JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 172, 274–288 (1993)

On the Convergence of Fuzzy Sets and the Completeness of the Space of Fuzzy Sets*

DAZHI ZHANG

Machine Intelligence Institute, Iona College, New Rochelle, New York 10801

HE OUYANG

Department of Mathematics, Washington University, St. Louis, Missouri 63130

E. STANLEY LEE†

Department of Industrial Engineering, Kanas State University, Manhattan, Kansas 66506

AND

RONALD R. YAGER

Machine Intelligence Institute, Iona College, New Rochelle, New York 10801

Received August 12, 1991

In this article, we first introduce several convergence concepts of fuzzy sets. Then we construct an embedding of the space of fuzzy sets in a normed linear space. We prove that the space of fuzzy sets is a complete metric space under the embedding. This framework enables us to study the calculus of fuzzy functions. © 1993 Academic Press, Inc.

1. INTRODUCTION

Convergence concepts are the foundation of mathematical analysis, while the convergence of fuzzy sets is the foundation of fuzzy analysis. Since the introduction of fuzzy sets by Zadeh [15], researchers have been concerned

* We thank Professor Peizhuang Wang of Beijing Normal University for his encouragement and comments on our preliminary drafts.

† Please address all correspondence to E. S. Lee.

0022-247X/93 $5.00

with the calculus of fuzzy functions and the definition and generalization of the convergence concepts in the fields of fuzzy sets and systems. Pioneers include Dubois end Prade [1], Lowen [5], Luo [6], Ouyang [9], Pu and Liu [10], and Puri and Ralescu [11]. For example, by using the Råström embedding theorem, Puri and Ralescu [11] studied the differential of a fuzzy function.

In this article, by treating a fuzzy subset as a *tower* of (classical) subsets, we introduce four convergence concepts of fuzzy sets. These include "convergent in measure," "convergent uniformly," "convergent μ-informly," and "L^p-convergent." The concept of L^p-convergence is defined based on an embedding of the space of fuzzy sets in a normed linear space. We prove that the space of fuzzy sets is a complete metric space under the proposed embedding. Therefore, we can study the differential calculus of fuzzy functions in a way similar to Puri and Ralescu [11]. The applications of fuzzy calculus is not new. For example, it is closely related to the analysis of evidence [14].

We proceed as follows. In Section 2, we define several convergence concepts for fuzzy sets. In Section 3, we construct an embedding of the space of fuzzy sets in a normed linear space and define the L^p-convergence of fuzzy sets. The completeness of the space of fuzzy sets will be discussed in Section 4 and a conclusion will be given in Section 5.

2. Convergence Concepts of Fuzzy Sets

In this section, we discuss several convergence concepts of fuzzy sets.

Definition 2.1. Let X be any nonempty set, and $\mathscr{P}(X)$ be the power set of X. Given any $\aleph \subseteq \mathscr{P}(X)$, a set-valued mapping

$$\omega: (0, 1] \to \aleph$$

$$\omega(t') \subseteq \omega(t) \qquad \forall t' > t$$

is called an $\aleph$-*fuzzy set*. If ω is required to satisfy a set of propostions $@$, then ω is called an $\aleph$-$@$-*fuzzy set*. The collection of all the $\aleph$-$@$-fuzzy sets is denoted by $\mathscr{F}(X, \aleph, @)$.

Example 2.1. Let $\aleph = \mathscr{P}(X)$ and let $@^*$ denote

$$\omega(t) = \bigcap_{t' < t} \omega(t').$$

Then by Negoita and Ralescu's representation theorem [8], $\mathscr{F}(X, \aleph, @^*)$ is the collection of all fuzzy sets of Zadeh, which usually denoted by $\mathscr{F}(X)$.

EXAMPLE 2.2. Let X be a metric space and 2^X be the collection of all the nonempty compact subsets of X. Let $@_0$ represent

(i) $@^*$; and

(ii) $\mathrm{cl}[\bigcup_{t \in (0,1]} \omega(t)] \in 2^X$.

Then $\mathscr{F}(X, 2^X, @_0)$ is the collection of all regular fuzzy sets studied in [9], where it was denoted by $\Omega_0(X)$.

EXAMPLE 2.3. Let X be a Banach space and $cc(X)$ be the collection of all the nonempty compact convex subsets of X. Then $\mathscr{F}(X, cc(X), @_0)$ (denoted by $\Omega_0^c(X)$) is the collection of all the regular convex fuzzy sets defined in [9].

DEFINITION 2.2. Let τ be a topology given on $\aleph$ ($\subseteq \mathscr{P}(X)$). Then we can always define a topology τ^* on $\mathscr{F}(X, \aleph, @) \subseteq \aleph^{(0,1]}$. That is, τ^* is the subspace relative topology when $\mathscr{F}(X, \aleph, @)$ is regarded as a subspace of the product space $\aleph^{(0,1]}$.

