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T-duality, quotients and generalized Kähler geometry

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ABSTRACT

We reopen the discussion of gauging two-dimensional off-shell $(2,2)$ supersymmetric sigma models written in terms of semichiral superfields. The gauging is now done by coupling the semichiral superfields to the new $(2,2)$ semichiral vector multiplet. We show that the two moment maps together with a third function form the complete set of three Killing potentials which are associated with this gauging. These Killing potentials lead to generalized moment maps. Next we construct the T-duality map, while keeping $(2,2)$ supersymmetry manifest. In the T-dual description, a pair of left and right semichiral superfields is replaced by a pair of chiral and twisted chiral multiplets. We end with a discussion on quotient construction.

1. Introduction and summary

The geometry of the target space of two-dimensional sigma models is dictated by the amount of preserved world-sheet supersymmetry and by the representation of the sigma model fields. In the physics literature it has been known for quite a while [1] that $(2,2)$ supersymmetric sigma models give rise to special geometry manifolds. These are called bi-hermitean manifolds, and are endowed with a Riemannian metric g, a closed three-form $H = 3dB$, and two complex structures $J^{(\pm)}$. The metric is hermitean with respect to both complex structures, and $J^{(\pm)}$ are covariantly constant with respect to connections that have torsion determined by H. More recently, it has been shown that the superfield representations needed for a complete description of the $(2,2)$ supersymmetric sigma model include, besides the better known chiral and twisted chiral superfields, are the semichiral superfields [2]. With only chiral and twisted chiral among the sigma model fields, the bi-hermitean geometry acquires an almost product structure, with the two complex structures commuting. In the case when the sigma model fields include the left and right semichiral superfields, the commutator of the two complex structures no longer vanishes. It is this latter case that we address in this Letter.

In the mathematics literature, the study of the generalized Calabi–Yau manifolds, which include a non-trivial B-field, lead to the development of generalized complex geometry [3]. Its main object is the generalized complex structure defined on the direct sum of the tangent and cotangent bundles $T \oplus T^*$. A special case of generalized complex geometry is the generalized Kähler geometry, which has two commuting generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$, and a positive definite metric $G = -\mathcal{J}_1\mathcal{J}_2$. Gualtieri [4] has shown the equivalence of the data which define the bi-hermitean geometry with those of the generalized Kähler geometry. Therefore these two notions are interchangeable. Special cases of the generalized Kähler geometry include symplectic and Kähler geometry. For recent work on related topics see [5].

An interesting question arises in the presence of isometries. In the symplectic case it is possible to talk about a Hamiltonian reduction, by defining the moment map (a function which is preserved by the action of the isometry group and follows from the requirement that the symplectic form is preserved) and restricting to the subspace of constant moment map. Similarly it is possible to define a Kähler quotient. The basic object associated with the quotient construction is the moment map. There are several proposals for the moment map definition in the context of generalized complex geometry. On the other hand, from the sigma model perspective, there is a natural place to look for the moment map, and that is the gauged sigma model. The moment map (sometimes referred to as Killing potential) appears in the off-shell $(2,2)$ supersymmetric gauged sigma model action, multiplying the gauge superfield strengths. We will match the sigma model construction of the moment map with the appropriate mathematical definition.

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This Letter is a follow up to [6], and here we give the answer to the open questions of that previous work. The new ingredient is the use of the appropriate $(2, 2)$ semichiral vector multiplet [7,8] for the gauging of the $(2, 2)$ supersymmetric semichiral sigma model. This is the subject of Section 2. We reduce the gauged action to $(1, 1)$ superspace. From the manifestly $(1, 1)$ supersymmetric gauged action we identify a set of three Killing potentials. In Section 3 we relate the Killing potentials to the moment maps. We have done this by starting from the reduced $(1, 1)$ action, and inquiring what are the conditions which insure its invariance under the second set of non-manifest $(1, 1)$ supersymmetries. Besides the usual bi-hermitean geometry requirements, we found a set of conditions which express the two moment maps in terms of the Killing potentials. In the process, we discovered that the third Killing potential is instrumental in fixing a certain ambiguity in the definition of the moment maps (from general arguments, the moment maps are defined up to a function σ such that $d\sigma$ is invariant under the action of the isometry group). After this ambiguity was fixed in the way we described, then we were able to prove the existence of two generalized moment maps, defined for the twisted generalized Kähler structure in [9], one for each generalized complex structure. In Section 4 we turn to the subject of T-duality. Our starting point is the gauged $(2, 2)$ semichiral sigma model action. We construct the duality functional in $(2, 2)$ superspace. Under T-duality, a pair of left and right semichiral superfields along the isometry direction and their anti-fields are replaced by a pair of chiral and twisted chiral superfields, and their anti-fields. Lastly, in Section 5, we discuss the quotient construction, and work out one explicit example.

