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Some new results on convolutions of heterogeneous gamma random variables☆

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ABSTRACT

Convolutions of independent random variables often arise in a natural way in many applied areas. In this paper, we study various stochastic orderings of convolutions of heterogeneous gamma random variables in terms of the majorization order [*p*-larger order, reciprocal majorization order] of parameter vectors and the likelihood ratio order [dispersive order, hazard rate order, star order, right spread order, mean residual life order] between convolutions of two heterogeneous gamma sets of variables wherein they have both differing scale parameters and differing shape parameters. The results established in this paper strengthen and generalize those known in the literature.

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1. Introduction

Convolutions of independent random variables often arise in a natural way in many applied areas including applied probability, reliability theory, actuarial science, nonparametric goodness-of-fit testing, and operations research. Since the distribution theory is quite complicated when the convolution involves independent and non-identical random variables, it is of great interest to investigate stochastic properties of convolutions and derive bounds and approximations on some characteristics of interest in this setup. Many results in this direction have appeared in the literature; see, for example, [3,20,4,11,14,9,16,23–29,12,13,2,5]. Because exponential distribution has a nice mathematical form and the unique memoryless property, most of these references treated only the convolutions of exponential random variables. It is well known that gamma distribution is one of the most commonly used distributions in statistics, reliability and life testing that includes exponential distribution as its special case (when its shape parameter is 1). Moreover, the gamma distribution can be widely applied in actuarial science as most total insurance claim distributions have quite similar shape to gamma distributions: non-negatively supported, skewed to the right and unimodal (see [7]). Let X be a gamma random variable with the shape parameter r and scale parameter λ. Then, in its standard form X has the probability density function

$$f(x; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} \exp(-\lambda x), \quad x > 0.$$

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It is an extremely flexible family of distributions with decreasing, constant, and increasing failure rates when $0 < r < 1$, $r = 1$ and $r > 1$, respectively. In this paper, various stochastic orders are studied for convolutions of heterogeneous gamma random variables.

We shall be using the concepts of majorization and related orders in our discussion. The notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \le \cdots \le x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 1.1. (i) A vector $\mathbf{x} = (x_1, \ldots, x_n) \in \Re^n$ is said to majorize another vector $\mathbf{y} = (y_1, \ldots, y_n) \in \Re^n$ (written as $\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$;

(ii) A vector $\mathbf{x} \in \Re^n$ is said to weakly supmajorize another vector $\mathbf{y} \in \Re^n$ (written as $\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n;$$

(iii) A vector $\mathbf{x} \in \Re_+^n$ is said to be p-larger than another vector $\mathbf{y} \in \Re_+^n$ (written as $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \le \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

Clearly, $\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y}$ implies $\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y}$, and $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$ is equivalent to $\log(\mathbf{x}) \overset{\text{w}}{\succeq} \log(\mathbf{y})$, where $\log(\mathbf{x})$ is the vector of logarithms of the coordinates of $\mathbf{x}$. Also, Khaledi and Kochar [8] showed that $\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y}$ implies $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \Re_+^n$. The converse is, however, not true. For example, $(1, 5.5) \overset{\text{p}}{\succeq} (2, 3)$, but clearly the majorization order does not hold.

For more details on majorization and p-larger orders and their applications, see [15,4,8]. Recently, Zhao and Balakrishnan [25] introduced a new partial order, called reciprocal majorization order.

Definition 1.2. The vector $\mathbf{x} \in \Re_+^n$ is said to reciprocal majorize another vector $\mathbf{y} \in \Re_+^n$ (written as $\mathbf{x} \overset{\text{rm}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \ge \sum_{i=1}^{j} \frac{1}{y_{(i)}}$$

for $j = 1, \ldots, n$.

From [12], the following implication holds:

$$\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{rm}}{\succeq} \mathbf{y}$$

for any two non-negative vectors $\mathbf{x}$ and $\mathbf{y}$. On the other hand, the $\overset{\text{rm}}{\succeq}$ order does not imply the $\overset{\text{p}}{\succeq}$ order. For example, from the definition of the $\overset{\text{rm}}{\succeq}$ order, it follows that $(1, 4) \overset{\text{rm}}{\succeq} \left(\frac{4}{3}, 2\right)$, but clearly the $\overset{\text{p}}{\succeq}$ order does not hold between these two vectors.

Let us first recall some results in the literature that are most pertinent to the main results of this paper. Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$, and let $X_{\lambda_1^*}, \ldots, X_{\lambda_n^*}$ be another set of independent exponential random variables with respective hazard rates $\lambda_1^*, \ldots, \lambda_n^*$. Boland et al. [3] showed that

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \ge_{\text{lr}} \sum_{i=1}^{n} X_{\lambda_i^*}; \tag{1.1}$$

see [22,18] for a comprehensive discussion on various stochastic orders. Bon and Păltănea [4] subsequently showed that

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{p}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \ge_{\text{hr}} \sum_{i=1}^{n} X_{\lambda_i^*}, \tag{1.2}$$

and they also focused on the special case when one convolution involved identically distributed random variables. Kochar and Ma [11] established that

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \ge_{\text{disp}} \sum_{i=1}^{n} X_{\lambda_i^*}. \tag{1.3}$$

Let $Y_{(r,\,\lambda_1)}, \ldots, Y_{(r,\,\lambda_n)}$ be independent gamma random variables with respective shape parameter vector $\boldsymbol{r} = (r, \ldots, r)$ $(r \geq 1)$ and scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and let $Y_{(r,\,\lambda_1^*)}, \ldots, Y_{(r,\,\lambda_n^*)}$ be another set of independent gamma random variables with respective shape parameter vector $\boldsymbol{r}$ and scale parameter vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Korwar [14] extended the above results in (1.1) and (1.3) to the case of gamma random variables with different scale parameters but with a common shape parameter (≥ 1) that

$$\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^{n} Y_{(r,\,\lambda_i)} \geq_{\text{lr}} \sum_{i=1}^{n} Y_{(r,\,\lambda_i^*)}; \tag{1.4}$$

and

$$\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^{n} Y_{(r,\,\lambda_i)} \geq_{\text{disp}} [\geq_{\text{hr}}] \sum_{i=1}^{n} Y_{(r,\,\lambda_i^*)}. \tag{1.5}$$

Khaledi and Kochar [9] further improved the result in (1.5) by relaxing majorization order to p-larger order:

$$\boldsymbol{\lambda} \overset{\text{p}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^{n} Y_{(r,\,\lambda_i)} \geq_{\text{disp}} [\geq_{\text{hr}}] \sum_{i=1}^{n} Y_{(r,\,\lambda_i^*)}. \tag{1.6}$$

One of the basic criteria for comparing variability in probability distributions is dispersive order.

Definition 1.3. A random variable X is said to be less dispersed than another random variable Y (denoted by $X \leq_{\text{disp}} Y$) if

$$F^{-1}(v) - F^{-1}(u) \leq G^{-1}(v) - G^{-1}(u)$$

for $0 \leq u \leq v \leq 1$, where F^{-1} and G^{-1} are the right continuous inverses of the distribution functions F and G of X and Y, respectively.

A weaker order, which was called the *right spread order* in [6] and the *excess wealth order* in [21] is defined as below.

Definition 1.4. X is said to be less right spread than Y (denoted by $X \leq_{\text{RS}} Y$) if

$$\int_{F^{-1}(p)}^{\infty} \overline{F}(t)\mathrm{d}t \leq \int_{G^{-1}(p)}^{\infty} \overline{G}(t)\mathrm{d}t, \quad 0 \leq p \leq 1.$$

The following implication is well known,

$$X \leq_{\text{disp}} Y \Longrightarrow X \leq_{\text{RS}} Y \Longrightarrow \text{Var}(X) \leq \text{Var}(Y).$$

The right spread order is closely related to the *NBUE order* comparing the relative aging property.

Definition 1.5. X is said to be more *NBUE* (new better than used in expectation) than Y (denoted by $X \leq_{\text{NBUE}} Y$) if

$$\frac{1}{\mathsf{E}(X)} \int_{F^{-1}(p)}^{\infty} \overline{F}(t)\mathrm{d}t \leq \frac{1}{\mathsf{E}(Y)} \int_{G^{-1}(p)}^{\infty} \overline{G}(t)\mathrm{d}t, \quad 0 \leq p \leq 1.$$

It is obvious that the NBUE order is equivalent to the right spread order when $\mathsf{E}(X) = \mathsf{E}(Y)$, however, they are distinct when $\mathsf{E}(X) \neq \mathsf{E}(Y)$ (cf. [10]).

Definition 1.6. X is said to be smaller than Y in the star order (denoted by $X \leq_* Y$) if $G^{-1}F(x)/x$ is increasing in x on the support of X.

Also, it is known that the star order implies NBUE order.

Recently, Kochar and Xu [13] investigated the star order and right spread order and obtained that

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow Y_{(r,\,\lambda_1)} + Y_{(r,\,\lambda_2)} \geq_* Y_{(r,\,\lambda_1^*)} + Y_{(r,\,\lambda_2^*)} \tag{1.7}$$

and

$$(1/\lambda_1, 1/\lambda_2) \overset{\text{m}}{\succeq} (1/\lambda_1^*, 1/\lambda_2^*) \Longrightarrow Y_{(r,\,\lambda_1)} + Y_{(r,\,\lambda_2)} \geq_* Y_{(r,\,\lambda_1^*)} + Y_{(r,\,\lambda_2^*)}. \tag{1.8}$$

With the aid of (1.8), they also proved that

$$\boldsymbol{\lambda} \overset{\text{rm}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^{n} Y_{(r,\,\lambda_i)} \geq_{\text{RS}} \sum_{i=1}^{n} Y_{(r,\,\lambda_i^*)}. \tag{1.9}$$

All the results in (1.4)–(1.9) are conditioned to the case when gamma random variables involved in convolutions have common shape parameters. However, convolutions of independent gamma random variables with different shape parameters often occur naturally in many problems, and especially in reliability theory. Let us consider a reliability scenario wherein there is a redundant standby system without repair consisting of n gamma components with different scale parameters and also different shape parameters. After the first failure, one standby component is put into operation at once; next, after the second failure, another standby component is put into operation, and so on. Finally, the system fails at the failure of the last component. Clearly, the lifetime of the system is just a convolution of n gamma lifetimes. It will be of great interest to investigate ordering properties of convolutions in this setup.

In this paper we shall further pursue this problem under gamma framework and establish some more general results in which the convolutions involved have different scale parameters and also different shape parameters. Let $Y_{(r_1,\,\lambda_1)}, \ldots, Y_{(r_n,\,\lambda_n)}$ be independent gamma random variables with respective shape parameter vector $\boldsymbol{r} = (r_1, \ldots, r_n)$ ($r_i \geq 1$) and scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and let $Y_{(r_1^*,\,\lambda_1^*)}, \ldots, Y_{(r_n^*,\,\lambda_n^*)}$ be another set of independent gamma random variables with respective shape parameter vector $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*)$ ($r_i^* \geq 1$) and scale parameter vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose there exists some permutation π such that $\pi\boldsymbol{r} = \boldsymbol{r}_\uparrow, \pi\boldsymbol{r}^* = \boldsymbol{r}_\uparrow^*, \pi\boldsymbol{\lambda} = \boldsymbol{\lambda}_\downarrow$ and $\pi\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*{}_\downarrow$, where the components of $\boldsymbol{\lambda}_\downarrow$ and $\boldsymbol{\lambda}^*{}_\downarrow$ are in descending order, and the components of $\boldsymbol{r}_\uparrow$ and $\boldsymbol{r}_\uparrow^*$ are in ascending order. We then establish that

$$\boldsymbol{r} \overset{\mathrm{m}}{\succeq} \boldsymbol{r}^* \quad \text{and} \quad \boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^n Y_{(r_i,\,\lambda_i)} \geq_{\mathrm{lr}} \sum_{i=1}^n Y_{(r_i^*,\,\lambda_i^*)}, \tag{1.10}$$

$$\boldsymbol{\lambda} \overset{\mathrm{p}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^n Y_{(r_i,\,\lambda_i)} \geq_{\mathrm{disp}} \sum_{i=1}^n Y_{(r_i,\,\lambda_i^*)} \tag{1.11}$$

and

$$\boldsymbol{r} \overset{\mathrm{m}}{\succeq} \boldsymbol{r}^* \quad \text{and} \quad \boldsymbol{\lambda} \overset{\mathrm{p}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^n Y_{(r_i,\,\lambda_i)} \geq_{\mathrm{hr}} \sum_{i=1}^n Y_{(r_i^*,\,\lambda_i^*)}. \tag{1.12}$$

Yu [24] focused on the special case when one set of gamma random variables is i.i.d. and obtained some similar results to those in (1.10)–(1.12). It should be remarked here that the results of Yu [24] are not direct consequences of our results obtained here since they have a less restrictive condition on the parameters.

