Nonlinear problem of active sound control

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We consider the problem of active sound control, in which some domain is protected from the field generated outside. The active shielding is realized via the implementation of additional sources in such a way that the total contribution of all sources leads to the wanted effect. Mathematically the problem is reduced to seeking the source terms satisfying some *a priori* described requirements and belongs to the class of inverse source problems. From the application standpoint, this problem can be closely related to active noise shielding and active vibration. It is important that along with unwanted field to be shielded a wanted field is accepted in the analysis. The solution to the problem requires only the knowledge of the total field at the perimeter of the shielded domain. For the first time the active shielding sources are obtained for the nonlinear statement of the problem. It is obtained via the theory of potentials, and the solution is represented in the form of a simple layer. For this purpose, the theory of Calderón–Ryaben'kii potentials is first extended to nonlinear formulations. In the solution, we also take into account the feedback of the secondary sources on the input data.

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1. Introduction

In the problem of active shielding (AS) some domain is to be protected from the field (noise) generated outside. It is realized via implementation of additional sources in such a way that the total contribution of all sources leads to the wanted effect. Mathematically the problem is reduced to the search of the source terms, which satisfy some *a priori* described requirements to the solution to an appropriate boundary value problem (BVP). Thus, the problem in question belongs to the class of inverse source problems [1]. Reviews of the theoretical and experimental methods related to this problem can be found in [2–8]. Most theoretical approaches assume some quite detailed information about the unwanted sources and the properties of the medium. The JMC method by Jessel, Mangiante and Canévet, see, e.g, [5,9], based on Huygens' source reconstruction, requires only the information on the unwanted field at the perimeter of the shielded domain. Yet this method is not used in case if a wanted field, generated in the domain to be protected, has to be taken into account. In addition, the JMC method has only been applied to linear problems formulated in unbounded domains.

In [10,11], the linear AS problem is solved in a finite-difference formulation via the difference potentials. The solution requires only the knowledge of the total field at the grid boundary of the shielded domain. This solution is extended to hyperbolic systems of equations in [8]. The general solution for the case of Helmholtz-type equation with variable coefficients is obtained in [12] and studied in detail in [7,13,14]. In [15], the problem of AS in composite domains is formulated for the first time and its general finite-difference solution is obtained. The multi-domain problem of AS in a continuous space is solved in [16]. In [17], the solution to the AS problem is extended to a continuous space for quite arbitrary linear BVPs with constant and variable coefficients. A nonstationary AS problem is tackled via generalized nonstationary potentials in [18]. As proven in [19], the obtained solution is even applicable to resonance regimes. A sensitivity analysis,

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done in [19], shows that the solution is robust. Obviously, this result is important for practical applications. A variable degree of noise attenuation has recently been considered in paper [20]. The formulation of the problem corresponds to quite general active sound control in composite domains.

In contrast to the previous studies, in the current paper we consider a nonlinear formulation. For the first time, the nonlinear AS problem is solved under quite general conditions. The solution to the problem is based on the total field supposed to be measured on the boundary of the domain to be shielded. In a practical realization this field would be disturbed by the field generated by the secondary (additional) sources. An approach to resolve this diffraction problem is suggested. It requires solving an auxiliary BVP. The BVP can efficiently be formulated and solved using the Difference Potential Method in [10]. The application of the AS solution obtained in the general formulation is demonstrated on the examples of the Linearized Euler Equations for aeroacoustics and the nonlinear Euler equations.

The general solution to the linear problem is obtained via the theory of potentials developed for linear BVPs in [21,22,10,23]. It was Calderón who introduced the potentials for general elliptic problems. Ryaben'kii developed the theory of potentials for general BVPs in finite-difference spaces. There is a deep analogy between these two theories, which were historically elaborated completely independently for a long time. The discussion of this subject is beyond the scope of the current paper. Some details can be found in [10,23]. To tackle the nonlinear AS problem, the theory of the potentials is extended to nonlinear formulations. One should note that the definition of the nonlinear potential and its properties proved in this paper have a general meaning and can be applied beyond the AS problem.