Let ω_α be a net in $\mathscr{F}(X, \aleph, @)$. The essence of this topology is that

$$\begin{aligned} &\omega_\alpha \to \alpha \text{ (in topology } \tau^*) \\ &\quad \Leftrightarrow \omega_\alpha(t) \to \omega(t) \text{ (in topology } \tau) \qquad \forall t \in (0, 1]. \end{aligned}$$

In this way we have further generalized the topology defined in [9].

EXAMPLE 2.4. Consider $\Omega_0(X)$ of Example 2.2. Let 2^X be equipped with the Hausdorff metric, i.e., for any $A, B \in 2^X$

$$d(A, B) = \inf\{\varepsilon \mid N(A, \varepsilon) \supseteq B, N(B, \varepsilon) \supseteq A\},$$

where

$$N(A, \varepsilon) = \{x \in X \mid \exists a \in A, \ni d(x, a) < \varepsilon\}$$

is the neighborhood of A. Then 2^X becomes a metric space. The following are some of its properties [9]:

(i) If 2^X is arcwise connected, then $\Omega_0(X)$ is connected;

(ii) $\Omega_0(X)$ satisfies the first axiom of countability;

(iii) If X is a compact connected metric space, then $\Omega_0(X)$ is connected;

(iv) $\Omega_0(R^n)$ is separable.

In general, given a metric d (therefore the topology τ_d) on $\aleph$, we can always obtain the topology τ_d^* on $\mathscr{F}(X, \aleph, @)$ according to Definition 2.2.

Topology τ_d^* is usually no longer a metric topology; however, it is a uniform topology. More generally. if τ is a uniform topology on $\aleph$, then τ^* is a uniform topology on $\mathscr{F}(X, \aleph, @)$, since a uniform topology is preserved by product topology and subspace relative topology [3].

DEFINITION 2.3. Let $\mathscr{U}$ be uniformity on $\aleph$ and $E \subseteq (0, 1]$. A net ω_α in $F(X, \aleph, @)$ is said to converge to $\omega \in \mathscr{F}(X, \aleph, @)$ uniformly on E if for any $U \in \mathscr{U}$, there exists α_0, such that for any $\alpha > \alpha_0$,

$$(\omega_\alpha(t), \omega(t)) \in U, \qquad \forall t \in E.$$

It is easy to verify that this definition satisfies all four axioms of the Moore–Smith convergence.

EXAMPLE 2.5. The convergence in the space $\mathscr{F}_0(X)$ of [11] is a uniform convergence.

Noticing the analytical structure of (0, 1], we may wich to define "converges μ-uniformly" and "converges in measure." To this end, we first assume that $((0, 1], \mathscr{A}, \mu)$ is a measure space, where $\mathscr{A}$ is a σ-algebra on (0, 1] and μ is any measure ($\mathscr{A}$ may consist of all the Borel sets on (0, 1] and μ may be the Borel measure, for example); then we generalize the space $\mathscr{F}(X, \aleph, @)$ to include those elements which satisfy $\omega(t') \subseteq \omega(t)$ $(\forall t' > t)$ almost everywhere. More specifically, the fuzzy set ω is in $\mathscr{F}(X, \aleph, @)$ if there exists an $E_\omega \subseteq (0, 1]$, $\mu(E_\omega) = 0$ such that

$$\omega(t') \subseteq \omega(t) \qquad \forall t' > t,\ t', t \in E_\omega^c.$$

DEFINITION 2.4. Net ω_α is said to converge to ω μ-uniformly if for any $\varepsilon > 0$, there exists a measurable set $E \subseteq (0, 1]$ such that $\mu(E) < \varepsilon$ and ω_α converges to ω uniformly on E^c.

DEFINITION 2.5. Let $\mathscr{U}$ be a uniformity on $\aleph$. A net ω_α in $\mathscr{F}(X, \aleph, @)$ is said to converge to $\omega \in \mathscr{F}(X, \aleph, @)$ in measure μ if for any $U \in \mathscr{U}$ and any $\varepsilon > 0$, there exists α_0, such that for any $\alpha > \alpha_0$,

$$\mu^*\{t \in (0, 1] \mid (\omega_\alpha(t), \omega(t)) \notin U\} < \varepsilon,$$

where μ^* is the extension of μ. (That is, $\mu^*(H) = \inf\{\mu(E) \mid E \supseteq H,\ E$ is measurable$\}$.)

The following result can be established without difficulty.

PROPOSITION 2.1. *Net ω_α converges to ω uniformly on* (0, 1] *implies that ω_α converges to ω μ-uniformly, which in turn implies that w_α converges to w in measure μ.*

Three convergence concepts of fuzzy sets have been introduced in this section. Based on these concepts, we introduce the L^p-convergence of fuzzy sets and discuss the completeness of the space of fuzzy sets in the following sections.

3. The L^p-Convergence

Throughtout this section, X is assumed to be a reflexive Banach space (for example, R^n). Recall that $cc(X)$ denotes the collection of all the convex compact subsets of X.