Let $Y_{(r_i,\,\lambda_i)}$ $[Y_{(r_i,\,\lambda_i^*)}]$, $i = 1, 2$, be two independent gamma random variables with $Y_{(r_i,\,\lambda_i)}[Y_{(r_i,\,\lambda_i^*)}]$ having the shape parameter r_i and scale parameter λ_i $[\lambda_i^*]$. We prove that, if $\lambda_1 \geq \lambda_2,\ \lambda_1^* \geq \lambda_2^*$ and $r_1 \leq r_2$, then

$$(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2)} \geq_* Y_{(r_1,\,\lambda_1^*)} + Y_{(r_2,\,\lambda_2^*)}, \tag{1.13}$$

and if $\lambda_1 \leq \lambda_2,\ \lambda_1^* \leq \lambda_2^*$, then

$$(1/\lambda_1, 1/\lambda_2) \overset{\mathrm{m}}{\succeq} (1/\lambda_1^*, 1/\lambda_2^*) \Longrightarrow Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2)} \geq_* Y_{(r_1,\,\lambda_1^*)} + Y_{(r_2,\,\lambda_2^*)}. \tag{1.14}$$

With the aid of (1.14), we also show that

$$(1/\lambda_1, \ldots, 1/\lambda_n) \overset{\mathrm{m}}{\succeq} (1/\lambda_1^*, \ldots, 1/\lambda_n^*) \Longrightarrow \sum_{i=1}^n Y_{(r_i,\,\lambda_i)} \geq_{\mathrm{RS}} \sum_{i=1}^n Y_{(r_i,\,\lambda_i^*)} \tag{1.15}$$

and

$$\boldsymbol{r} \overset{\mathrm{m}}{\succeq} \boldsymbol{r}^* \quad \text{and} \quad (1/\lambda_1, \ldots, 1/\lambda_n) \overset{\mathrm{m}}{\succeq} (1/\lambda_1^*, \ldots, 1/\lambda_n^*) \Longrightarrow \sum_{i=1}^n Y_{(r_i,\,\lambda_i)} \geq_{\mathrm{mrl}} \sum_{i=1}^n Y_{(r_i^*,\,\lambda_i^*)}. \tag{1.16}$$

Zhao and Balakrishnan [27] obtained some results similar to those in (1.10)–(1.12) and (1.14)–(1.16) for convolutions of Erlang random variables (i.e., gamma random variables with integer valued shape parameters). It is apparent that the results in (1.10)–(1.16) established in this paper generalize and strengthen the corresponding ones listed in (1.1)–(1.9) known in the literature.

2. Likelihood ratio ordering

The following result gives the density function of a convolution of two gamma distributions with different scale and shape parameters.

Theorem 2.1. *Let $Y_{(r_i,\ \lambda_i)}$ $(i = 1, 2)$ be two independent gamma random variables with $Y_{(r_i,\ \lambda_i)}$ having the density function*

$$f(y; r_i, \lambda_i) = \frac{y^{r_i-1}\lambda_i^{r_i}}{\Gamma(r_i)} \exp(-\lambda_i y), \quad y > 0.$$

Suppose that $\lambda_1 > \lambda_2$ and $r_1 < r_2$. Then, the density function of $Y_{(r_1,\ \lambda_1)} + Y_{(r_2,\ \lambda_2)}$ is given by

$$f(y; r_1, r_2, \lambda_1, \lambda_2) = k(y; r_1, r_2, \lambda_1, \lambda_2) \int_0^1 g(w, y; r_1, r_2, \lambda_1, \lambda_2)\mathrm{d}w, \tag{2.1}$$

where

$$k(y; r_1, r_2, \lambda_1, \lambda_2) = \frac{\lambda_1^{r_1}\lambda_2^{r_2}}{2^{r_1+r_2-1}\Gamma(r_1)\Gamma(r_2)} y^{r_1+r_2-1} \exp\left(-\frac{cy}{2}\right)$$
$$g(w, y; \lambda_1, \lambda_2) = (1 - w^2)^{r_1-1}\left[(1 - w)^{r_2-r_1} \exp(-\theta yw) + (1 + w)^{r_2-r_1} \exp(\theta yw)\right],$$

and where $y > 0,\ 0 < w < 1,\ c = \lambda_1 + \lambda_2,\ \theta = (\lambda_1 - \lambda_2)/2$.

Proof. By the convolution formula, the density function of $Y_{(r_1,\ \lambda_1)} + Y_{(r_2,\ \lambda_2)}$ can be written as

$$\begin{aligned} f(y; r_1, r_2, \lambda_1, \lambda_2) &= \int_0^y f(x; r_1, \lambda_1) f(y - x; r_2, \lambda_2)\mathrm{d}x \\ &= \int_0^y \frac{1}{\Gamma(r_1)} x^{r_1-1}\lambda_1^{r_1} \exp(-\lambda_1 x) \frac{1}{\Gamma(r_2)} (y - x)^{r_2-1}\lambda_2^{r_2} \exp[-\lambda_2(y - x)]\mathrm{d}x. \end{aligned}$$

Changing the variable x of integration to $u = x/y$ yields that

$$f(y; r_1, r_2, \lambda_1, \lambda_2) = \frac{\lambda_1^{r_1}\lambda_2^{r_2} y^{r_1+r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \int_0^1 u^{r_1-1}(1 - u)^{r_2-1} \exp\{-[\lambda_1 u + \lambda_2(1 - u)]y\}\mathrm{d}u. \tag{2.2}$$

By making the transform $u \to z = (2u - 1)\theta$, it follows that

$$f(y; r_1, r_2, \lambda_1, \lambda_2) = \frac{\lambda_1^{r_1}\lambda_2^{r_2} y^{r_1+r_2-1} \exp\left(-\frac{cy}{2}\right)}{\Gamma(r_1)\Gamma(r_2)2^{r_1+r_2-1}\theta} \int_{-\theta}^{\theta} \left(1 + \frac{z}{\theta}\right)^{r_1-1} \left(1 - \frac{z}{\theta}\right)^{r_2-1} \exp(-yz)\mathrm{d}z.$$

Split the interval $(-\theta, \theta)$ of integration into $(-\theta, 0)$ and $[0, \theta)$ and make the change of variable $z = -w$ for the interval $(-\theta, 0)$ to give

$$\begin{aligned} &\int_{-\theta}^{\theta} \left(1 + \frac{z}{\theta}\right)^{r_1-1} \left(1 - \frac{z}{\theta}\right)^{r_2-1} \exp(-yz)\mathrm{d}z \\ &= \int_{-\theta}^{0} \left(1 + \frac{z}{\theta}\right)^{r_1-1} \left(1 - \frac{z}{\theta}\right)^{r_2-1} \exp(-yz)\mathrm{d}z + \int_0^{\theta} \left(1 + \frac{z}{\theta}\right)^{r_1-1} \left(1 - \frac{z}{\theta}\right)^{r_2-1} \exp(-yz)\mathrm{d}z \\ &= \int_0^{\theta} \left(1 - \frac{w}{\theta}\right)^{r_1-1} \left(1 + \frac{w}{\theta}\right)^{r_2-1} \exp(yw)\mathrm{d}w + \int_0^{\theta} \left(1 + \frac{w}{\theta}\right)^{r_1-1} \left(1 - \frac{w}{\theta}\right)^{r_2-1} \exp(-yw)\mathrm{d}w \\ &= \int_0^{\theta} \left(1 - \frac{w^2}{\theta^2}\right)^{r_1-1} \left[\left(1 - \frac{w}{\theta}\right)^{r_2-r_1} \exp(-yw) + \left(1 + \frac{w}{\theta}\right)^{r_2-r_1} \exp(yw)\right]\mathrm{d}w \\ &= \int_0^1 \theta(1 - w^2)^{r_1-1}\left[(1 - w)^{r_2-r_1} \exp(-\theta yw) + (1 + w)^{r_2-r_1} \exp(\theta yw)\right]\mathrm{d}w, \end{aligned}$$

the last equality can be deduced by changing the variable w of integration to w/θ. Thus, we obtain the required result. □

Before stating our main result, we first present two useful lemmas. The first one turns out to be a useful tool for showing the monotonicity of a fraction whose numerator and denominator are integrals or summations.

Lemma 2.2 ([17]). *Let Θ be a subset of real line and U be a non-negative random variable having a cdf belonging to a stochastically ordered family $\mathcal{P} = \{H(\cdot|\theta), \theta \in \Theta\}$, that is, for $\theta_1, \theta_2 \in \Theta$, $H(\cdot|\theta_1) \le_{\mathrm{st}}(\ge_{\mathrm{st}})H(\cdot|\theta_2)$ whenever $\theta_1 < \theta_2$. Suppose a real function $\psi(u, \theta)$ on $\mathcal{R} \cdot \Theta$ is measurable in u for each θ such that $\mathsf{E}_\theta[\psi(U, \theta)]$ exists. Then,*

(i) *$\mathsf{E}_\theta[\psi(U, \theta)]$ is increasing in θ if $\psi(u, \theta)$ is increasing in θ and increasing (decreasing) in u;*
(ii) *$\mathsf{E}_\theta[\psi(U, \theta)]$ is decreasing in θ if $\psi(u, \theta)$ is decreasing in θ and decreasing (increasing) in u.*

Lemma 2.3. (a) *For $0 < w_1 < w_2$ and $\alpha \geq 1$, the function*

$$g(y) = \frac{\exp(-w_2 y) + \alpha \exp(w_2 y)}{\exp(-w_1 y) + \alpha \exp(w_1 y)}$$

is increasing in $y \in (0, \infty)$;

(b) *For $a > 0$ and $\theta_2 > \theta_1 > 0$, the function*

$$\zeta(w) = \frac{(1-w)^a \exp(-\theta_2 w) + (1+w)^a \exp(\theta_2 w)}{(1-w)^a \exp(-\theta_1 w) + (1+w)^a \exp(\theta_1 w)}$$

is increasing in $w \in (0, 1)$.

The proof of the above lemma is given in the Appendix.

Theorem 2.4. *Let $Y_{(r_i,\lambda_i)}$ $(Y_{(r_i,\lambda_i^*)})$, $i = 1, 2$ be two independent gamma random variables with $Y_{(r_i,\lambda_i)}$ $(Y_{(r_i,\lambda_i^*)})$ having the shape parameter r_i and scale parameter λ_i (λ_i^*). If $\lambda_1 \geq \lambda_2$, $\lambda_1^* \geq \lambda_2^*$ and $1 \leq r_1 \leq r_2$, then*

$$(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow Y_{(r_1,\lambda_1)} + Y_{(r_2,\lambda_2)} \geq_{\mathrm{lr}} Y_{(r_1,\lambda_1^*)} + Y_{(r_2,\lambda_2^*)}.$$

Proof. To prove the required result, it follows from A.8.a of [15] that we have to show that the result holds for majorization and convolution is decreasing in λ_i $(i = 1, 2)$ according to likelihood ratio order, which is actually true from Theorem 1.C.9 of [22] and the fact that a gamma random variable $Y_{(r,\lambda)}$ is decreasing in λ in the sense of the likelihood ratio order. Thus, it is enough to prove that

$$(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow Y_{(r_1,\lambda_1)} + Y_{(r_2,\lambda_2)} \geq_{\mathrm{lr}} Y_{(r_1,\lambda_1^*)} + Y_{(r_2,\lambda_2^*)}.$$

Now assume that $(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \lambda_2^*)$. We then have $\lambda_2 \leq \lambda_2^* \leq \lambda_1^* \leq \lambda_1$. The proof will be done by distinguishing three cases.

Case (i): $\lambda_1 = \lambda_1^*$

In this case, we have $\lambda_1 = \lambda_1^*$ and $\lambda_2 = \lambda_2^*$, and the result is trivially true.

