\Gamma,h}[\cdot]$. If $\phi = +1$ or $\phi = -1$, the density $\mu_{\Gamma,h,\phi}$ will be denoted by $\mu^+_{\Gamma,h}$ or $\mu^-_{\Gamma,h}$, respectively, and the corresponding expectations by $E^+_{\Gamma,h}$ or $E^-_{\Gamma,h}$, respectively. Straightforward arguments show that

$$\left|\frac{1}{|\Gamma|} \log Z(\Gamma, h, +1) - \frac{1}{|\Gamma|} \log Z(\Gamma, h, -1)\right| \leqslant \beta \frac{|\partial\Gamma|}{|\Gamma|}$$
$$|Z(\Gamma, h, +1)| \leqslant 2^{|\Gamma|} \exp(|\Gamma|\,(\beta d + h)) \tag{3.2}$$

where $\partial\Gamma$ is the boundary of $\Gamma$. Moreover, the set of configurations $S_\Gamma$ can be considered a lattice whereby $\eta \geqslant \xi$, $\eta, \xi \in S_\Gamma$, is interpreted pointwise. As such it makes sense to speak of increasing functions $f$ on $S_\Gamma$. It is known (cf. [5] or [7]) that if $f$ is increasing on $S_\Gamma$ and $-1 \leqslant \phi \leqslant \psi \leqslant +1$ for $\phi, \psi \in {}^*S$ then

$$E^-_{\Gamma,h}[f] \leqslant E^{\phi}_{\Gamma,h}[f] \leqslant E^{\psi}_{\Gamma,h}[f] \leqslant E^+_{\Gamma,h}[f]. \tag{3.3}$$

It is also known (cf. [5] or [7]) that if $J \subset \Sigma \subset \Gamma$, where $J \in \mathscr{D}$ and $\Sigma$ is an internal subset of $\Gamma$, and $f \in \mathscr{T}(J)$ is increasing on $S_\Gamma$, then

$$E^-_{\Sigma,h}[f] \leqslant E^-_{\Gamma,h}[f] \leqslant E^+_{\Gamma,h}[f] \leqslant E^+_{\Sigma,h}[f]. \tag{3.4}$$

It is also easily seen that the probability densities $\mu^\pm_{\Sigma,h}$ are translation invariant; that is, $\mu^\pm_{\Sigma,h}(\eta) = \mu^\pm_{\Sigma+y,h}(\tau_y\eta)$, $y \in {}^*\Lambda$, where $\tau_y$ is the translation operator mapping $S_\Sigma$ onto $S_{\Sigma+y}$ defined by $\tau_y\eta(x) = \eta(x-y)$, $x \in \Sigma + y$. The following lemma has a standard version in terms of an expanding sequence of standard rectangles (cf. [2] or [7]).

LEMMA 7. *If $\beta$ and $h$ are standard real numbers, then* $E^\pm_{\Gamma,h}[\eta(0)] \simeq E^\pm_{\Gamma,h}[(1/|\Gamma|)\sum_{y\in\Sigma}\eta(y)]$.

*Proof.* Consider ${}^\circ E^+_{\Gamma,h}[\eta(0)]$ and standard $\epsilon > 0$. We know that there is an internal rectangle $\Sigma \subset {}^*\Lambda$ (namely $\Gamma$) such that $E^+_{\Sigma,h}[\eta(0)]$ is within $\epsilon$ of ${}^\circ E^+_{\Gamma,h}[\eta(0)]$. By the transfer principle there is a standard rectangle $\Sigma \subset \Lambda$ for which $E^+_{\Sigma,h}[\eta(0)]$ is within $\epsilon$ of ${}^\circ E^+_{\Gamma,h}[\eta(0)]$; that is

$$E^+_{\Sigma,h}[\eta(0)] < E^+_{\Gamma,h}[\eta(0)] + \epsilon.$$

Letting $\Gamma_0 = \{y \in \Gamma : \Sigma + y \in \Gamma\}$, it follows from (3.4) that for $y \in \Gamma_0$

$$E^+_{\Gamma,h}[\eta(y)] \leqslant E^+_{\Sigma+y,h}[\eta(y)] = E^+_{\Sigma,h}[\eta(0)] < E^+_{\Gamma,h}[\eta(0)] + \epsilon.$$