3.1. *Preparation*

Definition 3.1. (i) The addition and non-negative scalar multiplication of $cc(X)$ are defined as

$$A+B=\{x=a+b \mid a\in A \text{ and } b\in B\}, \qquad \forall A, B\in cc(X).$$

$$\lambda A=\{x=\lambda a \mid \in A\}, \qquad \forall h\geqslant 0 \text{ and } \forall A, B\in cc(X).$$

(ii) $cc(X)$ is equipped with the Hausdorff metric, i.e., for any two compact convex subsets A and B of X, the distance between A and B is given by

$$d(A, B)=\inf\{r \mid \forall x\in A, \exists y\in B, \ni \|x-y\|<r, \text{ and } \forall y\in B, \exists x\in A, \ni \|x-y\|<r\}.$$

Proposition 3.1. [4]. *$cc(X)$ is a complete metric space.*

The following theorem is due to Rådström [12].

The Rådström embedding Theorem for $cc(X)$. *$cc(X)$ can be embedded in a normed linear space $Rcc(X)$ (we can, therefore, regard $cc(X)$ as a subspace of $Rcc(X)$) in such a way that the operations of $cc(X)$ as defined in Definition 2.1 are just the restrictions of the respective operations of $Rcc(X)$. The distance on $cc(X)$ induced from the norm on $Rcc(X)$ coincides with the Hausdorff distance. In addition, $Rcc(X)$ can be chosen to be minimal in the sense that if H is any normed linear space in which $cc(X)$ is embedded in the above fashion, then H contains a subspace containing $cc(X)$ and isomorphic to $Rcc(X)$. Furthermore, $cc(X)$ is a convex cone in $Rcc(X)$ which generates $Rcc(X)$, i.e., $Rcc(X)=cc(X)-cc(X)$. Finally, any linear operator from $cc(X)$ to a linear normed space Y can be uniquely extend to $Rcc(X)$.*

For the convenience of the reader, in the following we briefly summarize some results related to the μ-L^p-integral, which can be found in standard functional analysis. Refer to, for example, Dunford and Schwartz [2].

Let (M, Σ, μ) be a finite measure space and W be a linear normed space. The collection of all the mappings $f\colon M \to W$ is denoted by $F(M, W)$ (without counting μ-null differences). $F(M, W)$ is a linear space. Define $|f| = \inf\{\alpha + \mu^*(\|f\| > \alpha) \mid \alpha > 0\}$ where $\|\cdot\|$ is the norm in W and μ^* is the extension of μ, i.e., $\mu^*(E) = \inf\{\mu(F) \mid E \subseteq F$ and $F \in \Sigma\}$. Denote $\rho(f, g) = |f - g|$, then $(F(M, W), \rho)$ is a metric space.

It is readily verified that the convergence in the metric space $(F(M, W), \rho)$ is the convergence in measure in the following sense : Let $\{f_p\}$ be a sequence on $F(M, W)$, $f \in F(M, W)$. $\{f_p\}$ is said to converge to f in measure μ if and only if for any $\varepsilon > 0$, $\mu^*\{m \in M \mid \|f_p(m) - f(m)\| > \varepsilon\} \to 0$ $(p \to \infty)$.

Let $f \in F(M, W)$. Function f is called a simple function if f only takes on a finite number of values, say $x_1, x_2, ..., x_n$, and $f^{-1}(x_i) \in \Sigma$, $i = 1, 2, ..., n$. Thus $f = \Sigma x_i \chi_{M_i}$, where $M_i = f - 1(x_i)$ and χ is the characteristic function of sets. Function f is called a measurable function, if it can be expressed as the limit (converge in measure) of a sequence of simple functions.

The integral of a simple fuction $f = \Sigma x_i \chi_{M_i}$ is defined to be

$$\int_M f\, d\mu = \sum_{i=1}^{n} x_i \mu(M_i).$$

For $E \in \Sigma$, define the integral of f on E as

$$\int_E f\, d\mu = \sum_{i=1}^{n} x_i \mu(E \cap M_i).$$

The collection of all the simple functions is a linear subspace of $F(M, W)$ and the integral is a linear mapping from this subspace to W.

If $\{f_p^1\}$ and $\{f_p^2\}$ are sequences of simple functions both converging in measure μ to the same limit and if

$$\lim_{p,q} \int_M \|f_p^i(m) - f_q^i(m)\|\, d\mu = 0,$$

then the limits

$$\lim_p \int_E f_p^i(m)\, d\mu$$

$(i = 1, 2)$ exist uniformly with respect to E in Σ and are equal. Therefore, we can define $f \in F(M, W)$ to be μ-integrable if there is a sequence $\{f_p\}$ of simple functions converging to f in measure μ and satisfying in addition the equation

$$\lim_{p,q} \int_M \|f_p(m) - f_q(m)\| \, d\mu = 0.$$

When f is μ-integrable, we denote

$$\int_E f \, d\mu = \lim_p \int_E f_p \, d\mu, \qquad E \in \Sigma.$$

For $f \in F(M, W)$ measurable and $1 \leqslant p < \infty$, we define f to be μ-L^p-integrable if $\|f(\cdot)\|^p$ is μ-intgrable. The collection of all the μ-L^p-integrable functions is denoted by $L^p(M, W)$, which is a normed linear space with norm defined by

$$|f| = \left[\int_M \|f(m)\|^p \, d\mu \right]^{1/p}.$$

The following are two basic properties of this integral.