Case (ii): $\lambda_1 \neq \lambda_1^*$ *and* $\lambda_1^* \neq \lambda_2^*$

Let $f(1, y) = f(y; r_1, r_2, \lambda_1, \lambda_2)$ and $f(2, y) = f(y; r_1, r_2, \lambda_1^*, \lambda_2^*)$. It suffices to prove that

$$\begin{aligned}
\Delta(y) &= \frac{f(1, y)}{f(2, y)} \\
&\propto \frac{\int_0^1 (1-w^2)^{r_1-1}\left[(1-w)^{r_2-r_1}\exp(-\theta y w) + (1+w)^{r_2-r_1}\exp(\theta y w)\right]\mathrm{d}w}{\int_0^1 (1-w^2)^{r_1-1}\left[(1-w)^{r_2-r_1}\exp(-\theta^* y w) + (1+w)^{r_2-r_1}\exp(\theta^* y w)\right]\mathrm{d}w} \\
&= \mathsf{E}_y \psi(W, y)
\end{aligned}$$

is increasing in $y \in (0, \infty)$, where $\theta = (\lambda_1 - \lambda_2)/2 > (\lambda_1^* - \lambda_2^*)/2 = \theta^*$ and

$$\psi(w, y) = \frac{(1-w)^{r_2-r_1}\exp(-\theta y w) + (1+w)^{r_2-r_1}\exp(\theta y w)}{(1-w)^{r_2-r_1}\exp(-\theta^* y w) + (1+w)^{r_2-r_1}\exp(\theta^* y w)}$$

for $w \in (0, 1)$. Here, the distribution function of the random variable W belongs to the family $\mathscr{P} = \{H(\cdot|y), y \in \Re_+\}$ with densities

$$h(w|y) = c(y)(1-w^2)^{r_1-1}\left[(1-w)^{r_2-r_1}\exp(-\theta^* y w) + (1+w)^{r_2-r_1}\exp(\theta^* y w)\right]$$

and a normalizing constant $c(y)$ such that $\int_0^1 h(w|y)\mathrm{d}w = 1$. From Lemma 2.3, it follows that $\psi(w, y)$ is increasing both in $y \in (0, \infty)$ and $w \in (0, 1)$. On the other hand, note that, for $y_2 \geq y_1 > 0$,

$$\frac{h(w|y_2)}{h(w|y_1)} \propto \frac{(1-w)^{r_2-r_1}\exp(-\theta^* y_2 w) + (1+w)^{r_2-r_1}\exp(\theta^* y_2 w)}{(1-w)^{r_2-r_1}\exp(-\theta^* y_1 w) + (1+w)^{r_2-r_1}\exp(\theta^* y_1 w)}$$

is also increasing in $w \in (0, 1)$ from Lemma 2.3(b). This means that $H(\cdot|y_1) \leq_{\mathrm{lr}} H(\cdot|y_2)$ which in turn implies that $H(\cdot|y_1) \leq_{\mathrm{st}} H(\cdot|y_2)$ whenever $0 < y_1 \leq y_2$. By using Lemma 2.2 now, $\mathsf{E}_y \psi(W, y)$ is increasing in $y \in (0, \infty)$, which completes the proof of this case.

Case (iii): $\lambda_1 \neq \lambda_1^*$ *and* $\lambda_1^* = \lambda_2^*$

In this case, we have

$$f(2, y) = \frac{(c/2)^{r_1+r_2} y^{r_1+r_2-1}}{\Gamma(r_1+r_2)} \exp\left(-\frac{cy}{2}\right),$$

where $c = \lambda_1^* + \lambda_2^*$. Thus, it follows that

$$\begin{aligned}\Delta(y) &= \frac{f(1,\ y)}{f(2,\ y)} \\ &\propto \int_0^1 (1-w^2)^{r_1-1}\left[(1-w)^{r_2-r_1}\exp(-\theta yw) + (1+w)^{r_2-r_1}\exp(\theta yw)\right]\mathrm{d}w \\ &= \int_0^1 (1-w^2)^{r_1-1}\{(1-w)^{r_2-r_1}\cosh(\theta yw) + [(1+w)^{r_2-r_1} - (1-w)^{r_2-r_1}]\exp(\theta yw)\}\mathrm{d}w\end{aligned}$$

is increasing in $y \in (0,\ \infty)$ since both $\cosh(\theta yw)$ and $\exp(\theta yw)$ are increasing in $y \in (0,\ \infty)$. □

The following theorem is a natural extension of Theorem 2.4.

Theorem 2.5. *Let $Y_{(r_1,\ \lambda_1)}, \ldots, Y_{(r_n,\ \lambda_n)}$ be independent gamma random variables with respective shape parameter vector $\boldsymbol{r} = (r_1, \ldots, r_n)$ where each component is greater than or equal to 1 and scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and let $Y_{(r_1,\ \lambda_1^*)}, \ldots, Y_{(r_n,\ \lambda_n^*)}$ be another set of independent gamma random variables with respective shape parameter vector $\boldsymbol{r}$ and scale parameter vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose there exists some permutation π such that $\pi\boldsymbol{\lambda} = \boldsymbol{\lambda}_\downarrow$, $\pi\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*{}_\downarrow$ and $\pi\boldsymbol{r} = \boldsymbol{r}_\uparrow$, where the components of $\boldsymbol{\lambda}_\downarrow$ and $\boldsymbol{\lambda}^*{}_\downarrow$ are in descending order, and the components of $\boldsymbol{r}_\uparrow$ are in ascending order. Then,*

$$\boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^n Y_{(r_i,\ \lambda_i)} \geq_{\mathrm{lr}} \sum_{i=1}^n Y_{(r_i,\ \lambda_i^*)}.$$

Proof. Without loss of generality, let us assume that $\lambda_1 \leq \cdots \leq \lambda_n$, $\lambda_1^* \leq \cdots \leq \lambda_n^*$ and $r_1 \geq \cdots \geq r_n$. From the definition of the $\overset{\mathrm{w}}{\succeq}$ order, it is known that $\boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\lambda}^*$ is equivalent to $\sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \lambda_i^*,\ 1 \leq j \leq n$. Note that there must exist some λ_n' such that

$$\lambda_n' \geq \max\{\lambda_n,\ \lambda_n^*\} \quad \text{and} \quad (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n') \overset{\mathrm{m}}{\succeq} \boldsymbol{\lambda}^*.$$

Let $Y_{(r_n,\ \lambda_n')}$ be a gamma random variable with the shape parameter r_n and scale parameter λ_n', which is independent of $Y_{(r_i,\ \lambda_i)}$ $(1 \leq i \leq n-1)$, it then holds that $Y_{(r_n,\ \lambda_n)} \geq_{\mathrm{lr}} Y_{(r_1,\ \lambda_n')}$. Since the convolution of gamma distributions whose shape parameter is greater than or equal to 1 has a logconcave density, it follows that

$$\sum_{i=1}^n Y_{(r_i,\ \lambda_i)} \geq_{\mathrm{lr}} \sum_{i=1}^{n-1} Y_{(r_i,\ \lambda_i)} + Y_{(r_n,\ \lambda_n')}$$

by applying Theorem 1.C.9 of [22]. Now we find that it is enough to prove that

$$\boldsymbol{\lambda} \overset{\mathrm{m}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^n Y_{(r_i,\ \lambda_i)} \geq_{\mathrm{lr}} \sum_{i=1}^n Y_{(r_i,\ \lambda_i^*)}.$$

By the nature of majorization, it suffices to prove the result for the case that $(\lambda_1,\ \lambda_2) \overset{\mathrm{m}}{\succeq} (\lambda_1^*,\ \lambda_2^*)$ and $\lambda_i = \lambda_i^*,\ i = 3, \ldots, n$. From Theorem 2.4, it follows that

$$Y_{(r_1,\ \lambda_1)} + Y_{(r_2,\ \lambda_2)} \geq_{\mathrm{lr}} Y_{(r_1,\ \lambda_1^*)} + Y_{(r_2,\ \lambda_2^*)}.$$

Since $\sum_{i=3}^n Y_{(r_i,\ \lambda_i)}$ $\left(\sum_{i=3}^n Y_{(r_i,\ \lambda_i^*)}\right)$ has a logconcave density, applying Theorem 1.C.9 of [22] once again yields that

$$\sum_{i=1}^n Y_{(r_i,\ \lambda_i)} = Y_{(r_1,\ \lambda_1)} + Y_{(r_2,\ \lambda_2)} + \sum_{i=3}^n Y_{(r_i,\ \lambda_i)} \geq_{\mathrm{lr}} Y_{(r_1,\ \lambda_1^*)} + Y_{(r_2,\ \lambda_2^*)} + \sum_{i=3}^n Y_{(r_i,\ \lambda_i^*)} = \sum_{i=1}^n Y_{(r_i,\ \lambda_i^*)}. \quad \square$$

Lemma 2.6. (a) *For $w > 0$ and $\beta \geq \alpha \geq 1$, the function*

$$\vartheta(y) = \frac{1 + \beta\exp(wy)}{1 + \alpha\exp(wy)}$$

is increasing in $y \in (0, \infty)$;

(b) *For $y > 0$ and $0 < r_1 \leq r_1^* \leq r_2^* \leq r_2$ and $r_1 + r_2 = r_1^* + r_2^*$, the function*

$$\kappa(w) = \frac{(1-w)^{r_2}(1+w)^{r_1}\exp(-yw) + (1-w)^{r_1}(1+w)^{r_2}\exp(yw)}{(1-w)^{r_2^*}(1+w)^{r_1^*}\exp(-yw) + (1-w)^{r_1^*}(1+w)^{r_2^*}\exp(yw)}$$

is increasing in $w \in (0,\ 1)$.

The proof of the above lemma is given in the Appendix.

Theorem 2.7. *Let* $Y_{(r_i,\,\lambda_i)}$ $(Y_{(r_i^*,\,\lambda_i)})$, $i = 1, 2$ *be two independent gamma random variables with* $Y_{(r_i,\,\lambda_i)}$ $(Y_{(r_i^*,\,\lambda_i)})$ *having the shape parameter* r_i (r_i^*) *and scale parameter* λ_i*. If* $r_1 \le r_2$, $r_1^* \le r_2^*$ *and* $\lambda_1 \ge \lambda_2$*, then*

$$(r_1, r_2) \overset{\text{m}}{\succeq} (r_1^*, r_2^*) \Longrightarrow Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2)} \ge_{\text{lr}} Y_{(r_1^*,\,\lambda_1)} + Y_{(r_2^*,\,\lambda_2)}.$$

Proof. Assume $(r_1, r_2) \overset{\text{m}}{\succeq} (r_1^*, r_2^*)$ to hold, we then have $r_1 \le r_1^* \le r_2^* \le r_2$. If $r_2 = r_2^*$, then $r_1 = r_1^*$ and hence the result is trivially true. In the following, we assume that $r_2 > r_2^*$. Let $g(1, y) = f(y; r_1, r_2, \lambda_1, \lambda_2)$ and $g(2, y) = f(y; r_1^*, r_2^*, \lambda_1, \lambda_2)$. From (2.1), we have to prove that

$$\begin{aligned}\Xi(y) &= \frac{g(1,\ y)}{g(2,\ y)} \\ &\propto \frac{\int_0^1 (1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta y w) + (1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta y w)\mathrm{d}w}{\int_0^1 (1-w)^{r_2^*-1}(1+w)^{r_1^*-1}\exp(-\theta y w) + (1-w)^{r_1^*-1}(1+w)^{r_2^*-1}\exp(\theta y w)\mathrm{d}w} \\ &= \mathsf{E}_y \varphi_1(W, y)\end{aligned}$$

is increasing in $y \in (0,\ \infty)$, where $\theta = (\lambda_1 - \lambda_2)/2$ and

$$\varphi_1(w, y) = \frac{(1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta y w) + (1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta y w)}{(1-w)^{r_2^*-1}(1+w)^{r_1^*-1}\exp(-\theta y w) + (1-w)^{r_1^*-1}(1+w)^{r_2^*-1}\exp(\theta y w)}$$

for $w \in (0,\ 1)$. Here, the distribution function of the random variable W belongs to the family $\mathscr{P} = \{H_1(\cdot|y), y \in \Re\}$ with densities

$$h_1(w|y) = c_1(y)\left[(1-w)^{r_2^*-1}(1+w)^{r_1^*-1}\exp(-\theta y w) + (1-w)^{r_1^*-1}(1+w)^{r_2^*-1}\exp(\theta y w)\right]$$

and a normalizing constant $c_1(y)$ such that $\int_0^1 h_1(w|y)\mathrm{d}w = 1$. For fixed $w \in (0,\ 1)$, it can be seen that

$$\begin{aligned}\varphi_1(w, y) &\propto \frac{(1-w)^{r_2-r_1}\exp(-\theta y w) + (1+w)^{r_2-r_1}\exp(\theta y w)}{(1-w)^{r_2^*-r_1^*}\exp(-\theta y w) + (1+w)^{r_2^*-r_1^*}\exp(\theta y w)} \\ &\propto \frac{1 + \left(\frac{1+w}{1-w}\right)^{r_2-r_1}\exp(2\theta y w)}{1 + \left(\frac{1+w}{1-w}\right)^{r_2^*-r_1^*}\exp(2\theta y w)},\end{aligned}$$

which is increasing in $y \in (0,\ \infty)$ for $w \in (0,\ 1)$ and $\theta > 0$ according to Lemma 2.6(a). On the other hand, $\varphi_1(w, y)$ is also increasing in $w \in (0,\ 1)$ for $\theta y > 0$ from Lemma 2.6(b). In addition, for $y_2 \ge y_1 > 0$,

$$\frac{h_1(w|y_2)}{h_1(w|y_1)} \propto \frac{(1-w)^{r_2^*-r_1^*}\exp(-\theta y_2 w) + (1+w)^{r_2^*-r_1^*}\exp(\theta y_2 w)}{(1-w)^{r_2^*-r_1^*}\exp(-\theta y_1 w) + (1+w)^{r_2^*-r_1^*}\exp(\theta y_1 w)}$$

is also increasing in $w \in (0,\ 1)$ from Lemma 2.3(b). From this one gets that $H_1(\cdot|y_1) \le_{\text{lr}} H_1(\cdot|y_2)$ which in turn implies that $H_1(\cdot|y_1) \le_{\text{st}} H_1(\cdot|y_2)$ whenever $0 < y_1 \le y_2$. By using Lemma 2.2 now, $\mathsf{E}_y\varphi_1(W, y)$ is increasing in $y \in (0,\ \infty)$, which completes the entire proof. □

Upon using a proof quite similar to that of Theorem 2.5, we can obtain the following result.