2. The gauged $(2, 2)$ sigma model reduced to $(1, 1)$ superspace

We begin by recalling the new gauged $(2, 2)$ supersymmetry algebra, which defines the new semichiral vector multiplet [7,8] (our notation follows [7]):

$$\{\nabla_+, \nabla_+\} = \{\nabla_-, \nabla_-\} = 0, \qquad \{\nabla_+, \nabla_-\} = -4i\lambda \bar{T}\xi,$$

$$\{\nabla_+, \bar{\nabla}_-\} = 2\lambda(-S + iP)\xi, \qquad \{\nabla_-, \bar{\nabla}_+\} = 2\lambda(S + iP)\xi,$$

$$\{\nabla_+, \bar{\nabla}_+\} = 2i\nabla_{\ddagger}, \qquad \{\nabla_-, \bar{\nabla}_-\} = 2i\nabla_=,$$

$$[\nabla_=, \nabla_{\ddagger}] = [\nabla_=, \nabla_=] = 0, \qquad [\nabla_{\ddagger}, \nabla_=] = -\lambda\mathcal{W}\xi, \tag{1}$$

where the gauged supercovariant derivatives are defined as $\nabla_\alpha = D_\alpha - \lambda\Gamma_\alpha\xi$, and where Γ_α is the superconnection. The constraints preserving the semichiral representation $\gamma_a^{\alpha\beta}[\nabla_a, \nabla_\beta] = 0$ are solved by $\Gamma_+ = D_+\bar{V}_1$, $\Gamma_- = D_-\bar{V}_2$. The standard constraints $\gamma_a^{\alpha\beta}[\nabla_\alpha, \bar{\nabla}_\beta] = -4i\nabla_a$ allow for solving the vector superfield gauge connection, Γ_a in terms of Γ_α. The bosonic gauge-covariant derivatives are denoted by $\nabla_{\ddagger} = 2(\nabla_0 + \nabla_1)$, $\nabla_= = 2(\nabla_0 - \nabla_1)$. ξ is the generator of the $U(1)$ gauge transformation.[1]

As discussed in [6,10], the gauging of the sigma model can be done most straightforwardly at the level of $(2, 2)$ superspace. Here the sigma-model is defined entirely by the Kähler potential, which is a functional of the $(2, 2)$ superfields. The $(2, 2)$ superfields needed for a complete description of the two-dimensional off-shell $(2, 2)$ supersymmetric sigma models are [2]:

chiral: $\bar{D}_\pm\phi = 0$,   anti-chiral: $D_\pm\bar{\phi} = 0$,

twisted chiral: $\bar{D}_+\psi = D_-\psi = 0$,   twisted anti-chiral: $D_+\bar{\psi} = \bar{D}_-\bar{\psi} = 0$,

left semichiral: $\bar{D}_+X = 0$,   left anti-semichiral: $D_+\bar{X} = 0$,

right semichiral: $\bar{D}_-Y = 0$,   right anti-semichiral: $D_-\bar{Y} = 0$.   (2)

In the case we are interested in, the Kähler potential depends on left and right semichiral superfields and their anti-fields[2]

$$\mathcal{S} = \int d^2\bar{\theta}\, d^2\theta\, K(X, Y, \bar{X}, \bar{Y}). \tag{3}$$

Next, one uses that the Grassmann integration is equivalent to differentiation. In order to couple the matter fields to the vector superfield, the supercovariant derivatives $D_\alpha, \bar{D}_\alpha$ are replaced by the gauged supercovariant derivatives $\nabla_\alpha, \bar{\nabla}_\alpha$. Lastly, we descend to the level of $(1, 1)$ superspace by replacing the $(2, 2)$ gauged supercovariant derivatives by two copies of $(1, 1)$ derivatives. The final step is to keep only one of the two $(1, 1)$ supersymmetries manifest, by reducing along the direction of the other $(1, 1)$. This will give the manifestly $(1, 1)$ supersymmetric gauged sigma model.

More concretely, the two $(1, 1)$ gauge supercovariant derivatives are defined by

$$\hat{\nabla}_\alpha = \frac{1}{\sqrt{2}}(\nabla_\alpha + \bar{\nabla}_\alpha), \qquad \tilde{\nabla}_\alpha = \frac{i}{\sqrt{2}}(\nabla_\alpha - \bar{\nabla}_\alpha). \tag{4}$$

It is important to keep in mind that from the point of view of the $(1, 1)$ gauged sigma model, the $\tilde{\nabla}_\alpha$ derivatives act as the generators of the additional, non-manifest $(1, 1)$ supersymmetry.

The $(2, 2)$ fermionic measure is evaluated using the $(1, 1)$ gauge supercovariant derivatives

$$\int d^2\bar{\theta}\, d^2\theta = \frac{1}{16}\hat{\nabla}^\alpha\hat{\nabla}_\alpha\tilde{\nabla}^\beta\tilde{\nabla}_\beta. \tag{5}$$

The implicit assumption here is that the Kähler potential that we are gauging is invariant under the symmetry transformation (there is of course the possibility that the Kähler potential is invariant up to general Kähler transformations; the extension to this case, though relatively straightforward, is not addressed in this Letter).

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[1] Since we will be gauging a $U(1)$ isometry of target space associated to a sigma model we have replaced the usual anti-hermitian $U(1)$ generator denoted t with $t = -i\xi$, where ξ is the Killing vector for the isometry.