(i) A measurable function f is μ-integrable if and only if the numerical function $\|f(\cdot)\|$ is integrable.

(ii) If $T: W \to V$ is a bounded linear operator from W to a Banach space V, and $f \in L^1(M, W)$, then $Tf \in L^1(M, V)$. Furhtermore,

$$\int_E Tf \, d\mu = T \int_E f \, d\mu \qquad \forall E \in \Sigma.$$

3.2. *The Space $\Omega^p(X)$ and the L^p-Convergence*

Since $cc(X)$ can be embedded in $Rcc(X)$, any ω: $(0, 1] \to cc(X)$ can be regarded as a mapping from $(0, 1]$ to $Rcc(X)$. Note that $(0, 1]$ (equipped with the Borel measure) is a finite measure space and $Rcc(X)$ is a normed linear space. Therefore, $L^p((0, 1], Rcc(X))$ is well defined. We can now give the following definition.

DEFINITION 3.2. Denote by $\Omega^p(X)$ the fuzzy set space $\mathscr{F}(X, cc(X), (a)_p)$, where $(a)_p$ means

$$\omega \in L^p((0, 1], Rcc(X)).$$

Definition 3.2 says that $\Omega^p(X)$ consists of all the $cc(X)$-fuzzy sets which are μ-L^p-integrable. Thus $\Omega^p(X)$ is a subspace of the normed linear space $L^p((0, 1), Rcc(X))$. Note that the convergence in normed linear space $L^p((0, 1], Rcc(X))$ is the one according to the L^p-norm.

DEFINITION 3.3. Net $\omega_\alpha \in \Omega^p(X)$ is said to L^p-converge to $\omega \in \Omega^p(X)$ if ω_α, regarded as a net in $L^p((0, 1), Rcc(X))$, converges to ω.

Topological properties of the space $\Omega^p(X)$ and the L^p-convergence are discussed in the next section.

4. THE COMPLETENESS OF THE SPACE OF FUZZY SETS

In matematical analysis, the completeness of a space is one of the most fundamental properties. Puri and Ralescu [11] identified a collection $\mathscr{F}_0(X)$ of fuzzy subsets of X and proved that $(\mathscr{F}_0(X), d)$ is a complete metric space, where distance d is defined as the supremum of the Hausdorff distance between the respective level-sets. In the following, we prove that fuzzy set space $\Omega^p(X)$, as a subspace of normed linear space $L^p((0, 1], Rcc(X))$, is a complete metric space. First we have the following.

PROPOSITION 4.1. *Space $\Omega^p(X)$ is closed in $L^p((0, 1], Rcc(X))$.*

Proof. Let $\{\omega_n\}$ be a sequence in $\Omega^p(X)$ which L^p-converges to ξ in $L^p((0, 1], Rcc(X))$. Then there is a subsequence $\{\omega_k\}$ which converges to ξ almost everywhere. Let E be such that $\mu(E) = 0$ and $\omega_k(t) \to \xi(t)$ $\forall t \in E^c$, here the convergence is in the normed linear space $Rcc(X)$. But $cc(X)$ is complete and $\{\omega_k(t)\}$ is a Cauchy sequence for any $t \in E^c$, we conclude that $\xi(t) \in cc(X)$. We need to show that $\xi \in \Omega^p(X)$.

For any k define E_k to be such that $\mu(E_k) = 0$ and $\omega_k(t') \subseteq \omega_k(t)$ $(\forall t, t' \in E_k^c, t < t')$. Denote $E_1 = E \cup (\bigcup E_k)$. Then $\mu(E_1) = 0$. We need only to show that for any $t, t' \in E_1^c$, $t < t'$, $\xi(t') \subseteq \xi(t)$. Note that $\xi(t) = \underline{\lim}\, \omega_k(t) = \{x \mid \text{any neigborhood of } x \text{ has a nonempty intersection with almost all } \omega_k(t)\}$ [7]. Let $x \in \xi(t')$ and V be any neighborhood of x. Then there exists a positive integer K such that $\omega_k(t') \cap V \neq \emptyset$, $\forall k > K$. Since $\omega_k(t') \cap V \subseteq \omega_k(t) \cap V$, we see that $\omega_k(t) \cap V \neq \emptyset$. This implies that $x \in \xi(t)$, as desired. ∎

Since $Rcc(X)$ is generally not complete, so L^p is not. However, we have the following. First denote by d_{L_p} the distance measure in normed linear space $L^p((0, 1], Rcc(X))$.