Theorem 2.8. *Let* $Y_{(r_1,\,\lambda_1)}, \ldots, Y_{(r_n,\,\lambda_n)}$ *be independent gamma random variables with respective shape parameter vector* $\boldsymbol{r} = (r_1, \ldots, r_n)$ *where each component is greater than or equal to* 1 *and scale parameter vector* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$*, and let* $Y_{(r_1^*,\,\lambda_1)}, \ldots, Y_{(r_n^*,\,\lambda_n)}$ *be another set of independent gamma random variables with respective shape parameter vector* $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*)$ *where each component is greater than or equal to* 1 *and scale parameter vector* $\boldsymbol{\lambda}$*. Suppose there exists some permutation* π *such that* $\pi\boldsymbol{r} = \boldsymbol{r}_\uparrow$, $\pi\boldsymbol{r}^* = \boldsymbol{r}^*_\uparrow$ *and* $\pi\boldsymbol{\lambda} = \boldsymbol{\lambda}_\downarrow$*. Then,*

$$\boldsymbol{r} \overset{\text{m}}{\succeq} \boldsymbol{r}^* \Longrightarrow \sum_{i=1}^n Y_{(r_i,\,\lambda_i)} \ge_{\text{lr}} \sum_{i=1}^n Y_{(r_i^*,\,\lambda_i)}.$$

Finally, we give a result more general than those in Theorems 2.5 and 2.8, which can be used to compare heterogeneous gamma convolutions in terms of the likelihood ratio order wherein both shape parameter vectors and scale parameter vectors are different.

Theorem 2.9. *Let* $Y_{(r_1,\,\lambda_1)}, \ldots, Y_{(r_n,\,\lambda_n)}$ *be independent gamma random variables with respective shape parameter vector* $\boldsymbol{r} = (r_1, \ldots, r_n)$ *where each component is greater than or equal to* 1 *and scale parameter vector* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$*, and let* $Y_{(r_1^*,\,\lambda_1^*)}, \ldots, Y_{(r_n^*,\,\lambda_n^*)}$ *be another set of independent gamma random variables with respective shape parameter vector* $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*)$

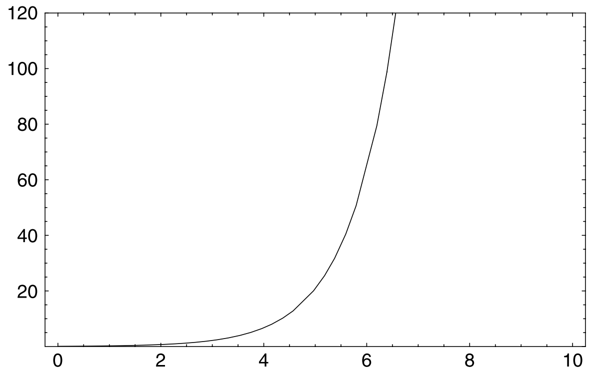

Fig. 1. Plot of the ratio $f(t; 3, 1, 1, 3)/f(t; 2, 2, 2, 3)$ between the densities of convolutions.

where each component is greater than or equal to 1 *and scale parameter vector* $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. *Suppose there exists some permutation* π *such that* $\pi\boldsymbol{r} = \boldsymbol{r}_\uparrow, \pi\boldsymbol{r}^* = \boldsymbol{r}^*_\uparrow, \pi\boldsymbol{\lambda} = \boldsymbol{\lambda}_\downarrow$ *and* $\pi\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*{}_\downarrow$. *Then,*

$$\boldsymbol{r} \overset{\mathrm{m}}{\succeq} \boldsymbol{r}^* \quad \text{and} \quad \boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^n Y_{(r_i,\,\lambda_i)} \ge_{\mathrm{lr}} \sum_{i=1}^n Y_{(r_i^*,\,\lambda_i^*)}.$$

Proof. Let $Y_{(r_1,\lambda_1^*)}, \ldots, Y_{(r_n,\lambda_n^*)}$ be independent gamma random variables with respective shape parameter vector $\boldsymbol{r} = (r_1, \ldots, r_n)$ and scale parameter vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Upon using Theorems 2.5 and 2.8, we have

$$\sum_{i=1}^n Y_{(r_i,\,\lambda_i)} \ge_{\mathrm{lr}} \sum_{i=1}^n Y_{(r_i,\,\lambda_i^*)} \ge_{\mathrm{lr}} \sum_{i=1}^n Y_{(r_i^*,\,\lambda_i^*)}. \quad \square$$

In order to illustrate the result obtained in Theorem 2.9, we provide the following numerical example. Let (X_1, X_2) be a vector of independent heterogeneous gamma random variables with shape parameter vector $(r_1, r_2) = (3, 1)$ and scale parameter vector $(\lambda_1, \lambda_2) = (1, 3)$. Let (X_1^*, X_2^*) be an another vector of independent heterogeneous gamma random variables with shape parameter vector $(r_1^*, r_2^*) = (2, 2)$ and scale parameter vector $(\lambda_1^*, \lambda_2^*) = (2, 3)$. Obviously, it holds that $r_1 \ge r_2, r_1^* \ge r_2^*, \lambda_1 \le \lambda_2, \lambda_1^* \le \lambda_2^*, (r_1, r_2) \overset{\mathrm{m}}{\succeq} (r_1^*, r_2^*)$ and $(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*)$. It can be seen from Fig. 1 that the ratio $f(t; 3, 1, 1, 3)/f(t; 2, 2, 2, 3)$ between the densities of convolutions is increasing in $t \in \Re_+$ which is in accordance with the result of Theorem 2.9.

3. Dispersive ordering and hazard rate ordering

Lemma 3.1 ([19]). *Let* $\{F_a | a \in \Re\}$ *be a class of distribution functions, such that* F_a *is supported on some interval* $(x_-^{(a)}, x_+^{(b)}) \subseteq (0, \infty)$ *and has a density* f_a *which does not vanish on any subinterval of* $(x_-^{(a)}, x_+^{(b)})$, *where* $x_-^{(a)}$ *and* $x_+^{(b)}$ *mean the left and right end points, respectively. Then,*

$$F_a \le_{\mathrm{disp}} F_{a^*}, \quad a, a^* \in \Re, \; a \le a^*$$

if and only if $F_a'(x)/f_a(x)$ *is decreasing in* x, *where* F_a' *is the derivative of* F_a *with respect to* a.

Theorem 3.2. *Let* $Y_{(r_i,\,\lambda_i)}$ $(Y_{(r_i,\,\lambda_i^*)})$, $i = 1, 2$ *be two independent gamma random variables with* $Y_{(r_i,\,\lambda_i)}$ $(Y_{(r_i,\,\lambda_i^*)})$ *having the shape parameter* r_i *and scale parameter* λ_i (λ_i^*). *If* $\lambda_1 \ge \lambda_2$, $\lambda_1^* \ge \lambda_2^*$ *and* $1 \le r_1 \le r_2$, *then*

$$(\lambda_1, \lambda_2) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2)} \ge_{\mathrm{disp}} Y_{(r_1,\,\lambda_1^*)} + Y_{(r_2,\,\lambda_2^*)}.$$

Proof. Suppose $(\lambda_1, \lambda_2) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \lambda_2^*)$. We then have that $\lambda_2 \le \lambda_2^*$ and $\lambda_1\lambda_2 \le \lambda_1^*\lambda_2^*$. There must exist some λ_1' such that $\lambda_1' \ge \lambda_1$ and $\lambda_1'\lambda_2 = \lambda_1^*\lambda_2^*$. Let $Y_{(r_1,\,\lambda_1')}$ be a gamma random variable with the shape parameter r_1 and scale parameter λ_1', independent of $Y_{(r_2,\,\lambda_2)}$. Since a gamma random variable $Y_{(r,\,\lambda)}$ is decreasing with the scale parameter λ in the sense of the dispersive order, it follows that $Y_{(r_1,\,\lambda_1)} \ge_{\mathrm{disp}} Y_{(r_1,\,\lambda_1')}$. Moreover, it is known that the gamma distribution whose shape parameter is greater than or equal to 1 has logconcave density. Applying this and Theorem 3.B.9 of [22], it follows that

$$Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2)} \ge_{\mathrm{disp}} Y_{(r_1,\,\lambda_1')} + Y_{(r_2,\,\lambda_2)}.$$

Thus, it reduces to proving the theorem under the conditions $\lambda_2 \le \lambda_2^* \le \lambda_1^* \le \lambda_1$ and $\lambda_1\lambda_2 = \lambda_1^*\lambda_2^*$. If $\lambda_1 = \lambda_1^*$, then $\lambda_2 = \lambda_2^*$ and hence the result is trivially true. In what follows we only need to give the proof for the case when $\lambda_1 \neq \lambda_1^*$ and $\lambda_1^* \neq \lambda_2^*$ as a limiting argument can be used to prove the result when $\lambda_1^* = \lambda_2^*$. Let $a_1 = \log\lambda_1,\ a_2 = \log\lambda_2,\ a_1^* = \log\lambda_1^*,\ a_2^* = \log\lambda_2^*$ and $a_1 + a_2 = a_1^* + a_2^* = d$. We then have $a_1 > a_2$ and $a_1^* > a_2^*$. Also, the following relation holds:

$$(a_1, a_2) \overset{\mathrm{m}}{\succeq} (a_1^*, a_2^*).$$

Upon setting $a_1 = a$ and using (2.2), we get

$$f(y; r_1, r_2, a) = \frac{e^{r_1 a} e^{r_2(d-a)} y^{r_1+r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \int_0^1 u^{r_1-1}(1-u)^{r_2-1} \exp\{-[e^a u + e^{d-a}(1-u)]y\} du.$$

Taking the derivative with respect to a for $f(y; r_1, r_2, a)$ and after simplifications, we have

$$\begin{aligned} f'(y; r_1, r_2, a) &= \frac{(r_1-r_2) e^{r_1 a} e^{r_2(d-a)} y^{r_1+r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \int_0^1 u^{r_1-1}(1-u)^{r_2-1} \exp\{-[e^a u + e^{d-a}(1-u)]y\} du \\ &\quad - \frac{r_1 e^{(r_1+1)a} e^{r_2(d-a)} y^{(r_1+1)+r_2-1}}{\Gamma(r_1+1)\Gamma(r_2)} \int_0^1 u^{r_1}(1-u)^{r_2-1} \exp\{-[e^a u + e^{d-a}(1-u)]y\} du \\ &\quad + \frac{r_2 e^{r_1 a} e^{(r_2+1)(d-a)} y^{r_1+(r_2+1)-1}}{\Gamma(r_1)\Gamma(r_2+1)} \int_0^1 u^{r_1-1}(1-u)^{r_2} \exp\{-[e^a u + e^{d-a}(1-u)]y\} du \\ &= (r_1-r_2) f(y; r_1, r_2, a) - r_1 f(y; r_1+1, r_2, a) + r_2 f(y; r_1, r_2+1, a). \end{aligned} \tag{3.1}$$

Note that the Laplace transform of $f(y; r_1, r_2, a)$ is given by

$$L(f(y; r_1, r_2, a)) = \left(\frac{e^a}{s+e^a}\right)^{r_1} \left(\frac{e^{d-a}}{s+e^{d-a}}\right)^{r_2}, \tag{3.2}$$

and taking the Laplace transform of both sides of (3.1) and applying (3.2) yields that

$$\begin{aligned} L(f'(y; r_1, r_2, a)) &= r_1[L(f(y; r_1, r_2, a)) - L(f(y; r_1+1, r_2, a))] + r_2[L(f(y; r_1, r_2+1, a)) - L(f(y; r_1, r_2, a))] \\ &= \left[\frac{s r_1(s+e^{d-a})}{e^d} - \frac{s r_2(s+e^a)}{e^d}\right] \left(\frac{e^a}{s+e^a}\right)^{r_1+1} \left(\frac{e^{d-a}}{s+e^{d-a}}\right)^{r_2+1}. \end{aligned} \tag{3.3}$$