Therefore,

$$E^+_{\Gamma,h}\left[\frac{1}{|\Gamma_0|}\sum_{y\in\Gamma_0}\eta(y)\right] < E^+_{\Gamma,h}[\eta(0)] + \epsilon.$$

Since

$$E^+_{\Gamma,h}\left[\frac{1}{|\Gamma_0|}\sum_{y\in\Gamma_0}\eta(y)\right] = \frac{|\Gamma|}{|\Gamma_0|}E^+_{\Gamma,h}\left[\frac{1}{|\Gamma|}\sum_{y\in\Gamma}\eta(y)\right] - E^+_{\Gamma,h}\left[\frac{1}{|\Gamma_0|}\sum_{y\in\Gamma\sim\Gamma_0}\eta(y)\right]$$

and the internal cardinality of $\Gamma \sim \Gamma_0$ is at most $|\partial\Gamma|\,|\Sigma|$, where $\partial\Gamma$ is the boundary of $\Gamma$,

$$E^+_{\Gamma,h}\left[\frac{1}{|\Gamma_0|}\sum_{y\in\Gamma_0}\eta(y)\right] \simeq E^+_{\Gamma,h}\left[\frac{1}{|\Gamma|}\sum_{y\in\Gamma}\eta(y)\right]$$

and

$$E^+_{\Gamma,h}\left[\frac{1}{|\Gamma|}\sum_{y\in\Gamma}\eta(y)\right] < E^+_{\Gamma,h}[\eta(0)] + 2\epsilon.$$

On the other hand consider $y \in \Gamma$ and the number $E^+_{\Gamma-y,h}[\eta(0)]$. Using the transfer principle, there is a standard rectangle $\Sigma_y$ such that $E^+_{\Sigma_y,h}[\eta(0)] < E^+_{\Gamma-y,h}[\eta(0)] + \epsilon$. Since $\Sigma_y \subset \Gamma$, by (3.4)

$$E^+_{\Gamma,h}[\eta(0)] \leqslant E^+_{\Sigma_y h}[\eta(0)] < E^+_{\Gamma-y,h}[\eta(0)] + \epsilon = E^+_{\Gamma,h}[\eta(y)] + \epsilon$$

and

$$E^+_{\Gamma,h}\left[\frac{1}{|\Gamma|}\sum_{y\in\Gamma}\eta(y)\right] \geqslant E^+_{\Gamma,h}[\eta(0)] - \epsilon.$$

This completes the proof.

Consider now the internal functions $F_+(h)$ and $F_-(h)$ defined on $^*R$ by

$$F_\pm(h) = \frac{1}{|\Gamma|}\log Z(\Gamma, h, \pm 1).$$

Both $F_+$ and $F_-$ are continuous functions on $^*R$. According to (3.2), $F_+(h) \simeq F_-(h)$ for each $h \in {}^*R$. It is also easily seen that

$$F'_\pm(h) = E^\pm_{\Gamma,h}\left[\frac{1}{|\Gamma|}\sum_{y\in\Gamma}\eta(y)\right] \simeq E^\pm_{\Gamma,h}[\eta(0)] \tag{3.5}$$

and that

$$F''_\pm(h) = |\Gamma| \operatorname{Var}^\pm_{\Gamma,h}\left[\frac{1}{|\Gamma|}\sum_{y\in\Gamma}\eta(y)\right] \geqslant 0$$