THEOREM 4.1. $(\Omega^p(X), d_{L_p})$ is a complete metric space.

To prove this theorem, we need some preparations. The following lemmas are of interest themselves. First denote by $\Omega_M(X)$ the collection of all the measurable towers in $\mathscr{F}(X, cc(X), \emptyset)$.

LEMMA 4.1. *Let $\omega_n \in \Omega_M(X)$, and $\omega_n - \omega_n$ converges to 0 in measure μ*

(here the operator "$-$" is the one in the normed linear space $Rcc(X)$). Then there is a subsequence ω_k and $\omega \in \Omega_M(X)$ such that ω_k converges to ω μ-uniformly.

Proof. For any $\varepsilon > 0$ and any $\delta > 0$, there exists a positive integer $n(\varepsilon, \delta)$ such that $\mu\{t \mid d(\omega_n(t), \omega_m(t)) > \varepsilon\} < \delta$, $\forall m, n > n(\varepsilon, \delta)$. Therefore, we can find a subsequence ω_{n_i} of ω_n and $E_i \subseteq (0, 1]$, such that $\mu(E_i) < 1/2^i$ and

$$d(\omega_{n_i}(t), \omega_{n_{i+1}}(t)) < 1/2^i \qquad \forall t \in E_i^c.$$

Set

$$F_k = \bigcup_{i=k}^{\infty} E_i,$$

then $\mu(F_k) < 1/2^{k-1}$ and for any $t \in F_k^c$, as long as $j > i \geqslant k$, we have

$$d(\omega_{n_i}(t), \omega_{n_j}(t)) \leqslant \sum_{m=k}^{\infty} d(\omega_{n_m}(t), \omega_{n_{m+1}}(t)) < 1/2^{k-1}.$$

Therefore,

$$\forall t \in \bigcup_{k=1}^{\infty} F_k^c = \left(\bigcap_{k=1}^{\infty} F_k\right)^c,$$

$\{\omega_{n_i}(t)\}$ is a Cauchy sequence in $cc(X)$. But $cc(X)$ is complete, thus $\omega_{n_i}(t)$ converges to $\omega(t)$ which belongs to $cc(X)$. Clearly, $\omega_{n_i}(t)$ converges to $\omega(t)$ uniformly on F_k^c for every k; this implies that ω_{n_i} converges to ω μ-uniformly. Similar to the proof of Proposition 4.1, we can prove that $\omega \in \Omega_M(X)$. ∎

COROLLARY 4.1. *Assume that $\omega_n - \omega_m$ converges to 0 in measure μ. Then ω_n converges to some ω $(\in \Omega_M(X))$ in measure μ.*

Proof. Let ω_k be a subsequence which converges to ω μ-uniformly. Then ω_k converges to ω is measure μ. Given any $\varepsilon > 0$,

$$\begin{aligned}
&\mu\{t \mid d(\omega(t), \omega_n(t)) > \varepsilon\} \\
&\quad \leqslant \mu\{t \mid d(\omega(t), \omega_k(t)) + d(\omega_k(t), \omega_n(t)) > \varepsilon\} \\
&\quad \leqslant \mu\{t \mid d(\omega(t), \omega_k(t)) > \varepsilon/2\} + \mu\{t \mid d(\omega_k(t), \omega_n(t)) > \varepsilon/2\} \\
&\quad \rightarrow 0 \qquad (n, k \rightarrow \infty).
\end{aligned}$$ ∎

COROLLARY 4.2. *Let $\omega_n \in \Omega_M(X)$. If ω_n converges to ω in measure μ, then there is subsequence ω_k which converges to ω μ-uniformly.*

Proof. Given any $\varepsilon > 0$,

$$\begin{aligned}
&\mu\{t \mid d(\omega_m(t), \omega_n(t)) > \varepsilon\} \\
&\quad \leqslant \mu\{t \mid d(\omega_m(t), \omega(t)) + d(\omega(t), \omega_n(t)) > \varepsilon\} \\
&\quad \leqslant \mu\{t \mid d(\omega_m(t), \omega(t)) > \varepsilon/2\} + \mu\{t \mid d(\omega(t), \omega_n(t)) > \varepsilon/2\} \\
&\quad \to 0 \qquad (n, m \to \infty).
\end{aligned}$$

From Lemma 4.1, there is a subsequence ω_k which converges to $\bar{\omega} \in \Omega_M(X)$ μ-uniformly. Thus ω_k converges to $\bar{\omega}$ in measure μ. Note that ω_k also converges to ω in measure μ and that the limit of converging in measure is unique without counting a null-measure difference. Thus $\bar{\omega} = \omega$ and ω_k converges to ω μ-uniformly. ∎

LEMMA 4.2. *Let $\omega_n \in \Omega^p(X)$ be a Cauchy sequence. Then $\omega_n - \omega_m$ converges to 0 in measure μ.*

Proof. (cf. [2, III 3.6]). From the assumption, $|\omega_n - \omega_m|_{L^p} \to 0$ $(n, m \to \infty)$. Thus for any $\varepsilon > 0$, there is a positive integer n_0 such that for any $n, m \geqslant n_0$,

$$\int_{(0,1]} g_{n,m}(t)\, d\mu < \varepsilon,$$

where $g_{n,m}(\cdot) = \|\omega_n(\cdot) - \omega_m(\cdot)\|^p$.