Now apply the relation $L\left(\int_0^y f'(u; r_1, r_2, a) du\right) = L(f'(y; r_1, r_2, a))/s$ and (3.3) to give that

$$\begin{aligned} L(F'(y; r_1, r_2, a)) &= \frac{r_1(s+e^{d-a}) - r_2(s+e^a)}{e^d} \left(\frac{e^a}{s+e^a}\right)^{r_1+1} \left(\frac{e^{d-a}}{s+e^{d-a}}\right)^{r_2+1} \\ &= -\frac{r_1(e^a - e^{d-a})}{e^d} \left(\frac{e^a}{s+e^a}\right)^{r_1+1} \left(\frac{e^{d-a}}{s+e^{d-a}}\right)^{r_2+1} - \frac{(r_2-r_1)}{e^{d-a}} \left(\frac{e^a}{s+e^a}\right)^{r_1} \left(\frac{e^{d-a}}{s+e^{d-a}}\right)^{r_2+1}. \end{aligned} \tag{3.4}$$

Upon taking the inverse Laplace transforms of both sides of (3.4) and dividing it by $f(y; r_1, r_2, a)$, we obtain

$$\frac{F'(y; r_1, r_2, a)}{f(y; r_1, r_2, a)} = -\frac{r_1(e^a - e^{d-a})}{e^d} \frac{f(y; r_1+1, r_2+1, a)}{f(y; r_1, r_2, a)} - \frac{(r_2-r_1)}{e^{d-a}} \frac{f(y; r_1, r_2+1, a)}{f(y; r_1, r_2, a)}. \tag{3.5}$$

It can be readily verified that both

$$\frac{f(y; r_1+1, r_2+1, a)}{f(y; r_1, r_2, a)} \quad \text{and} \quad \frac{f(y; r_1, r_2+1, a)}{f(y; r_1, r_2, a)}$$

are increasing in $y > 0$ for $1 \le r_1 \le r_2$, and hence the left hand side of (3.5) is decreasing in y since $a > d - a$ and $r_1 \le r_2$. Therefore, the desired result follows from Lemma 3.1. □

The following theorem is a natural extension of Theorem 3.2.

Theorem 3.3. *Let $Y_{(r_1,\lambda_1)}, \ldots, Y_{(r_n,\lambda_n)}$ be independent gamma random variables with respective shape parameter vector $\boldsymbol{r} = (r_1, \ldots, r_n)$ where each component is greater than or equal to 1 and scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and let $Y_{(r_1,\lambda_1^*)}, \ldots, Y_{(r_n,\lambda_n^*)}$ be another set of independent gamma random variables with respective shape parameter vector $\boldsymbol{r}$ and scale parameter vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose there exists some permutation π such that $\pi\boldsymbol{\lambda} = \boldsymbol{\lambda}_\downarrow$, $\pi\boldsymbol{\lambda}^* = \boldsymbol{\lambda}_\downarrow$ and $\pi\boldsymbol{r} = \boldsymbol{r}_\uparrow$. Then,*

$$\boldsymbol{\lambda} \overset{\mathrm{p}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^n Y_{(r_i,\lambda_i)} \ge_{\mathrm{disp}} \sum_{i=1}^n Y_{(r_i,\lambda_i^*)}.$$

Proof. Without loss of generality, let us assume that $\lambda_1 \le \cdots \le \lambda_n$, $\lambda_1^* \le \cdots \le \lambda_n^*$ and $r_1 \ge \cdots \ge r_n$. $\boldsymbol{\lambda} \overset{\text{p}}{\succeq} \boldsymbol{\lambda}^*$ is equivalent to $\prod_{i=1}^{j} \lambda_i \le \prod_{i=1}^{j} \lambda_i^*,\ 1 \le j \le n$. It is easy to see that there exists some $\lambda_n' \ge \max\{\lambda_n,\ \lambda_n^*\}$ such that

$$\prod_{i=1}^{j} \lambda_i \le \prod_{i=1}^{j} \lambda_i^*, \quad 1 \le j \le n-1 \quad \text{and} \quad \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} \lambda_i^*.$$

Let $Y_{(r_n,\ \lambda_n')}$ be a gamma random variable with the shape parameter r_n and scale parameter λ_n', independent of $Y_{(r_i,\ \lambda_i)}$ $(1 \le i \le n-1)$, it then follows that $Y_{(r_n,\ \lambda_n)} \ge_{\text{disp}} Y_{(r_n,\ \lambda_n')}$. Since the convolution of gamma distributions with shape parameter greater than or equal to 1 has a logconcave density, it follows that

$$\sum_{i=1}^{n} Y_{(r_i,\ \lambda_i)} \ge_{\text{disp}} \sum_{i=1}^{n-1} Y_{(r_i,\ \lambda_i)} + Y_{(r_n,\ \lambda_n')}$$

by applying Theorem 3.B.9 of [22]. Denote

$$\boldsymbol{a} = (a_1, \ldots, a_n) = (\log \lambda_1, \ldots, \log \lambda_n), \quad \boldsymbol{a}^* = (a_1^*, \ldots, a_n^*) = (\log \lambda_1^*, \ldots, \log \lambda_n^*),$$

we then find that it is enough to prove that

$$\boldsymbol{a} \overset{\text{m}}{\succeq} \boldsymbol{a}^* \Longrightarrow \sum_{i=1}^{n} Y_{(r_i,\ e^{a_i})} \ge_{\text{disp}} \sum_{i=1}^{n} Y_{(r_i,\ e^{a_i^*})}.$$

By the nature of majorization, it suffices to prove the result for the case when $(a_1,\ a_2) \overset{\text{m}}{\succeq} (a_1^*,\ a_2^*)$ and $a_i = a_i^*,\ i = 3, \ldots, n$. From Theorem 3.2, it follows that

$$Y_{(r_1,\ e^{a_1})} + Y_{(r_2,\ e^{a_2})} \ge_{\text{disp}} Y_{(r_1,\ e^{a_1^*})} + Y_{(r_2,\ e^{a_2^*})}.$$

Since $\sum_{i=3}^{n} Y_{(r_i,\ e^{a_i})}$ $\left(\sum_{i=3}^{n} Y_{(r_i,\ e^{\lambda_i^*})}\right)$ has a logconcave density, applying Theorem 3.B.9 of [22] once again yields that

$$\sum_{i=1}^{n} Y_{(r_i,\ \lambda_i)} = Y_{(r_1,\ e^{a_1})} + Y_{(r_2,\ e^{a_2})} + \sum_{i=3}^{n} Y_{(r_i,\ e^{a_i})} \ge_{\text{disp}} Y_{(r_1,\ e^{a_1^*})} + Y_{(r_2,\ e^{a_2^*})} + \sum_{i=3}^{n} Y_{(r_i,\ e^{a_i^*})} = \sum_{i=1}^{n} Y_{(r_i,\ \lambda_i^*)}. \quad \square$$

Finally, we give a general result which can be used to compare heterogeneous gamma convolutions in terms of the hazard rate order wherein both shape parameter vectors and scale parameter vectors are different. Let X and Y be two random variables with distribution functions F and G, respectively. Bagai and Kochar [1] then showed that $X \le_{\text{disp}} Y$ and F or G being IFR (increasing failure rate) implies that $X \le_{\text{hr}} Y$. Using this and the fact that the convolution of IFR distributions is still IFR, we immediately get the following result from Theorems 2.5 and 3.3.

Theorem 3.4. *Let $Y_{(r_1,\ \lambda_1)}, \ldots, Y_{(r_n,\ \lambda_n)}$ be independent gamma random variables with respective shape parameter vector $\boldsymbol{r} = (r_1, \ldots, r_n)$ where each component is greater than or equal to 1 and scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and let $Y_{(r_1^*,\ \lambda_1^*)}, \ldots, Y_{(r_n^*,\ \lambda_n^*)}$ be another set of independent gamma random variables with respective shape parameter vector $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*)$ where each component is greater than or equal to 1 and scale parameter vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose there exists some permutation π such that $\pi\boldsymbol{r} = \boldsymbol{r}_\uparrow$, $\pi\boldsymbol{r}^* = \boldsymbol{r}_\uparrow^*$, $\pi\boldsymbol{\lambda} = \boldsymbol{\lambda}_\downarrow$ and $\pi\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*{}_\downarrow$. Then,*

$$\boldsymbol{r} \overset{\text{m}}{\succeq} \boldsymbol{r}^* \quad \textit{and} \quad \boldsymbol{\lambda} \overset{\text{p}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^{n} Y_{(r_i,\ \lambda_i)} \ge_{\text{hr}} \sum_{i=1}^{n} Y_{(r_i^*,\ \lambda_i^*)}.$$

Let (X_1, X_2) be a vector of independent heterogeneous gamma random variables with shape parameter vector $(r_1, r_2) = (3, 1)$ and scale parameter vector $(\lambda_1, \lambda_2) = (1, 3)$. Denote by $h(t; 3, 1, 1, 3)$ the hazard rate function of convolution between X_1 and X_2. Let (X_1^*, X_2^*) be another vector of independent heterogeneous gamma random variables with shape parameter parameter $(r_1^*, r_2^*) = (2, 2)$ and scale parameter vector $(\lambda_1^*, \lambda_2^*) = (1.5, 2)$, and denote by $h(t; 2, 2, 1.5, 2)$ the hazard rate function of convolution between X_1^* and X_2^*. It is clear that $r_1 \ge r_2$, $r_1^* \ge r_2^*$, $\lambda_1 \le \lambda_2$, $\lambda_1^* \le \lambda_2^*$, $(r_1, r_2) \overset{\text{m}}{\succeq} (r_1^*, r_2^*)$ and $(\lambda_1, \lambda_2) \overset{\text{p}}{\succeq} (\lambda_1^*, \lambda_2^*)$ (but the $\overset{\text{w}}{\succeq}$ order does not hold between these two vectors). It can be seen from Fig. 2 that $h(t; 3, 1, 1, 3) \le h(t; 2, 2, 1.5, 2)$ for all $t \in \Re_+$ which is in accordance with the result of Theorem 3.4.

4. Star ordering and right spread ordering

We shall need the following lemma for proving the main result, which is originally due to [19, p. 429].

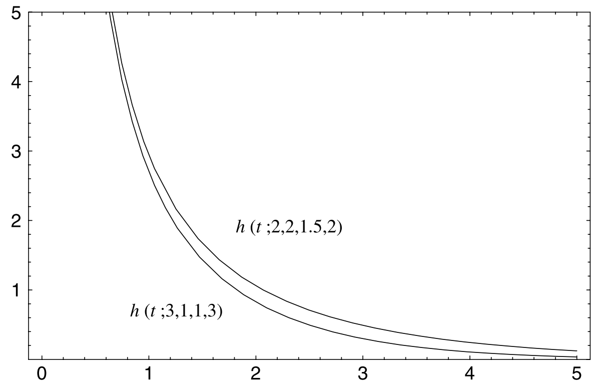

Fig. 2. Plots of hazard rate functions of two gamma convolutions.

Lemma 4.1. *Let $\{F_a | a \in \Re\}$ be a class of distribution functions, such that F_a is supported on some interval $(x_-^{(a)}, x_+^{(b)}) \subseteq (0, \infty)$ and has a density f_a which does not vanish on any subinterval of $(x_-^{(a)}, x_+^{(b)})$. Then,*

$$F_a \le_* F_{a^*}, \quad a, a^* \in \Re,\ a \le a^*$$

if and only if $F_a'(x)/xf_a(x)$ is decreasing in x, where F_a' is the derivative of F_a with respect to a.