where $\operatorname{Var}^\pm_{\Gamma,h}$ is the variance relative to the probability density $\mu^\pm_{\Gamma,h}$. The functions $F_+$ and $F_-$ are therefore continuous convex functions on $^*R$. It is also known from the Yang and Lee Circle Theorem (cf. [11] or [13]) that both $F_+$ and $F_-$ can be continued analytically from $(0, \infty)$ to $\operatorname{Re} z > 0$ and from $(-\infty, 0)$ to $\operatorname{Re} z < 0$ in the hypercomplex plane. By (3.2), $|F_\pm(h)| \leqslant \log 2 + \beta d + h$ and $|F_\pm(h)|$ is finite for finite $\beta$ and $h$. Consider now standard $h \neq 0$ and a fixed standard $\beta$. An application of the Cauchy integral formula shows that $F''_\pm(h)$ is finite for such $h$. Making use of this fact and the convexity of $F_\pm$, an elementary argument shows that

$$F'_\pm(h) \simeq S - \lim_{\Delta h\to 0}\frac{F_\pm(h+\Delta h) - F_\pm(h)}{\Delta h} \tag{3.6}$$

where the limit is taken over standard $\Delta h$. Since $F_+(h)$ and $F_-(h)$ are finite and $F_+(h) \simeq F_-(h)$, we can define a standard function $G(h) = {}^\circ F_\pm(h)$. Clearly $G$ is convex, differentiable at any standard $h \neq 0$, and $G'(h) \simeq F'_\pm(h)$ for such $h$. It follows from (3.5) that $E^+_{\Gamma,h}[\eta(0) \simeq E^-_{\Gamma,h}[\eta(0)]$ for $h \in R \sim \{0\}$. Since the same argument holds for any standard $x \in \Gamma$, $E^+_{\Gamma,h}[\eta(x)] \simeq E^-_{\Gamma,h}[\eta(x)]$ for all such $x$. By (3.3), $E^+_{\Gamma,h}[\eta(x)] \simeq E^-_{\Gamma,h}[\eta(x)] \simeq E^\phi_{\Gamma,h}[\eta(x)]$ for all $\phi \in {}^*S$. Letting $\mu_h{}^\phi = (\mu^\phi_{\Gamma,h})_S$, the measure on $\mathscr{B}(S)$ induced by $\mu^\phi_{\Gamma,h}$ via the standard part map, $E_h{}^+[\eta(x)] =$

$E_h^-[\eta(x)] = E_h^\phi[\eta(x)]$ for all $x \in \Lambda$ and $\phi \in {}^*S$. This is enough to show that $\mu_h^+ = \mu_h^- = \mu_h^\phi$ for all $\phi \in {}^*S$ and $h \in R \sim \{0\}$ (cf. [7]).

Note that we have not shown that the $T_h(t)$ semigroup has a unique invariant measure, but only that $\mu_h^\phi$ is independent of $\phi \in {}^*S$ for $h \in R \sim \{0\}$. Uniqueness is known from the work of Holley [4].

We now consider the critical case $h = 0$ (no external field). In this case, the parameter $\beta$ plays a role and the notation will reflect the dependence upon $\beta$; e.g., $Z(\Gamma, \beta, h, \phi)$, $F_\pm(\beta, h)$, $E^\pm_{\Gamma,\beta,h}[\cdot]$, etc. Up to the point where it was asserted that $F_\pm(\beta, \cdot)$ can be continued analytically into the hypercomplex plane, everything still applies. In particular, $G(\beta, h) = {}^\circ F_\pm(\beta, h)$ is defined unambiguously for finite $\beta$, $h$ and we need only consider circumstances under which $F_\pm(\beta, \cdot)$ is complex analytic in a standard neighborhood of $h = 0$. The partition function $Z(\Gamma, \beta, h, +1)$ can be written

$$Z(\Gamma, \beta, h, +1) = \exp[(\beta d - h) \mid \Gamma \mid] \mathscr{P}_\Gamma(e^{2h})$$

where $\mathscr{P}_\Gamma$ is a polynomial (cf. [13]). Clearly analyticity of $\log Z(\Gamma, \beta, \cdot, +1)$ is determined by the zeros of the polynomial $\mathscr{P}_\Gamma$. According to a well-known result (see p. 82 of [13]), $\mathscr{P}_\Gamma(z)$ has no zeros inside the circle in the hypercomplex plane defined by