Now every $g_{n,m}$ can be expressed as the limit (of converging in measure μ) of a sequence of real-valued simple functions $\{h^k_{n,m}\}_k$; in addition,

$$\lim_{k \to \infty} \int_{(0,1]} h^k_{n,m}(t)\, d\mu = \int_{(0,1]} g_{n,m}(t)\, d\mu.$$

Thus when k is sufficiently large,

$$\int_{(0,1]} h^k_{n,m}(t)\, d\mu < 2\varepsilon.$$

There is no harm in assuming that the above inequality holds for all k. Since $h^k_{n,m}$ is a simple fuction, the set $E^k_{n,m} = \{t \mid h^k_{n,m}(t) > \gamma\}$ is measurable; furthermore,

$$\gamma\mu(E^k_{n,m}) \leqslant \int_{(0,1]} h^k_{n,m}(t)\, d\mu < 2\varepsilon,$$

which implies that $\mu(E^k_{n,m}) < 2\varepsilon/\gamma$.

Since $h^k_{n,m}$ converges to $g_{n,m}$ in measure μ, we can find, for every sufficiently large k, a measurable set $F^k_{n,m}$, such that $\mu(F^k_{n,m}) < \varepsilon/\gamma$ and for any $t \notin F^k_{n,m}$, $|h^k_{n,m}(t) - g_{n,m}(t)| < \gamma$. Hence $\|\omega_n(t) - \omega_m(t)\| < (2\gamma)^{1/p}$ whenever $t \notin F^k_{n,m} \cup E^k_{n,m}$.

Now let $\delta_1, \delta_2 > 0$. Choose γ and ε to be so small that $(2\gamma)^{1/p} < \delta_1$, and $3\varepsilon/\gamma < \delta_2$. Then by letting $G = F^k_{n,m} \cup E^k_{n,m}$, we have $\mu(G) < \delta_2$, and for $n, m \geqslant n_0$ and $t \notin G$, $\|\omega_n(t) - \omega_m(t)\| < \delta_1$.

LEMMA 4.3. *Assume that ω_n converges to ω in measure μ. Then for any $\lambda \in (0, 1)$, we can always find a subsequence g_{n_i} (of g_n) which converges to g in measure μ on the interval $[\lambda, 1]$, where $g_n(t) = \|\omega_n(t)\|^p$ and $g(t) = \|\omega(t)\|^p$.*

Proof. For $t \in [\lambda, 1]$, $\omega(t) \subseteq \omega(\lambda)$. Let 0 denote the subset of X which consists of only zero element of X, then 0 belongs to $cc(X)$. It is clear that 0 is also the zero element of $Rcc(X)$. Thus for any $t \in [\lambda, 1]$, $\|\omega(t)\| = \|\omega(t) - 0\| = d(\omega(t), 0) \leqslant d(\omega(\lambda), 0) = \|\omega(\lambda) - 0\| = \|\omega(\lambda)\|$.

Denote $E_{n,\varepsilon} = \{t \mid \omega_n(t) - \omega(t)\| > \varepsilon\}$. Then from the assumption that ω_n converges to ω in measure μ, we can find n_i such that $\mu(E_{n_i,1}) < 1/2^i$. Therefore,

$$\|\omega_{n_i}(t)\| \leqslant 1 + \|\omega(\lambda)\| \triangleq \alpha$$

$$\forall t \in \left[\left(\bigcup_{i=i_0}^{\infty} E_{n_i,1}\right) \cup (0, \lambda)\right]^c \qquad \text{and} \qquad i \geqslant i_0.$$

Also note that

$$\{t \in [\lambda, 1] \mid |g_{n_i}(t) - g(t)| > a\}$$

$$\subseteq \left(\bigcup_{i=i_0}^{\infty} E_{n_i,1}\right) \cup \{t \mid \|\omega_{n_i}(t) - \omega(t)\| \, p\alpha^{p-1} > a\}.$$

Given $\delta > 0$, since ω_{n_i} converges to ω in measure μ, there exists a positive integer i_1, such that for $i \geqslant i_1$,

$$\mu\{t \mid \|\omega_{n_i}(t) - \omega(t)\| \, p\alpha^{p-1} > a\} < \delta/2.$$

Note that

$$\mu\left(\bigcup_{i=i_0}^{\infty} E_{n_i,1}\right) \leqslant \frac{1}{2^{i_0-1}},$$

let us choose i_0 such that $1/2^{i_0-1} < \delta/2$. Thus if $i_2 \geqslant \max\{i_0, i_1\}$, then

$$\mu\{t \in [\lambda, 1] \mid |g_{n_i}(t) - g(t)| > a\} < \delta, \qquad \forall i \geqslant i_2. \quad \blacksquare$$

LEMMA 4.4. *Let $\omega_n \in \Omega^p(X)$ be a Cauchy sequence. Then there is a subsequence g_{n_i} (of g_n) which is a Cauchy sequence in L^1. Here g_n is defined as in Lemma* 4.3.