Theorem 4.2. *Let $Y_{(r_i,\,\lambda_i)}$ $(Y_{(r_i,\,\lambda_i^*)})$, $i = 1, 2$ be two independent gamma random variables with $Y_{(r_i,\,\lambda_i)}$ $(Y_{(r_i,\,\lambda_i^*)})$ having the shape parameter $r_i > 0$ and scale parameter λ_i (λ_i^*). If $\lambda_1 \ge \lambda_2$, $\lambda_1^* \ge \lambda_2^*$ and $r_1 \le r_2$, then*

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2)} \ge_* Y_{(r_1,\,\lambda_1^*)} + Y_{(r_2,\,\lambda_2^*)}.$$

Proof. Assume that $(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*)$. We then have $\lambda_2 \le \lambda_2^* \le \lambda_1^* \le \lambda_1$ and $\lambda_1 + \lambda_2 = \lambda_1^* + \lambda_2^* = c$. If $\lambda_1 = \lambda_1^*$ then $\lambda_2 = \lambda_2^*$, and the result is trivially true. We only need to prove the result for the case when $\lambda_1 \ne \lambda_1^*$ and $\lambda_1^* \ne \lambda_2^*$ since the case when $\lambda_1^* = \lambda_2^*$ can be readily obtained by a limiting argument. Letting $\lambda_1 = \lambda \in (c/2, c]$, we can rewrite $Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2)}$ as $Y_{(r_1,\,\lambda)} + Y_{(r_2,\,c-\lambda)}$ and from (2.2) its density is given by

$$f(y; r_1, r_2, \lambda) = \frac{\lambda^{r_1}(c-\lambda)^{r_2} y^{r_1+r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \int_0^1 u^{r_1-1}(1-u)^{r_2-1} \exp\{-[\lambda u + (c-\lambda)(1-u)]y\} du.$$

Taking the derivative with respect λ for $f(y;\ r_1,\ r_2,\ \lambda)$ and after simplifications, we have

$$\begin{aligned} f'(y; r_1, r_2, \lambda) &= \left(\frac{r_1}{\lambda} - \frac{r_2}{c-\lambda}\right) \frac{\lambda^{r_1}(c-\lambda)^{r_2} y^{r_1+r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \int_0^1 u^{r_1-1}(1-u)^{r_2-1} \exp\{-[\lambda u + (c-\lambda)(1-u)]y\} du \\ &\quad - \frac{r_1}{\lambda} \frac{\lambda^{(r_1+1)}(c-\lambda)^{r_2} y^{(r_1+1)+r_2-1}}{\Gamma(r_1+1)\Gamma(r_2)} \int_0^1 u^{r_1}(1-u)^{r_2-1} \exp\{-[\lambda u + (c-\lambda)(1-u)]y\} du \\ &\quad + \frac{r_2}{c-\lambda} \frac{\lambda^{r_1}(c-\lambda)^{(r_2+1)} y^{r_1+(r_2+1)-1}}{\Gamma(r_1)\Gamma(r_2+1)} \int_0^1 u^{r_1-1}(1-u)^{r_2} \exp\{-[\lambda u + (c-\lambda)(1-u)]y\} du \\ &= \left(\frac{r_1}{\lambda} - \frac{r_2}{c-\lambda}\right) f(y; r_1, r_2, \lambda) - \frac{r_1}{\lambda} f(y; r_1+1, r_2, \lambda) + \frac{r_2}{c-\lambda} f(y; r_1, r_2+1, \lambda). \end{aligned} \tag{4.1}$$

It is known that the Laplace transform of $f(y; r_1, r_2, \lambda)$ can be written as

$$L(f(y; r_1, r_2, \lambda)) = \left(\frac{\lambda}{s+\lambda}\right)^{r_1} \left(\frac{c-\lambda}{s+c-\lambda}\right)^{r_2}, \tag{4.2}$$

and taking the Laplace transform of both sides of (4.1) and applying (4.2) yields that

$$\begin{aligned} L(f'(y; r_1, r_2, \lambda)) &= \frac{r_1}{\lambda}[L(f(y; r_1, r_2, \lambda)) - L(f(y; r_1+1, r_2, \lambda))] \\ &\quad + \frac{r_2}{c-\lambda}[L(f(y; r_1, r_2+1, \lambda)) - L(f(y; r_1, r_2, \lambda))] \\ &= \frac{s}{\lambda^2(c-\lambda)^2} [r_1(s+c-\lambda)(c-\lambda) - r_2(s+\lambda)\lambda] \left(\frac{\lambda}{s+\lambda}\right)^{r_1+1} \left(\frac{c-\lambda}{s+c-\lambda}\right)^{r_2+1} \end{aligned}$$

$$= \frac{sr_1}{\lambda^2}\left(\frac{\lambda}{s+\lambda}\right)^{r_1+1}\left(\frac{c-\lambda}{s+c-\lambda}\right)^{r_2} - \frac{sr_2}{(c-\lambda)^2}\left(\frac{\lambda}{s+\lambda}\right)^{r_1}\left(\frac{c-\lambda}{s+c-\lambda}\right)^{r_2+1}. \tag{4.3}$$

Now apply the relation $L\left(\int_0^y f'(u; r_1, r_2, \lambda)\mathrm{d}u\right) = L(f'(y; r_1, r_2, \lambda))/s$ and (4.3) to give that

$$L(F'(y; r_1, r_2, \lambda)) = \frac{r_1}{\lambda^2}\left(\frac{\lambda}{s+\lambda}\right)^{r_1+1}\left(\frac{c-\lambda}{s+c-\lambda}\right)^{r_2} - \frac{r_2}{(c-\lambda)^2}\left(\frac{\lambda}{s+\lambda}\right)^{r_1}\left(\frac{c-\lambda}{s+c-\lambda}\right)^{r_2+1}. \tag{4.4}$$

Upon taking the inverse Laplace transforms of both sides of (4.4) and dividing it by $yf(y; r_1, r_2, \lambda)$, we obtain

$$\frac{F'(y; r_1, r_2, \lambda)}{yf(y; r_1, r_2, \lambda)} = \frac{r_1}{\lambda^2}\frac{f(y; r_1+1, r_2, \lambda)}{yf(y; r_1, r_2, \lambda)} - \frac{r_2}{(c-\lambda)^2}\frac{f(y; r_1, r_2+1, \lambda)}{yf(y; r_1, r_2, \lambda)}. \tag{4.5}$$

Hence, according to Lemma 4.1, it suffices to prove that

$$\frac{f(y; r_1+1, r_2, \lambda)}{yf(y; r_1, r_2, \lambda)}$$

is decreasing while

$$\frac{f(y; r_1, r_2+1, \lambda)}{yf(y; r_1, r_2, \lambda)}$$

is increasing in $y > 0$. It is seen from Theorem 2.1 that

$$\begin{aligned}\frac{f(y; r_1+1, r_2, \lambda)}{yf(y; r_1, r_2, \lambda)} &\propto \frac{\int_0^1 \left[(1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta yw) + (1+w)^{r_2-1}(1-w)^{r_1}\exp(\theta yw)\right]\mathrm{d}w}{\int_0^1 \left[(1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta yw) + (1+w)^{r_2-1}(1+w)^{r_1-1}\exp(\theta yw)\right]\mathrm{d}w} \\ &= \mathsf{E}_y\psi_2(W, y),\end{aligned}$$

where $\theta = (\lambda_1 - \lambda_2)/2$ and

$$\varphi_2(w, y) = \frac{(1-w)^{r_2-1}(1+w)^{r_1}\exp(-\theta yw) + (1-w)^{r_1}(1+w)^{r_2-1}\exp(\theta yw)}{(1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta yw) + (1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta yw)}$$

for $w \in (0,\ 1)$. Here, the distribution function of the random variable W belongs to the family $\mathcal{P} = \{H_2(\cdot|y), y \in \Re\}$ with densities

$$h_2(w|y) = c_2(y)\left[(1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta yw) + (1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta yw)\right]$$

and a normalizing constant $c_2(y)$ such that $\int_0^1 h_2(w|y)\,\mathrm{d}w = 1$. It can be verified that

$$\begin{aligned}\varphi_2(w, y) &= 1 - w\frac{(1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta yw) - (1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta yw)}{(1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta yw) + (1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta yw)} \\ &= 1 - w\left[1 - \frac{2}{\left(\frac{1+w}{1-w}\right)^{r_2-r_1}\exp(2\theta yw) + 1}\right]\end{aligned}$$

are decreasing both in $w \in (0,\ 1)$ and $y \in (0,\ \infty)$. In addition, for $y_2 \ge y_1 > 0$,

$$\frac{h_2(w|y_2)}{h_2(w|y_1)} \propto \frac{(1-w)^{r_2-r_1}\exp(-\theta y_2 w) + (1+w)^{r_2-r_1}\exp(\theta y_2 w)}{(1-w)^{r_2-r_1}\exp(-\theta y_1 w) + (1+w)^{r_2-r_1}\exp(\theta y_1 w)}$$

is increasing in $w \in (0,\ 1)$ from Lemma 2.3(b). From this one gets that $H_2(\cdot|y_1) \le_{\mathrm{lr}} H_2(\cdot|y_2)$ which in turn implies that $H_2(\cdot|y_1) \le_{\mathrm{st}} H_2(\cdot|y_2)$ whenever $0 < y_1 \le y_2$. By using Lemma 2.2 now, $\mathsf{E}_y\varphi_2(W, y)$ is decreasing in $y \in (0,\ \infty)$. To conclude, we finally need to prove that

$$\begin{aligned}\frac{f(y; r_1, r_2+1, \lambda)}{yf(y; r_1, r_2, \lambda)} &\propto \frac{\int_0^1 \left[(1-w)^{r_2}(1+w)^{r_1-1}\exp(-\theta yw) + (1+w)^{r_2}(1-w)^{r_1-1}\exp(\theta yw)\right]\mathrm{d}w}{\int_0^1 \left[(1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta^* yw) + (1+w)^{r_2-1}(1+w)^{r_1-1}\exp(\theta^* yw)\right]\mathrm{d}w} \\ &= \mathsf{E}_y\psi_3(W, y)\end{aligned}$$

is increasing in $y \in (0,\ \infty)$, where

$$\varphi_3(w, y) = \frac{(1-w)^{r_2}(1+w)^{r_1-1}\exp(-\theta yw) + (1-w)^{r_1-1}(1+w)^{r_2}\exp(\theta yw)}{(1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta yw) + (1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta yw)}$$

for $w \in (0, 1)$. Here, the distribution function of the random variable W belongs to the family $\mathcal{P} = \{H_3(\cdot|y), y \in \Re\}$ with densities

$$h_3(w|y) = c_3(y)\left[(1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta y w) + (1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta y w)\right]$$

and a normalizing constant $c_3(y)$ such that $\int_0^1 h_3(w|y)\,\mathrm{d}w = 1$. Note that

$$\begin{aligned}\varphi_3(w, y) &= 1 + w\frac{(1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta y w) - (1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta y w)}{(1-w)^{r_1-1}(1+w)^{r_2-1}\exp(\theta y w) + (1-w)^{r_2-1}(1+w)^{r_1-1}\exp(-\theta y w)} \\ &= 1 + w\left[1 - \frac{2}{\left(\frac{1+w}{1-w}\right)^{r_2-r_1}\exp(2\theta y w) + 1}\right]\end{aligned}$$

are increasing both in $w \in (0, 1)$ and $y \in (0, \infty)$. Moreover, it is known from the discussion above that $H_3(\cdot|y_1) \le_{\mathrm{st}} H_3(\cdot|y_2)$ whenever $0 < y_1 \le y_2$. By using Lemma 2.2 once again, $\mathsf{E}_y\varphi_3(W, y)$ is increasing in $y \in (0, \infty)$. Thus, we finish the entire proof. □

In the next result we shall present a different condition on the scale parameter vectors of gamma convolutions for star ordering to hold.