2. General formulation of the AS problem

We consider the following generalized mathematical formulation of the AS problem. Assume that some field (sound) W is described by a BVP supposed to be well posed in a domain $D \subseteq \mathbb{R}^m$:

$$L(W) \equiv \sum_1^m \frac{\partial F^i}{\partial x^i} + b = f, \tag{1}$$

$$W \in \Xi_D, \tag{2}$$

where $\{x^i\}$ is a Cartesian coordinate system; W, f, b and F^i are vector-functions with the dimension of p; $b = b(W) \in C$, $F^i = F^i(W) \in C^1, F^i(0) \equiv 0, (i = 1, \ldots, m)$; Ξ_D is some space of functions specified below. Here, operator L can be, generally speaking, nonlinear.

The solution to BVP (1), (2) is considered in the generalized sense [24]. For this purpose we introduce the space of basis functions $\Phi \in C_0^\infty(D)$. Eq. (1) is then considered in the weak sense:

$$\langle L(W), \Phi \rangle = \langle f, \Phi \rangle$$

for any $\Phi \in C_0^\infty(D)$, where $\langle \cdot, \cdot \rangle$ means a distribution.

Along with a distribution Ψ, we introduce restriction (or local element [24]) Ψ_Ω of Ψ to a domain $\Omega \subset D$: for any test function Φ with supp $\Phi \subset \Omega$, the following equality holds

$$\langle \Psi_\Omega, \Phi \rangle = \langle \Psi, \Phi \rangle.$$

For our further consideration, we also introduce characteristic function θ_Ω of a domain Ω: $\Omega \subset D, \theta_\Omega(x) = 1$ if $x \in \Omega$, $\theta_\Omega(x) = 0$ if $x \in D \setminus \Omega$. Thus, e.g., $\theta_{\overline{\Omega}} g$ means the part of g supported on $\overline{\Omega}$.

Next, we consider a bounded domain D^+: $\overline{D^+} \subset D$, having smooth enough boundary Γ. The sources on the right-hand side of (1) can be situated both in D^+ and outside D^+:

$$f = f^+ + f^-, \tag{3}$$
$$\text{supp} f^+ \subset D^+,$$
$$\text{supp} f^- \subset D^- \equiv D \setminus \overline{D^+}.$$

Here, f^+ is the source of a wanted field (sound), while f^- is the source of an unwanted field (noise). It is supposed that f is a regular function at least in some vicinity of the interface boundary $\Gamma = \overline{D^+} \cap \overline{D^-}$.

Assume that $f \in F_D$, where F_D is a linear space of functions f for which the solution to BVP (1), (2) exists and unique. We also suppose that: $f \in F_D \implies \theta_{D^+} f \in F_D$; $W \in \Xi_D \implies L(W) \in F_D$ and $L(0) = 0$. In addition, in the domain D^+ we require that if each of any two external sources does not generate any field (in D^+), then their superposition does not generate either. In other words, functions $L(\theta_{D^-} U) : U \in \Xi_D$ create a linear manifold in F_D.

We define the functional space Ξ_D as follows. The functions from Ξ_D are piece-wise smooth, bounded and satisfy some boundary conditions on ∂D that may be inherited from physics. More precisely, we assume that

$$\Xi_D \subset H^s(D^+) \cap H^s(D^-),$$
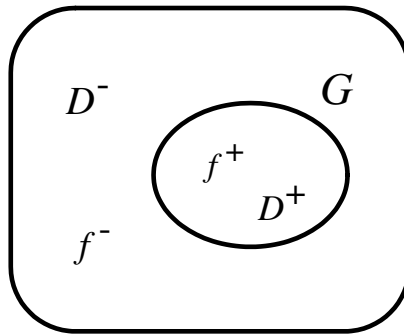
where $s > 1/2$, H^s is the Sobolev space.

Fig. 1. Domain sketch.

Suppose that we know the field $W(\Gamma)$ on the boundary Γ. We assume that only this information is available from, say, measurements. In particular, the distribution of the sources f on the right-hand side is unknown.