Proof. For any $\varepsilon > 0$, since ω_n is a Cauchy sequence, there is N such that for any $n, m > N$,

$$\int_{(0,1]} \|\omega_n(t) - \omega_m(t)\|^p \, d\mu < \varepsilon.$$

Since function $g_N(t) = \|\omega_N(t)\|^p$ is integrable, there exists $\delta > 0$, such that when $\mu(E) < \delta$,

$$\int_E g_N(t) \, d\mu < \varepsilon.$$

Now note that by Lemma 4.2 and Corollary 4.1, ω_n converges to ω in measure μ. Let λ be such that $\mu[(0, \lambda)] < \delta/2$, by Lemma 4.3, there is a subsequence g_{n_i} which converges on the interval $[\lambda, 1]$ to g in measure μ. Thus for the above δ, there exists i_0, such that for all $i, j \geqslant i_0$, .

$$\mu\{t \mid |g_{n_i}(t) - g_{n_j}(t)| > \varepsilon\} < \delta/2.$$

Denote

$$E_{i,j,\varepsilon} = \{t \mid |g_{n_i}(t) - g_{n_j}(t)| > \varepsilon\}.$$

If i_1 is such that $i_1 \geqslant i_0$, and $n_{i_1} \geqslant N$, then for all $i, j \geqslant i_1$, we have

$$\begin{aligned}
|g_{n_i} - g_{n_j}|_{L^1} &= \int_{(0,1]} |g_{n_i}(t) - g_{n_j}(t)| \, d\mu \\
&= \int_{(0,\lambda)} |g_{n_i}(t) - g_{n_j}(t)| \, d\mu + \int_{[\lambda,1]} |g_{n_i}(t) - g_{n_j}(t)| \, d\mu \\
&\leqslant \int_{(0,\lambda)} |g_{n_i}(t) - g_{n_j}(t)| \, d\mu + \int_{E_{i,j,\varepsilon}} |g_{n_i}(t) - g_{n_j}(t)| \, d\mu \\
&\quad + \int_{[\lambda,1] \setminus E_{i,j,\varepsilon}} |g_{n_i}(t) - g_{n_j}(t)| \, d\mu \\
&\leqslant \int_{(0,\lambda) \cup E_{i,j,\varepsilon}} g_{n_i}(t) \, d\mu + \int_{(0,\lambda) \cup E_{i,j,\varepsilon}} g_{n_j}(t) \, d\mu + \varepsilon\mu((0,1])
\end{aligned}$$

$$\leqslant \left\{ \left[\int_{(0,\lambda)\cup E_{t,j,\varepsilon}} \|\omega_{n_i}(t) - \omega_N(t)\|^p \, d\mu \right]^{1/p} + \left[\int_{(0,\lambda)\cup E_{t,j,\varepsilon}} \|\omega_N(t)\|^p \, d\mu \right]^{1/p} \right\}^p$$

$$+ \left\{ \left(\int_{(0,\lambda)\cup E_{t,j,\varepsilon}} \|\omega_{n_j}(t) - \omega_N(t)\|^p \, d\mu \right]^{1/p} + \left[\int_{(0,\lambda)\cup E_{t,j,\varepsilon}} \|\omega_N(t)\|^p \, d\mu \right]^{1/p} \right\}^p + \varepsilon\mu((0,1])$$

$$\leqslant (\varepsilon^{1/p} + \varepsilon^{1/p})^p + (\varepsilon^{1/p} + \varepsilon^{1/p})^p + \varepsilon\mu((0,1])$$

$$= \{2^{p+1} + \mu((0,1])\} \cdot \varepsilon. \quad \blacksquare$$

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. Let $\omega_n \in \Omega^p(X)$ be a Cauchy sequence. From Lemma 4.2, and Corollary 4.1, there is an $\omega \in \Omega_M(X)$ such that ω_n converges to ω in measure μ. Let g_n and g be defined the same as in Lemma 4.3. We proceed as follows: first we show that $g \in L^1$ (therefore $\omega \in L^p$); then we prove that ω_n converges to ω when regarded as elements of $L^p((0,1], Rcc(X))$; and finally, since $\Omega^p(X)$ is closed in $L^p((0,1], Rcc(X))$ (Proposition 4.1), we conclude that $\omega \in \Omega^p(X)$.

Now let us show that $g \in L^1$.