Theorem 4.3. *Let $Y_{(r_i, \theta_i)}$ ($Y_{(r_i, \theta_i^*)}$), $i = 1, 2$ be two independent gamma random variables with $Y_{(r_i, \theta_i)}$ ($Y_{(r_i, \theta_i^*)}$) having the shape parameter r_i and scale parameter $\theta_i = 1/\lambda_i$ ($\theta_i^* = 1/\lambda_i^*$). If $\lambda_1 \ge \lambda_2$ and $\lambda_1^* \ge \lambda_2^*$, then*

$$(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow Y_{(r_1, \theta_1)} + Y_{(r_2, \theta_2)} \ge_* Y_{(r_1, \theta_1^*)} + Y_{(r_2, \theta_2^*)}.$$

Proof. Following a similar argument to the proof in Theorem 4.2, we only need to prove the result for the case when $\lambda_1^* \ne \lambda_2^*$. Assume $(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \lambda_2^*)$ to hold and let $\lambda = \lambda_1, \lambda^* = \lambda_1^*$ and $\lambda_1 + \lambda_2 = c$. We then have $\lambda \ge \lambda^* > c/2$. Thus, $Y_{(r_1, \theta_1)} + Y_{(r_2, \theta_2)}$ can be rewritten as $\lambda Y_{(r_1, 1)} + (c-\lambda)Y_{(r_2, 1)}$ and its distribution function is given by

$$F(t; r_1, r_2, \lambda) = \iint \frac{{x_1}^{r_1-1}\mathrm{e}^{-x_1}}{\Gamma(r_1)}\frac{{x_2}^{r_2-1}\mathrm{e}^{-x_2}}{\Gamma(r_2)}\mathrm{d}x_1\mathrm{d}x_2,$$

where the integration is over the region $x_1, x_2 \ge 0$ and $\lambda x_1 + (c-\lambda)x_2 \le t$, and hence it can be rewritten as

$$F(t; r_1, r_2, \lambda) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t/\lambda}\int_0^{(t-\lambda x_1)/(c-\lambda)} {x_1}^{r_1-1}\mathrm{e}^{-x_1}{x_2}^{r_2-1}\mathrm{e}^{-x_2}\mathrm{d}x_2\mathrm{d}x_1.$$

Making the transforms

$$r = x_1 + x_2, \qquad s = \frac{x_1}{x_1 + x_2}$$

one gets

$$F(t; r_1, r_2, \lambda) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^1 s^{r_1-1}(1-s)^{r_2-1}\int_0^{t/h(s)} r^{r_1+r_2-1}\mathrm{e}^{-r}\mathrm{d}r\mathrm{d}s, \tag{4.6}$$

where $h(s) = \lambda s + (c-\lambda)(1-s)$. Taking the derivative with respect to λ, we get

$$F'(t; r_1, r_2, \lambda) = \frac{t^{r_1+r_2}}{\Gamma(r_1)\Gamma(r_2)}\int_0^1 \frac{s^{r_1-1}(1-s)^{r_2-1}(1-2s)}{[h(s)]^{r_1+r_2+1}}\mathrm{e}^{-t/h(s)}\mathrm{d}s.$$

In addition, it can be seen from (4.6) that the density function of $Y_{(r_1, \theta_1)} + Y_{(r_2, \theta_2)}$ is given by

$$f(t; r_1, r_2, \lambda) = \frac{t^{r_1+r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\int_0^1 \frac{s^{r_1-1}(1-s)^{r_2-1}}{[h(s)]^{r_1+r_2}}\mathrm{e}^{-t/h(s)}\mathrm{d}s.$$

Upon applying Lemma 4.1, we find that it suffices to prove that

$$\frac{F'(t; r_1, r_2, \lambda)}{tf(t; r_1, r_2, \lambda)} = \frac{\int_0^1 \frac{s^{r_1-1}(1-s)^{r_2-1}(1-2s)}{[h(s)]^{r_1+r_2+1}}\mathrm{e}^{-t/h(s)}\mathrm{d}s}{\int_0^1 \frac{s^{r_1-1}(1-s)^{r_2-1}}{[h(s)]^{r_1+r_2}}\mathrm{e}^{-t/h(s)}\mathrm{d}s} = \mathsf{E}_t\psi(S, t)$$

is decreasing in $t \in (0, \infty)$, where

$$\varphi(s, t) = \frac{1-2s}{h(s)}$$

for $s \in (0,\ 1)$. The distribution function of the random variable S belongs to the family $\mathcal{P} = \{H(\cdot|t), t \in \Re\}$ with densities

$$h(s|t) = c(t)\frac{s^{r_1-1}(1-s)^{r_2-1}}{[h(s)]^{r_1+r_2}}e^{-t/h(s)}$$

and a normalizing constant $c(t)$ such that $\int_0^1 h(s|t)\,ds = 1$. It can be readily checked that $\varphi(s,t)$ are decreasing both in $s \in (0,\ 1)$ and $t \in (0,\ \infty)$. On the other hand, for $t_2 \geq t_1 > 0$, we have

$$\frac{h(s|t_2)}{h(s|t_1)} \propto \exp\left(-\frac{t_2-t_1}{h(s)}\right)$$

is increasing in $s \in (0,\ 1)$ which implies that $H(\cdot|t_1) \leq_{\rm st} H(\cdot|t_2)$ whenever $0 < t_1 \leq t_2$. By using Lemma 2.2, it follows that $\mathsf{E}_t\varphi(S,t)$ is decreasing in $t \in (0,\ \infty)$. □

Remark 4.4. Theorem 4.3 here extends Theorem 3.6 in [13] wherein they only gave the result for the special case when the gamma distributions have common shape parameters.

Next, we present the result on the right spread order.

Theorem 4.5. *Let $Y_{(r_i,\lambda_i)}$ $(Y_{(r_i,\lambda_i^*)})$, $i = 1, 2$ be two independent gamma random variables with $Y_{(r_i,\lambda_i)}$ $(Y_{(r_i,\lambda_i^*)})$ having the shape parameter r_i and scale parameter λ_i (λ_i^*). If $\lambda_1 \geq \lambda_2$, $\lambda_1^* \geq \lambda_2^*$ and $1 \leq r_1 \leq r_2$, then*

$$(\lambda_1,\lambda_2) \stackrel{\rm rm}{\succeq} (\lambda_1^*,\lambda_2^*) \Longrightarrow Y_{(r_1,\lambda_1)} + Y_{(r_2,\lambda_2)} \geq_{\rm RS} Y_{(r_1,\lambda_1^*)} + Y_{(r_2,\lambda_2^*)}.$$

Proof. From the definition of $\stackrel{\rm rm}{\succeq}$ order, $(\lambda_1,\lambda_2) \stackrel{\rm rm}{\succeq} (\lambda_1^*,\lambda_2^*)$ implies that

$$\frac{1}{\lambda_2} \geq \frac{1}{\lambda_2^*}, \qquad \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \geq \frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}.$$

To obtain the required result, we now need to distinguish two cases.

Case (a): $\frac{1}{\lambda_1} \geq \frac{1}{\lambda_1^*}$

In this case, we have

$$\frac{1}{\lambda_1} \geq \frac{1}{\lambda_1^*}, \qquad \frac{1}{\lambda_2} \geq \frac{1}{\lambda_2^*},$$

which implies that

$$Y_{(r_1,\lambda_1)} \geq_{\rm RS} Y_{(r_1,\lambda_1^*)}, \qquad Y_{(r_2,\lambda_2)} \geq_{\rm RS} Y_{(r_2,\lambda_2^*)}.$$

Since a gamma random variable with shape parameter greater than or equal to 1 has logconcave density function, the result follows from Theorem 3.C.7 of [22].

Case (b): $\frac{1}{\lambda_1} < \frac{1}{\lambda_1^*}$

In this case, we have

$$\frac{1}{\lambda_2} \geq \frac{1}{\lambda_2^*} \geq \frac{1}{\lambda_1^*} > \frac{1}{\lambda_1}.$$

It can be seen that there exists some λ_2' such that

$$\frac{1}{\lambda_2} \geq \frac{1}{\lambda_2'} \geq \frac{1}{\lambda_2^*} \quad \text{and} \quad \frac{1}{\lambda_1} + \frac{1}{\lambda_2'} = \frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}.$$

Note that

$$Y_{(r_1,\lambda_1)} + Y_{(r_2,\lambda_2)} \geq_{\rm RS} Y_{(r_1,\lambda_1)} + Y_{(r_2,\lambda_2')},$$

and hence it will be enough if we could prove that

$$Y_{(r_1,\lambda_1)} + Y_{(r_2,\lambda_2')} \geq_{\rm RS} Y_{(r_1,\lambda_1^*)} + Y_{(r_2,\lambda_2^*)}.$$

Since

$$\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2'}\right) \stackrel{\rm m}{\succeq} \left(\frac{1}{\lambda_1^*},\frac{1}{\lambda_2^*}\right),$$

it follows from Theorem 4.3 that

$$Y_{(r_1,\lambda_1)} + Y_{(r_2,\lambda_2')} \geq_* Y_{(r_1,\lambda_1^*)} + Y_{(r_2,\lambda_2^*)},$$

which in turn implies that

$$Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2')} \geq_{\text{NBUE}} Y_{(r_1,\,\lambda_1^*)} + Y_{(r_2,\,\lambda_2^*)}. \tag{4.7}$$

Let $Y = Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2')}$ with distribution and survival functions F and $\overline{F}$, respectively. Similarly, let $Y^* = Y_{(r_1,\,\lambda_1^*)} + Y_{(r_2,\,\lambda_2^*)}$ with distribution and survival functions G and $\overline{G}$, respectively. From (4.7) and the definition of $\geq_{\text{NBUE}}$ order, we then have

$$\frac{1}{\mu_F}\int_{F^{-1}(u)}^{\infty} \overline{F}(x)\,\mathrm{d}x \geq \frac{1}{\mu_G}\int_{G^{-1}(u)}^{\infty} \overline{G}(x)\,\mathrm{d}x \tag{4.8}$$

for all $u \in (0,\ 1]$, where μ_F (μ_G) denotes the mean of Y (Y^*). It can be readily seen that

$$\mu_F = \frac{r_1}{\lambda_1} + \frac{r_2}{\lambda_2'} = r_1\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2'}\right) + (r_2 - r_1)\frac{1}{\lambda_2'}$$

and

$$\mu_G = \frac{r_1}{\lambda_1^*} + \frac{r_2}{\lambda_2^*} = r_1\left(\frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}\right) + (r_2 - r_1)\frac{1}{\lambda_2^*},$$

which implies that $\mu_F \geq \mu_G$ as $r_1 \leq r_2$. Using this and (4.8) one gets

$$\int_{F^{-1}(u)}^{\infty} \overline{F}(x)\,\mathrm{d}x \geq \int_{G^{-1}(u)}^{\infty} \overline{G}(x)\,\mathrm{d}x,$$

that is,

$$Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2')} \geq_{\text{RS}} Y_{(r_1,\,\lambda_1^*)} + Y_{(r_2,\,\lambda_2^*)},$$

which completes the proof. □

Theorem 4.6. *Let $Y_{(r_1,\,\lambda_1)}, \ldots, Y_{(r_n,\,\lambda_n)}$ be independent gamma random variables with respective shape parameter vector $\mathbf{r} = (r_1, \ldots, r_n)$ where each component is greater than or equal to 1 and scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and let $Y_{(r_1,\,\lambda_1^*)}, \ldots, Y_{(r_n,\,\lambda_n^*)}$ be another set of independent gamma random variables with respective shape parameter vector $\mathbf{r}$ and scale parameter vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose there exists some permutation π such that $\pi\boldsymbol{\lambda} = \boldsymbol{\lambda}_\downarrow$, $\pi\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*{}_\downarrow$ and $\pi\mathbf{r} = \mathbf{r}_\uparrow$. Then,*

$$\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right) \overset{\text{m}}{\succeq} \left(\frac{1}{\lambda_1^*}, \ldots, \frac{1}{\lambda_n^*}\right) \Longrightarrow \sum_{i=1}^{n} Y_{(r_i,\,\lambda_i)} \geq_{\text{RS}} \sum_{i=1}^{n} Y_{(r_i,\,\lambda_i^*)}.$$

Proof. Without loss of generality, let us assume that $\lambda_1 \geq \cdots \geq \lambda_n,\ \lambda_1^* \geq \cdots \geq \lambda_n^*$ and $r_1 \leq \cdots \leq r_n$. By the nature of majorization, it suffices to prove the result for the case when

$$\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right) \overset{\text{m}}{\succeq} \left(\frac{1}{\lambda_1^*}, \frac{1}{\lambda_2^*}\right) \tag{4.9}$$

and $\lambda_i = \lambda_i^*,\ i = 3, \ldots, n$. Since (4.9) implies that

$$(\lambda_1,\ \lambda_2) \overset{\text{rm}}{\succeq} (\lambda_1^*,\ \lambda_2^*),$$

it follows from Theorem 4.5 that

$$Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2)} \geq_{\text{RS}} Y_{(r_1,\,\lambda_1^*)} + Y_{(r_2,\,\lambda_2^*)}.$$

Note that $\sum_{i=3}^{n} Y_{(r_i,\,\lambda_i)}$ $\left(\sum_{i=3}^{n} Y_{(r_i,\,\lambda_i^*)}\right)$ has a logconcave density, using this and Theorem 3.C.7 of [22] yields that

$$\sum_{i=1}^{n} Y_{(r_i,\,\lambda_i)} = Y_{(r_1,\,\lambda_1)} + Y_{(r_2,\,\lambda_2)} + \sum_{i=3}^{n} Y_{(r_i,\,\lambda_i)} \geq_{\text{RS}} Y_{(r_1,\,\lambda_1^*)} + Y_{(r_2,\,\lambda_2^*)} + \sum_{i=3}^{n} Y_{(r_i,\,\lambda_i^*)} = \sum_{i=1}^{n} Y_{(r_i,\,\lambda_i^*)}. \quad \square$$

Similar to Theorem 3.4, we also give a general result on the mean residual life order. As the convolution of DMRL (decreasing mean residual life) distributions is still DMRL, using Theorem 3.C.5 of [22], one can get the following result from Theorems 2.5 and 4.6.