The AS problem is reduced to the search of additional sources G on $\overline{D^-}$ (see Fig. 1) such that the solution to the following BVP

$$L(W') = f + G,\tag{4}$$
$$\text{supp}\, G \subset \overline{D^-},$$
$$W' \in \varXi_D$$

coincides on the subdomain D^+ with the solution to BVP (1), (2) if $f^- \equiv 0$:

$$L(W^+) = f^+,\tag{5}$$
$$W^+ \in \varXi_D.$$

Thus, we seek G such that

$$W'_{D^+} = W^+_{D^+}.\tag{6}$$

It is to be noted that the statement of the problem given above is a generalization of the statement of linear AS problem [10,7,17]. It appears that the AS problem in question has a solution in the form of a simple-layer source term under quite general conditions.

The general solution to the linear AS problem is obtained via the theory of the Calderón–Ryaben'kii potentials [10], [23]. Then, this approach can be extended to the nonlinear formulation.

3. Calderón–Ryaben'kii potentials

Let us consider a linear BVP (1), (2) such that

$$F^i = A^i(\mathbf{x})W \quad (i = 1, \ldots, m),\tag{7}$$

where $A^i(\mathbf{x}) \in C^1(D)$, $b = B(\mathbf{x})W$, $B(\mathbf{x}) \in C(D)$.

One can introduce an operator $P_{D^+} : \varXi_{D^+} \to \varXi_{D^+}$, $\varXi_{D^+} = \{U_{D^+} \mid U \in \varXi_D\}$ as follows. The classical definition [10] is given by:

Definition 1.

$$P_{D^+} V_{D^+}(\mathbf{x}) \stackrel{def}{=} V_{D^+} - \int_{D^+} Gr(\mathbf{x}, \mathbf{y}) LV(\mathbf{y})\mathrm{d}\mathbf{y}.$$

Here, Gr is Green's function of the linear BVP (1), (2).

We can also introduce the following definition of the operator P_{D^+} via distributions, which does not exploit Green's function:

Definition 2.

$$P_{D^+} V_{D^+} \stackrel{def}{=} L_{D^+}^{-1}(\theta_{\overline{D^-}} LV),\tag{8}$$

where $L_{D^+}^{-1} g \stackrel{def}{=} L^{-1} g_{|D^+}$.

It appears that in the case of regular functions these definitions are equivalent [23].

Assume now that $U^+ = L^{-1}f^+$ and $U^- = L^{-1}\overline{f}^-$; here, $U^+, U^- \in \Xi_D$, $\text{supp}\,\overline{f}^- \subset \overline{D^-}$. Then, one can immediately verify the following important properties of the potential [10,15]:

$$P_{D+}U_{D+}^+ = 0_{D+}, \tag{9}$$

and

$$P_{D+}U_{D+}^- = U_{D+}^-. \tag{10}$$

Hence, from the linearity of the problem we obtain:

$$P_{D+}U_{D+} = U_{D+}^-, \tag{11}$$

where $U = L^{-1}(f^+ + \overline{f}^-)$.

Thus, the field generated in D^+ does not contribute to the potential P_{D+} determined on this domain, while the field generated outside D^+ is projected by the operator P_{D+} onto itself. Hence, the operator P_{D+} operates as a projection [10].

Next, following [10,23] we introduce the notion of a clear trace $\text{Tr}(\Gamma)U_{D+}$:

$$\text{Tr}(\Gamma)V_{D+} = \text{Tr}(\Gamma)W_{D+} \Rightarrow P_{D+}V_{D+} = P_{D+}W_{D+}. \tag{12}$$

Here, $V, W \in \Xi_D$, $\text{Tr}(\Gamma)$ is a boundary operator: $\Xi_D \to \Xi_\Gamma$, $\Xi_\Gamma \subset H^{s-1/2}(\Gamma)$ [23].

Then, we arrive at the definition of the potential with density ξ_Γ:

$$P_{D+\Gamma}\xi_\Gamma \stackrel{def}{=} P_{D+}V_{D+}, \tag{13}$$

where $\xi_\Gamma = \text{Tr}(\Gamma)V$. Thus, the value of the potential $P_{D+\Gamma}\xi_\Gamma$ is fully determined by its density ξ_Γ [10].