From Lemma 4.4, there is subsequence g_k which is a Cauchy sequence in L^1. Let h_k $(k \geqslant 1)$ be integrable simple functions satisfying

$$\int_{(0,1]} |h_k(t) - g_k(t)| \, d\mu < 1/k$$

and

$$|h_k - g_k|_F < 1/k,$$

where the norm $|\cdot|_F$ is that of the space $F((0,1], R^1)$ (refer to Section 3). It is easy to see that h_k is also a Cauchy sequence in L^1. In fact,

$$|h_i - h_j|_{L^1} = \int_{(0,1]} |h_i(t) - h_j(t)| \, d\mu$$

$$\leqslant \int_{(0,1]} |h_i(t) - g_i(t)| \, d\mu + \int_{(0,1]} |g_i(t) - g_j(t)| \, d\mu$$

$$+ \int_{(0,1]} |g_j(t) - h_j(h)| \, d\mu$$

$$\to 0 \qquad (i, j \to \infty).$$

Also, h_k converges to g in measure μ since

$$|h_k - g|_F \leqslant |h_k - g_k|_F + |g_k - g|_F \to 0 \qquad (k \to \infty).$$

Therefore, g is integrable as expected.

Next we show that ω_n converges to ω in space $L^p((0, 1], Rcc(X))$. For any $\varepsilon > 0$, since ω_n converges to ω in measure μ, there exist A_n $(n = 1, 2, ...)$, which are subsets of $(0, 1]$, such that $\mu(A_n) \to 0$ $(n \to \infty)$, and

$$\|\omega_n(t) - \omega(t)\| < \varepsilon \qquad \forall t \notin A_n.$$

But ω_n is a Cauchy sequence in L^p and there is a positive integer N, such that whenever $n, m \geqslant N$,

$$\left[\int_{(0,1]} \|\omega_n(t) - \omega_m(t)\|^p \, d\mu\right]^{1/p} < \varepsilon.$$

Since real valued functions $\|\omega_N(t)\|^p$ and $\|\omega(t)\|^p$ are integrable, there exists $\delta > 0$, such that whenever $\mu(E) < \delta$, we have

$$\left[\int_E \|\omega_N(t)\|^p \, dt\right]^{1/p} < \varepsilon,$$

and

$$\left[\int_E \|\omega(t)\|^p \, dt\right]^{1/p} < \varepsilon.$$

From $\mu(A_n) \to 0$ $(n \to \infty)$, we can find N_1 such that for all $n \geqslant N_1$, $\mu(A_n) < \delta$. Therefore, if $n \geqslant \max\{N, N_1\}$, we have

$$\begin{aligned}
(|\omega_n - \omega|_{L^p})^p &= \int_{(0,1]} \|\omega_n(t) - \omega(t)\|^p \, d\mu \\
&= \int_{A_n} \|\omega_n(t) - \omega(t)\|^p \, d\mu + \int_{A_n^c} \|\omega_n(t) - \omega(t)\|^p \, d\mu \\
&\leqslant \left\{\left[\int_{A_n} \|\omega_n(t) - \omega_N(t)\|^p \, d\mu\right]^{1/p}\right. \\
&\quad \left. + \left[\int_{A_n} \|\omega_N(t) - \omega(t)\|^p \, d\mu\right]^{1/p}\right\}^p + \int_{A_n^c} \|\omega_n(t) - \omega(t)\|^p \, d\mu \\
&\leqslant \left\{\left[\int_{A_n} \|\omega_n(t) - \omega_N(t)\|^p \, d\mu\right]^{1/p} + \left[\int_{A_n} \|\omega_N(t)\|^p \, d\mu\right]^{1/p}\right. \\
&\quad \left. + \left[\int_{A_n} \|\omega(t)\|^p \, d\mu\right]^{1/p}\right\}^p + \int_{A_n^c} \|\omega_n(t) - \omega(t)\|^p \, d\mu \\
&\leqslant (\varepsilon + \varepsilon + \varepsilon)^p + \varepsilon^p \mu(A_n^c) \\
&\leqslant \{3^p + \mu((0, 1])\}\varepsilon^p.
\end{aligned}$$

Hence ω_n converges to ω in the space $L^p((0, 1], Rcc(X))$. But $\omega_n \in \Omega^p(X)$ and $\Omega^p(X)$ is closed in $L^p((0, 1], Rcc(X))$ (proposition 4.1), and we see that $\omega \in \Omega^p(X)$, which completes the proof. ▮

5. CONSLUSION

In this article, by treating a fuzzy set as a tower of (classical)subsets, we introduced several convergence concepts of fuzzy sets. These include "convergent in measure," "convergent uniformly," "convergence μ-uniformly," and "L^p-convergent." The L^p-convergence concept is defined based on an embedding of the space of fuzzy sets in a normed linear space. We proved that the space of fuzzy sets under the embedding is a complete metric space. Finally, using this framework, we can also study the calculus of fuzzy functions in a way similar to Puri and Ralescu [11].

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