$$\left| \frac{z}{z+1} \right| < [2 \exp(e^{2\beta d} - 1) - 1]^{-1}.$$

The point $z = 1$ will be in the interior of this circle provided $2d\beta < \log(1 + \log \frac{3}{2})$. If we consider only standard $\beta$ for which this inequality holds, then $\mathscr{P}_\Gamma(z)$ will have no zeros in a standard neighborhood of $z = 1$. This means that $\log Z(\Gamma, \beta, \cdot, +1)$ (and $\log Z(\Gamma, \beta, h, -1) = \log Z(\Gamma, \beta, -h, +1)$) has an analytic extension to a standard $\epsilon$ neighborhood of $h = 0$. This is all that is required to show that for sufficiently small standard $\beta$, $\mu^+_{\beta,0} = \mu^-_{\beta,0} = \mu^\phi_{\beta,0}$ for all $\phi \in {}^*S$. The usual argument using Peierl's inequality can be used to show that $\mu^+_{\beta,0} \neq \mu^-_{\beta,0}$ for sufficiently large $\beta$ (cf. [13]).

All of the results of this section are known, except for the use of nonstandard analysis to derive them. In this case, the significance lies not so much in new proofs of old theorems but rather in the use of the $S_{\Gamma,\phi}(t)$ semigroup as a model for a large but infinite number of particles with spins. In the standard approach to spin systems, one must let the number of sites in a finite system become infinite in order to exhibit phase transition (that is, a failure of $F_\pm$ to be analytic at a point) whereas in real life all spin systems are finite and do not exhibit phase transitions. Hyperfinite spin systems make more sense in this regard; the functions $F_\pm$ are internally differentiable but not $S$-differentiable at $h = 0$ for large $\beta$. Relative to the internal scale, phase transition cannot occur, but relative to the external standard scale, phase transition can occur.

## References


1. R. L. Dobrushin, Markov processes with a large number of locally interacting components—existence of the limiting process and its ergodicity, *Problemy Peredači Informacii* **7** (1971), 70–87.
2. R. Griffiths, A proof that the free energy of a spin system is extensive, *J. Mathematical Phys.* **5** (1964), 1215–1222.
3. R. Holley, A class of interactions in an infinite spin system, *Advances in Math.* **5** (1970), 291–309.
4. R. Holley, An ergodic theorem for interacting systems with attractive interactions, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **24** (1972), 325–334.
5. R. Holley, Recent results on the stochastic Ising model, *Rocky Mt. J. Math.* **4** (1974), 479–496.
6. R. Holley, Markovian interaction processes with finite range interactions, *Ann. Math. Statist.* **43** (1972), 1961–1967.
7. J. L. Lebowitz and A. Martin-Löf, On the uniqueness of the equilibrium state for the Ising spin models, *Commun. Math. Phys.* **25** (1972), 276–282.
8. T. M. Liggett, Existence theorems for infinite particle systems, *Trans. Amer. Math. Soc.* **165** (1972), 471–481.
9. P. A. Loeb, Conversion from nonstandard to standard measure spaces and applications in probability theory, *Trans. Amer. Math. Soc.* **211** (1975), 113–122.
10. P. A. Loeb, Weak limits of measures and the standard part map, *Proc. Amer. Math. Soc.*, in press.
11. C. J. Preston, "Gibbs States on Countable Sets," Cambridge Univ. Press, London, 1974.
12. A. Robinson, "Nonstandard Analysis," rev. ed. North-Holland, Amsterdam/London 1974.
13. D. Ruelle, "Statistical Mechanics," Benjamin, New York, 1969.
14. K. D. Stroyan and W. A. J. Luxemburg, "Introduction to the Theory of Infinitesimals," Academic Press, New York, 1976.
15. F. Spitzer, Interaction of Markov processes, *Adv. in Math.* **5** (1970), 246–290.