Next, let us introduce a trace operation as follows. Let Γ_ϵ^+ be smooth manifolds parallel to Γ in the sense of [25, Ch. 2]: $\Gamma_\epsilon^+ \subset D^+$, $\Gamma_\epsilon^+ \to \Gamma$ if $\epsilon \to 0$. Then, the trace operator $\text{Tr}_\Gamma^+ : H^s(D^+) \to H^{s-1/2}(\Gamma)$ is given by

$$\text{Tr}_\Gamma^+ U_{D+} \stackrel{def}{=} \lim_{\epsilon \to 0} \text{Tr}_{\Gamma_\epsilon^+} U_{D+}, \tag{14}$$

where

$$\text{Tr}_{\Gamma_\epsilon^+} U_{D+} \stackrel{def}{=} U_{D+}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_\epsilon^+.$$

Similar, in D^- we introduce the trace operator $\text{Tr}_\Gamma^- : H^s(D^-) \to H^{s-1/2}(\Gamma)$ and

$$\text{Tr}_\Gamma^- U_{D-} \stackrel{def}{=} \lim_{\epsilon \to 0} U_{D-}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_\epsilon^-. \tag{15}$$

One can see that for the case of operator L determined by (1), (7) the clear trace of potential P_{D+} is given by Tr_Γ^+.

The potential $P_{D+\Gamma}\xi_\Gamma$ can be found by the following proposition proven in [23].

Proposition 1.

$$P_{D+\Gamma}\xi_\Gamma = -L_{D+}^{-1}A_n\xi_\Gamma\delta(\Gamma), \tag{16}$$

where $\xi_\Gamma = \text{Tr}_\Gamma^+V_{D+}$, $V \in \Xi_D$, $A_n = \sum A^i n_i$, $\mathbf{n}$ is the outward normal to the boundary Γ.

4. Nonlinear potentials

Let us now consider the nonlinear formulation of the problem. For this purpose we introduce a nonlinear potential. It is easy to see that Definition 2, in fact, can immediately be applied to the nonlinear problem:

Definition 3.

$$P_{D+}(V_{D+}) \stackrel{def}{=} L_{D+}^{-1}(\theta_{\overline{D^-}}L(V)). \tag{17}$$

Definition 3 can be rewritten as the set of simultaneous equations:

$$\begin{cases} P_{D+}(V_{D+}) = \text{Gr}_{D+}(g - g_{D+}), \\ \text{Gr}_{D+}(g) = V_{D+}, \end{cases} \tag{18}$$

where $V \in \Xi_D$, $L(V) \in F_D$, $\text{Gr} := L^{-1}$ is Green's operator, $\text{Gr}_{D+}(g) \stackrel{def}{=} L_{D+}^{-1}(g)$.

Then, we arrive at the following identity:

$$P_{D+}(Gr_{D+}(g)) = Gr_{D+}(g - g_{D+}). \tag{19}$$

Equality (19) can be considered as the generalized nonlinear Green's identity.

Next, for any test function Φ we have

$$
\begin{aligned}
\langle L(V), \Phi \rangle &= -\sum_1^m \langle F^i, \nabla_i \Phi \rangle + \langle b, \Phi \rangle = -\sum_1^m \int_D F^i \nabla_i \Phi \, d\mathbf{x} + \langle b, \Phi \rangle \\
&= -\sum_1^m \int_{D^+} F^i \nabla_i \Phi \, d\mathbf{x} - \sum_1^m \int_{D^-} F^i \nabla_i \Phi \, d\mathbf{x} + \langle b, \Phi \rangle \\
&= -\sum_1^m \int_{D^+} \nabla_i (F^i \Phi) \, d\mathbf{x} + \sum_1^m \int_{D^+} \nabla_i F^i \Phi \, d\mathbf{x} - \sum_1^m \int_{D^-} \nabla_i (F^i \Phi) \, d\mathbf{x} + \sum_1^m \int_{D^-} \nabla_i F^i \Phi \, d\mathbf{x} + \langle b, \Phi \rangle \\
&= \langle \{L(V)\}, \Phi \rangle + \int_\Gamma [F_n(V)]_\Gamma \, \Phi = \langle \{L(V)\} + [F_n(V)]_\Gamma \, \delta(\Gamma), \Phi \rangle.
\end{aligned}
\tag{20}
$$