Theorem 4.7. *Let $Y_{(r_1,\,\lambda_1)}, \ldots, Y_{(r_n,\,\lambda_n)}$ be independent gamma random variables with respective shape parameter vector $\mathbf{r} = (r_1, \ldots, r_n)$ where each component is greater than or equal to 1 and scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and let*

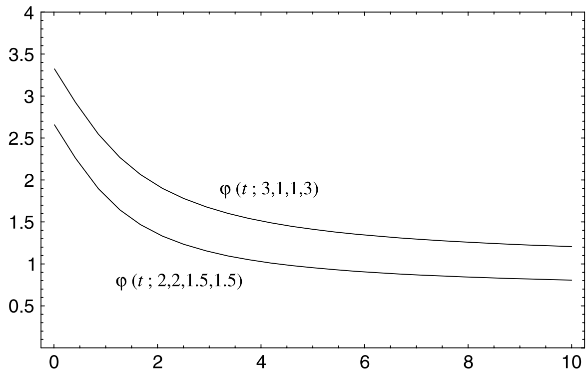

Fig. 3. Plots of mean residual life functions of two gamma convolutions.

$Y_{(r_1^*,\,\lambda_1^*)}, \ldots, Y_{(r_n^*,\,\lambda_n^*)}$ be another set of independent gamma random variables with respective shape parameter vector $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*)$ where each component is greater than or equal to 1 and scale parameter vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose there exists some permutation π such that $\pi\boldsymbol{r} = \boldsymbol{r}_\uparrow, \pi\boldsymbol{r}^* = \boldsymbol{r}^*_\uparrow, \pi\boldsymbol{\lambda} = \boldsymbol{\lambda}_\downarrow$ and $\pi\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*{}_\downarrow$. Then,

$$\boldsymbol{r} \overset{\text{m}}{\succeq} \boldsymbol{r}^* \quad \textit{and} \quad \left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right) \overset{\text{m}}{\succeq} \left(\frac{1}{\lambda_1^*}, \ldots, \frac{1}{\lambda_n^*}\right) \Longrightarrow \sum_{i=1}^n Y_{(r_i,\,\lambda_i)} \geq_{\text{mrl}} \sum_{i=1}^n Y_{(r_i^*,\,\lambda_i^*)}.$$

Let (X_1, X_2) be a vector of independent heterogeneous gamma random variables with shape parameter vector $(r_1, r_2) = (3, 1)$ and scale parameter vector $(\lambda_1, \lambda_2) = (1, 3)$. Denote by $\varphi(t; 3, 1, 1, 3)$ the mean residual life function of convolution between X_1 and X_2. Let (X_1^*, X_2^*) be another vector of independent heterogeneous gamma random variables with shape parameter parameter $(r_1^*, r_2^*) = (2, 2)$ and scale parameter vector $(\lambda_1^*, \lambda_2^*) = (1.5, 1.5)$, and denote by $\varphi(t; 2, 2, 1.5, 1.5)$ the mean residual life function of convolution between X_1^* and X_2^*. Note that $r_1 \geq r_2, r_1^* \geq r_2^*, \lambda_1 \leq \lambda_2, \lambda_1^* \leq \lambda_2^*, (r_1, r_2) \overset{\text{m}}{\succeq} (r_1^*, r_2^*)$ and $\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right) \overset{\text{m}}{\succeq} \left(\frac{1}{\lambda_1^*}, \frac{1}{\lambda_2^*}\right)$ (but the $\overset{\text{p}}{\succeq}$ order does not hold between (λ_1, λ_2) and $(\lambda_1^*, \lambda_2^*)$). It can be seen from Fig. 3 that $\varphi(t; 3, 1, 1, 3) \geq \varphi(t; 2, 2, 1.5, 1.5)$ for all $t \in \Re_+$ which is in accordance with the result of Theorem 4.7.

Remark 4.8. It is remarkable that the main results of this paper in Theorems 2.9, 3.3, 4.2 and 4.6 strengthen and generalize the corresponding those of Theorem 3.4 in [14], Theorem 2.1 in [9], Theorem 3.3 in [13] and Theorem 4.2 in [12] from convolutions of independent gamma random variables with common shape parameters to ones with different shape parameters. As stated in some counterexamples provided in the literature, these results cannot be extended in general to the case where convolutions have different shape parameters. However, here we have established the extension under the restriction that the components of the shape vector are ordered in an opposite way with those of the scale vector.

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Appendix

Proof of Lemma 2.3. (a) Derivative of g with respect to y is

$$\begin{aligned}
g'(y) &\overset{\text{sgn}}{=} [-w_2\exp(-w_2y) + \alpha w_2\exp(w_2y)][\exp(-w_1y) + \alpha\exp(w_1y)] \\
&\quad - [-w_1\exp(-w_1y) + \alpha w_1\exp(w_1y)][\exp(-w_2y) + \alpha\exp(w_2y)] \\
&= \{-w_2\exp[-(w_1+w_2)y] - \alpha w_2\exp[(w_1-w_2)y] + \alpha w_2\exp[(w_2-w_1)y] + \alpha^2 w_2\exp[(w_1+w_2)y]\} \\
&\quad - \{-w_1\exp[-(w_1+w_2)y] - \alpha w_1\exp[(w_2-w_1)y] + \alpha w_1\exp[(w_1-w_2)y] + \alpha^2 w_1\exp[(w_1+w_2)y]\} \\
&= (w_1-w_2)\exp[-(w_1+w_2)y] + \alpha(w_1+w_2)\exp[(w_2-w_1)y] - \alpha(w_1+w_2)\exp[(w_1-w_2)y] \\
&\quad + \alpha^2(w_2-w_1)\exp[(w_1+w_2)y] \\
&\geq (w_2-w_1)\{\exp[(w_1+w_2)y] - \exp[-(w_1+w_2)y]\} \\
&\quad + \alpha(w_1+w_2)\{\exp[(w_2-w_1)y] - \exp[-(w_2-w_1)y]\} \\
&\geq 0,
\end{aligned}$$

which implies that $g(y)$ is increasing in $y > 0$.

(b) Taking the derivative with respect to w for $\zeta(w)$, we have

$$\begin{aligned}
\zeta'(w) &\overset{\text{sgn}}{=} [-a(1-w)^{a-1}\exp(-\theta_2 w) - \theta_2(1-w)^a\exp(-\theta_2 w) + a(1+w)^{a-1}\exp(\theta_2 w) + \theta_2(1+w)^a\exp(\theta_2 w)] \\
&\quad \times [(1-w)^a\exp(-\theta_1 w) + (1+w)^a\exp(\theta_1 w)] - [-a(1-w)^{a-1}\exp(-\theta_1 w) - \theta_1(1-w)^a\exp(-\theta_1 w) \\
&\quad + a(1+w)^{a-1}\exp(\theta_1 w) + \theta_1(1+w)^a\exp(\theta_1 w)] \times [(1-w)^a\exp(-\theta_2 w) + (1+w)^a\exp(\theta_2 w)] \\
&= (\theta_2-\theta_1)(1+w)^{2a}\exp[(\theta_1+\theta_2)w] - (\theta_2-\theta_1)(1-w)^{2a}\exp[-(\theta_1+\theta_2)w] \\
&\quad + \left(\frac{a}{1-w} + \frac{a}{1+w} + \theta_1 + \theta_2\right)(1-w^2)^a\{\exp[(\theta_2-\theta_1)w] - \exp[(\theta_1-\theta_2)w]\} \\
&\geq (\theta_2-\theta_1)(1-w)^{2a}\{\exp[(\theta_1+\theta_2)w] - \exp[-(\theta_1+\theta_2)w]\} \\
&\quad + \left(\frac{a}{1-w} + \frac{a}{1+w} + \theta_1 + \theta_2\right)(1-w^2)^a\{\exp[(\theta_2-\theta_1)w] - \exp[-(\theta_2-\theta_1)w]\} \\
&\geq 0.
\end{aligned}$$

So we can conclude that the function $\zeta(w)$ is increasing in $w \in (0, 1)$. □

Proof of Lemma 2.6. (a) The required result can be readily obtained only by noting that

$$\vartheta'(y) \overset{\text{sgn}}{=} (\beta-\alpha)w\exp(wy) \geq 0.$$

(b) Take the derivative with respect to w for $\kappa(w)$ to give

$$\begin{aligned}
\kappa'(w) &\overset{\text{sgn}}{=} \left[\left(\frac{r_1}{1+w} - \frac{r_2}{1-w} - y\right)(1-w)^{r_2}(1+w)^{r_1}\mathrm{e}^{-wy} + \left(\frac{r_2}{1+w} - \frac{r_1}{1-w} + y\right)\cdot(1-w)^{r_1}(1+w)^{r_2}\mathrm{e}^{wy}\right] \\
&\quad \times [(1-w)^{r_2^*}(1+w)^{r_1^*}\mathrm{e}^{-wy} + (1-w)^{r_1^*}(1+w)^{r_2^*}\mathrm{e}^{wy}] \\
&\quad - \left[\left(\frac{r_1^*}{1+w} - \frac{r_2^*}{1-w} - y\right)(1-w)^{r_2^*}(1+w)^{r_1^*}\mathrm{e}^{-wy} + \left(\frac{r_2^*}{1+w} - \frac{r_1^*}{1-w} + y\right)\right. \\
&\quad \left.\cdot(1-w)^{r_1^*}(1+w)^{r_2^*}\mathrm{e}^{wy}\right] \times [(1-w)^{r_2}(1+w)^{r_1}\mathrm{e}^{-wy} + (1-w)^{r_1}(1+w)^{r_2}\mathrm{e}^{wy}].
\end{aligned}$$

Simplifying the above expression by routine calculations one gets

$$\kappa'(w) \overset{\text{sgn}}{=} \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4,$$

where

$$\begin{aligned}
\varepsilon_1 &= \left(\frac{r_1}{1+w} - \frac{r_2}{1-w} - \frac{r_1^*}{1+w} + \frac{r_2^*}{1-w}\right)(1-w)^{r_2+r_2^*}(1+w)^{r_1+r_1^*}\mathrm{e}^{-2yw}; \\
\varepsilon_2 &= \left(\frac{r_2}{1+w} - \frac{r_1}{1-w} - \frac{r_2^*}{1+w} + \frac{r_1^*}{1-w}\right)(1-w)^{r_1+r_1^*}(1+w)^{r_2+r_2^*}\mathrm{e}^{2yw}; \\
\varepsilon_3 &= \left(\frac{r_1}{1+w} - \frac{r_2}{1-w} - \frac{r_2^*}{1+w} + \frac{r_1^*}{1-w} - 2y\right)(1-w)^{r_1^*+r_2}(1+w)^{r_1+r_2^*}; \\
\varepsilon_4 &= \left(\frac{r_2}{1+w} - \frac{r_1}{1-w} - \frac{r_1^*}{1+w} + \frac{r_2^*}{1-w} + 2y\right)(1-w)^{r_1+r_2^*}(1+w)^{r_1^*+r_2}.
\end{aligned}$$

Now we find that it will be enough if we could prove that $\delta_1 = \varepsilon_1 + \varepsilon_2 \geq 0$ and $\delta_2 = \varepsilon_3 + \varepsilon_4 \geq 0$. Note that

$$\begin{aligned}
\delta_1 &= \varepsilon_1 + \varepsilon_2 \\
&\geq \left(\frac{r_2 - r_2^*}{1+w} + \frac{r_1^* - r_1}{1-w} + \frac{r_1 - r_1^*}{1+w} + \frac{r_2^* - r_2}{1-w}\right)(1-w)^{r_2+r_2^*}(1+w)^{r_1+r_1^*}\mathrm{e}^{-2yw} \\
&= \left[\frac{(r_1+r_2) - (r_1^*+r_2^*)}{1+w} + \frac{(r_1^*+r_2^*) - (r_1+r_2)}{1-w}\right](1-w)^{r_2+r_2^*}(1+w)^{r_1+r_1^*}\mathrm{e}^{-2yw} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\delta_2 &= \varepsilon_3 + \varepsilon_4 \\
&\geq \left(\frac{r_2 - r_1^*}{1+w} + \frac{r_2^* - r_1}{1-w} + \frac{r_1 - r_2^*}{1+w} + \frac{r_1^* - r_2}{1-w}\right)(1-w)^{r_1^*+r_2}(1+w)^{r_1+r_2^*}
\end{aligned}$$

$$= \left[\frac{(r_1 + r_2) - (r_1^* + r_2^*)}{1 + w} + \frac{(r_1^* + r_2^*) - (r_1 + r_2)}{1 - w}\right](1 - w)^{r_1^* + r_2}(1 + w)^{r_1 + r_2^*}$$
$$= 0.$$

Thus, the desired result follows. □

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