Here, $\{L(V)\}$ is the part of $L(V)$ supported on $D \setminus \Gamma$, $F_n = \sum_1^m F^i n_i$, $[.]_\Gamma$ means the discontinuity across the boundary Γ:

$$
[W]_\Gamma \overset{def}{=} \mathrm{Tr}_\Gamma^- W - \mathrm{Tr}_\Gamma^+ W.
$$

Hence,

$$
g - g_{D^+} = \theta_{D^-} L(V) + [F_n(V)]_\Gamma \, \delta(\Gamma).
\tag{21}
$$

Assume that $\xi_\Gamma = \mathrm{Tr}_\Gamma^+ V_{D^+}$. Then,

$$
g - g_{D^+} = L(\theta_{D^-} V) - F_n(\xi_\Gamma) \delta(\Gamma)
\tag{22}
$$

and the potential $P_{D\Gamma}(\xi_{\Gamma+})$ can be represented by

$$
P_{D^+ \Gamma}(\xi_\Gamma) = \mathrm{Gr}_{D^+} (L(\theta_{D^-} V) - F_n(\xi_\Gamma) \delta(\Gamma)),
\tag{23}
$$

where $V \in \varXi_D$.

It appears that, as in the linear case, the potential, in fact, only depends on its density.

Proposition 2. *The potential $P_{D^+}(U_{D^+})$ is fully determined by $\xi_\Gamma = \mathrm{Tr}_\Gamma^+ U_{D^+}$.*

Proof. The solution to BVP

$$
\begin{aligned}
&L(V) = g - g_{D^+} \equiv L(\theta_{D^-} U) - F_n(\xi_\Gamma)\delta(\Gamma), \\
&V \in \varXi_D
\end{aligned}
$$

is given by $V = U^-$, where $U^- = \mathrm{Gr}(f^-)$.

Next, consider $\widehat{V} = \mathrm{Gr}(-F_n(\xi_\Gamma)\delta(\Gamma))$. Then,

$$
L(V) = L(\theta_{D^-} U) + L(\theta_{D^-} \widehat{V}) + L(\theta_{D^+} \widehat{V}).
$$

Hence, $L(\theta_{D^-} V) = L(\theta_{D^-} U) + L(\theta_{D^-} \widehat{V})$ and $\theta_{D^+} V = \theta_{D^+} U^-$.

Let us change g by $\widetilde{g} \in F_D$ on D^- in such a way that $\mathrm{Tr}_\Gamma^+ \mathrm{Gr}(\widetilde{g}) = \mathrm{Tr}_\Gamma^+ \mathrm{Gr}(g) = \mathrm{Tr}_\Gamma^+ U_{D^+}$. Then,

$$
\widetilde{g} - g_{D^+} = L(\theta_{D^-} \widetilde{U}) - F_n(\xi_\Gamma)\delta(\Gamma),
$$

where $\widetilde{U} = \mathrm{Gr}(\widetilde{g})$. Since $\widetilde{g} - g_{D^+} \in F_D$, we obtain

$$
\mathrm{Gr}(\widetilde{g} - g_{D^+}) =
\begin{cases}
U_{D^+}^- & \text{if } \mathbf{x} \in D^+, \\
\widetilde{V}_{D^-}^- & \text{if } \mathbf{x} \in D^-,
\end{cases}
\tag{24}
$$

where $L(\theta_{D^-} \widetilde{V}) = L(\theta_{D^-} \widetilde{U}) + L(\theta_{D^-} \widehat{V})$, $\widetilde{V} \in \varXi_D$.

Thus, $P_{D^+ \Gamma}(\xi_\Gamma) = U_{D^+}^-$. $\quad \square$

Corollary 1. *We can now generalize Proposition 1.*

$$
P_{D^+ \Gamma}(\xi_\Gamma) = \mathrm{Gr}_{D^+}(-F_n(\xi_\Gamma)\delta(\Gamma)).
\tag{25}
$$

It is immediately obtained from (24) by setting $\widetilde{U}_{D^-} = 0_{D^-}$.

It is important to note that, similar to the linear case, the nonlinear potential is a projection.

One can see that if supp $L(V) \subset D^+$, $V \in \varXi_D$, then from Definition 3 it immediately follows that $P_{D+}(V_{D+}) = 0_{D+}$. In turn, if supp $L(V) \subset D^-$, then $P_{D+}(V_{D+}) = V_{D+}$.

In the case of BVP (1), (2),

$$P_{D+}(W) \stackrel{def}{=} L_{D+}^{-1}(f^-) = W_{D+}^-,$$
(26)

where $W^- = L^{-1}(f^-)$.

Thus, the nonlinear potential operator is also a projection.

Finally, let us consider requirements to the problem, under which functions $L(\theta_{D-}U) : U \in \varXi_D$ create a linear subspace. For this purpose, we introduce sets

$$F_D^- = \{f \,|\, \exists U \in \varXi_D : f = L(\theta_{D-}U)\}$$

and

$$F_\Gamma^- = \{f \,|\, \exists U \in \varXi_D : f = L(\theta_{D-}U), \text{supp} f \in \Gamma\}.$$

One can see that F_D^- is the factorspace of F_D with regard to F_Γ^-. Thus, F_D^- is a linear space provided that F_Γ^- is a linear space. Obviously, the latter statement is true if the operator L is linear in any vicinity of the boundary Γ. In a general case, we can linearize the operator L in a vicinity of Γ. It can be done if Γ nowhere coincides with the characteristic hypersurface. This follows from the Cauchy–Kovalevskaya Theorem; see, e.g., [26].

5. Solution to the nonlinear AS problem

Let us now consider the solution to the nonlinear AS problem. It can be obtained via the nonlinear potentials introduced in the previous section.

From Proposition 2 and the projection property, it follows that if W_Γ is the total field at the boundary Γ, then $P_{D+\Gamma}(W_\Gamma) = W_{D+}^-$. Hence, function $P_{D+\Gamma}(W_\Gamma)$ gives us the field to be annihilated in D^+.

Similar to the linear case, the condition of the noise cancelation on D^+ is the following:

$$P_{D+}(W') = L_{D+}^{-1}(\theta_{\overline{D-}}L(W')) = 0_{D+}.$$

Since

$$\theta_{\overline{D-}}L(W') = \theta_{\overline{D-}}L(W) + G = L(\theta_{\overline{D-}}W) - F_n(W_\Gamma)\delta(\Gamma) + G,$$

we arrive at the following condition:

$$L_{D+}^{-1}\left(L(\theta_{\overline{D-}}W) - F_n(W_\Gamma)\delta(\Gamma) + G\right) = 0_{D+}.$$

One can choose $G = -\theta_{D-}L(W) = F_n(W_\Gamma)\delta(\Gamma) - L(\theta_{D-}W)$, and this source term provides, of course, the exact noise cancelation in the entire domain D. Meanwhile, if we require noise cancelation only in D^+, then we can only retain the single-layer source term

$$G_0 = F_n(W_\Gamma)\delta(\Gamma).$$
(27)

It is clear that if the problem in question is linear, then

$$G_0 = A_n W_\Gamma \delta(\Gamma).$$

This solution to the AS problem coincides with the one obtained in [17].

It is to be noted that the AS solution (27) does not explicitly depend on the boundary conditions. Although the boundary conditions are not explicitly specified, we are able to obtain the AS source term if the BVP in question is well posed.

6. Feedback of the secondary sources. Nonstationary effects

The realization of the secondary source (27) is based on the knowledge (measurement) of W_Γ. Once the AS source is implemented, the field both in the shielded domain D^+ and outside changes. Moreover, the field W' is a piece-wise function having a discontinuity across the boundary Γ. Thus, the implementation of the AS source results in some uncertainty especially in a nonstationary case. To overcome this problem, we suggest the following procedure.

Let us consider the potential $P_{D+\Gamma}(W'_{\Gamma-})$ where $W'_{\Gamma-} \stackrel{def}{=} \text{Tr}_\Gamma^- W'$. It corresponds to consideration of domain $D_\epsilon^+ \supset D^+$ bounded by $\Gamma_\epsilon^- : \Gamma_\epsilon^- \subset D^-$, $\Gamma_\epsilon^- \to \Gamma$ if $\epsilon \to 0$. Then,

$$P_{D+\Gamma}(W'_{\Gamma-}) = \lim_{\epsilon \to 0} P_{D_\epsilon^+ \Gamma_\epsilon^-}(W'_{\Gamma_\epsilon^-}).$$

Thus, thanks to the projection property (26), we arrive at the following equality:

$$P_{D+\Gamma}(W'_{\Gamma-}) = W_{D+}^-.$$

Hence, the potential $P_{D+\Gamma}(W'_{\Gamma-})$ gives us the field to be canceled.

Next, we have

$$P_\Gamma(W'_\Gamma) \stackrel{def}{=} \mathrm{Tr}(\Gamma)P_{D+\Gamma}(W'_{\Gamma-}) = W_\Gamma^-.$$

Then, from (27), the simple-layer AS source term can be determined by

$$G'_0 = F_n(P_\Gamma(W'_{\Gamma-}))\delta(\Gamma) \equiv F_n(W_\Gamma^-)\delta(\Gamma). \tag{28}$$

This AS source term leads to the cancelation of the noise field W_{D+}^-. Indeed, let us now consider the following BVP:

$$L(\overline{W'}) = f^- + F_n(W_\Gamma^-)\delta(\Gamma),$$
$$\overline{W'} \in \Xi_D.$$

Its solution is given by

$$\overline{W'} = \begin{cases} 0 & \text{on } D^+, \\ W^- & \text{on } D^-. \end{cases} \tag{29}$$

Hence, $P_{D+}(W'_{D+}) = L_{D+}^{-1}(f^- + F_n(W_\Gamma^-)\delta(\Gamma)) = 0_{D+}$, and the field in D^+ is fully determined by the internal sources.

Thus, the measurements should be done on the external boundary, and the realization requires the solution of a BVP because the operator P_Γ is non-local.

In the linear case, to calculate the potential, we can apply a spectral method as used in the Method of Difference Potentials [10]. In this approach, a set of basis functions $\{\phi_j(\Gamma)\}$, $(j = 1, \ldots, N)$ is introduced on Γ. The potentials $P_\Gamma\phi_j(\Gamma)$ can be calculated in advance. Then, the potential $P_\Gamma W'_\Gamma$ can quickly be obtained as soon as we approximate W'_Γ by the basis functions. In the nonlinear case, of course, the problem becomes more complicated.

One should note that the AS solution (27) is also applicable to a nonstationary AS problem in $\mathbb{R}^{m+1}$ with homogeneous initial data in the cylinder $K_T = D \times (0, T)$ $(T > 0)$:

$$L(W) \equiv W_t + \sum_1^m \frac{\partial F^i}{\partial x^i} = f, \tag{30}$$
$$W \in \Xi_D,$$
$$W(\mathbf{x}, 0) = 0.$$

In addition to the stationary formulation, we assume that the space Ξ_D consists of the functions smooth enough with respect to the time variable: $\Xi_D \subset C^p(K_T)$ and satisfying homogeneous initial conditions. We consider the generalized solution to initial BVP (30):

$$\int_0^T \int_D (L(W) - f, \Phi)\mathrm{d}\mathbf{x}\mathrm{d}t = 0, \tag{31}$$

for any $\Phi \in C_0^\infty(K_T)$, where $(\cdot, \cdot)$ means a scalar product. The proof, then, mostly repeats the stationary case with the replacement of b by W_t.

Next, we consider two examples illustrating the application of the AS solution, obtained in the general formulation, to particular governing equations.

7. Linearized Euler equations

First, we consider a linear acoustic problem described by the Linearized Euler equations:

$$\frac{1}{\rho_0 c_0^2}(p'_t + (\mathbf{u}_0, \nabla)p') + \frac{1}{\rho_0 c_0^2}(\mathbf{u}', \nabla)p_0 + \nabla \cdot \mathbf{u}' = \frac{1}{\rho_0 c_0^2}f^{(p)} + q_{vol}, \tag{32}$$
$$\rho_0(\mathbf{u}'_t + (\mathbf{u}_0, \nabla)\mathbf{u}' + (\mathbf{u}', \nabla)\mathbf{u}_0) + \nabla p' = \mathbf{f}^{(u)} + \mathbf{f}_{vol},$$

where $(\mathbf{a}, \mathbf{c})$ denotes the scalar product of vectors $\mathbf{a}$ and $\mathbf{c}$, $\mathbf{u}'$ is the particle velocity, p' is the sound pressure, c_0 is the sound speed; the velocity $\mathbf{u}_0$ and the pressure $\mathbf{p}_0$ correspond to some main "reference" flow; q_{vol} is the volume velocity per a unit volume and $\mathbf{f}_{vol}$ is the force per a unit volume [3].

In this case, we have

$$W = (u'_1, u'_2, u'_3, p')^T, \tag{33}$$

where u'_j $(j = 1, 2, 3)$ are the coordinates of $\mathbf{u}'$ in a Cartesian coordinate system.

Then, the matrix A_n is given by

$$A_n = \begin{pmatrix} n_1 & n_2 & n_3 & \dfrac{v_n}{\rho_0 c_0^2} \\ \rho_0 v_n & 0 & 0 & n_1 \\ 0 & \rho_0 v_n & 0 & n_2 \\ 0 & 0 & \rho_0 v_n & n_3 \end{pmatrix}, \tag{34}$$

where $v_n = \mathbf{u}_0 \cdot \mathbf{n}$.

Thus, from (7) and (27) we obtain the following AS source terms:

$$q_{vol} = (\mathbf{u}' \cdot \mathbf{n}_{|\Gamma} + \frac{v_n}{\rho_0 c_0^2} p'_{|\Gamma}) \delta(\Gamma), \tag{35}$$

$$\mathbf{f}_{vol} = (p'_{|\Gamma}\mathbf{n} + \rho_0 v_n \mathbf{u}'_{|\Gamma}) \delta(\Gamma).$$

It can be seen that the AS terms (35) take into account flux through the boundary Γ.

The other example demonstrates the application of the general AS solution (27) to a nonlinear problem.

8. Euler equations

Let us now consider the nonlinear Euler equations:

$$L(U) = U_t + \sum_{1}^{3} F^i(U)_{x_i}, \tag{36}$$

where

$$U = (\rho, \rho u_1, \rho u_2, \rho u_3, E)^T, \tag{37}$$

$$F^i(U) = u_i U + p(0, \delta_{1i}, \delta_{2i}, \delta_{3i}, u_i)^T \tag{38}$$

where ρ is the density; u_1, u_2, u_3 are the velocity coordinates in a Cartesian coordinate system $\{x_i\}$ ($i = 1, 2, 3$); E is the total energy density; p is the pressure; $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$.

The AS solution is then given by:

$$G_0 = \delta(\Gamma) \tag{39}$$

$$\times (\rho V_n, \rho u_1 V_n + p n_1, \rho u_2 V_n + p n_2, \rho u_3 V_n + p n_3, H V_n)_\Gamma^T \tag{40}$$

where $V_n = \mathbf{u} \cdot \mathbf{n}, H = E + p$.

9. Conclusion

The theory of Calderón–Ryaben'kii potentials has been used to obtain the general solution to the linear AS problem. The solution only requires the knowledge of the total field (wanted and unwanted) at the perimeter of the shielded domain. The theory of the potentials has been extended to nonlinear BVPs. Green's identity has been derived for the nonlinear potential. On the basis of the nonlinear potentials, the solution to the inverse source nonlinear AS problem has been obtained in the form of a simple-layer source term. The solution is able to take into account the feedback of the secondary sources on the field measured on the boundary. The obtained AS solutions can be applied to the Linearized Euler Equations and the nonlinear Euler equations.

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