[2] Both types of semichiral superfields are needed to define a sigma-model [11].

We now reduce the manifestly $(2, 2)$ supersymmetric action to $(1, 1)$ superspace by evaluating the $(1, 1)$ derivatives $\tilde{\nabla}^\alpha \tilde{\nabla}_\alpha$ onto the Kähler potential. After some algebra, we obtain

$$
\tilde{\nabla}^\alpha \tilde{\nabla}_\alpha K = \frac{i}{2}\big[\hat{\nabla}_+\varphi^I m_{II'}\hat{\nabla}_-\chi^{I'} + \Upsilon_+^{I'} n_{I'I}\Psi_-^I + \Psi_-^I\big(2\omega_{IJ}\hat{\nabla}_+\varphi^J + ip_{II'}\hat{\nabla}_+\chi^{I'}\big) + \Upsilon_+^{I'}\big(2\omega_{I'J'}\hat{\nabla}_-\chi^{J'} - iq_{I'I}\hat{\nabla}_-\varphi^I\big)\big]
$$
$$
+ 8i\lambda\big[K_i\big(\xi X^i\big) - K_{\bar{i}}\big(\xi \bar{X}^{\bar{i}}\big) - K_{i'}\big(\xi Y^{i'}\big) + K_{\bar{i}'}\big(\xi \bar{Y}^{\bar{i}'}\big)\big]T_2 - 4i\lambda\big[K_i\big(\xi X^i\big) - K_{\bar{i}}\big(\xi \bar{X}^{\bar{i}}\big) + K_{i'}\big(\xi Y^{i'}\big) - K_{\bar{i}'}\big(\xi \bar{Y}^{\bar{i}'}\big)\big]S
$$
$$
+ 2\lambda\big[K_i\big(\xi X^i\big) + K_{\bar{i}}\big(\xi \bar{X}^{\bar{i}}\big) - K_{i'}\big(\xi Y^{i'}\big) - K_{\bar{i}'}\big(\xi \bar{Y}^{\bar{i}'}\big)\big](2T_1 + P),
$$

$$(6)$$

where we have kept the notation of [12]: $I = (i, \bar{i})$, $\varphi = X|$, $\chi = Y|$, etc. By inspecting the resulting $(1, 1)$ sigma model action, we see that, as expected, we have the same metric and NS–NS two-form obtained in [12]. However, there are some differences with respect to the case when the gauging of the $U(1)$ isometry is done by using the usual $(2, 2)$ super-Yang–Mills multiplet [6]. These differences are visible in the terms which depend on the superfield strengths. We shall focus on this aspect in the next section.

3. Moment maps

In the case of Kähler geometry, which is the target space geometry associated with a sigma model derived from a $(2, 2)$ chiral superfield-dependent Kähler potential [13], the gauging of an isometry requires that the generator of the isometry preserves not only the metric (i.e. it is Killing) but the complex structure as well (i.e. it is holomorphic). As a consequence, the isometry generator preserves the symplectic form $\omega = g J$. Therefore,

$$
\mathcal{L}_\xi \omega = i_\xi d\omega + d(i_\xi\omega) = 0 \tag{7}
$$

implies that $i_\xi\omega$ is locally exact. This defines the moment map

$$
i_\xi\omega = d\mu, \tag{8}
$$

also referred to as the Hamiltonian function for symplectic manifolds, and the Killing potential for Kähler manifolds [14]. In the latter case, by going to the holomorphic coordinate base which diagonalizes the complex structure, and using that $\omega = 2i\partial_{i\bar{j}}K\, d\phi^i \wedge d\bar{\phi}^{\bar{j}}$, one finds

$$
-i\xi^i\partial_{i\bar{j}}K = \partial_{\bar{j}}\mu, \qquad i\xi^{\bar{j}}\partial_{i\bar{j}}K = \partial_i\mu. \tag{9}
$$

This can be integrated in the case of an $U(1)$ isometry to yield

$$
\mu = -i\xi^i\partial_i K + i\xi^{\bar{j}}\partial_{\bar{j}}K. \tag{10}
$$

In the study of $(2, 2)$ supersymmetric two-dimensional sigma models, Gates et al. [1] showed that their target space admits a bi-hermitean metric (hermitean with respect to two complex structures). The complex structures are covariantly constant with respect to a torsion-full connection. The torsion is related to the field strength of a two-form potential, the B field. In the mathematics literature, the bi-hermitean geometry is known as generalized Kähler geometry [4].

If the $(2, 2)$ supersymmetric sigma model employs only chiral and twisted chiral superfields, the two complex structures commute. This type of geometry is referred to as an almost product structure space [1]. As in the previous case, the moment map follows from requiring that the isometry generator preserve the anti-symmetric two-forms $\omega^{(\pm)} = g J^{(\pm)}$. This means that

$$
\mathcal{L}_\xi \omega^{(\pm)} = 0. \tag{11}
$$

In the case of generalized Kähler geometry, $\omega^{(\pm)}$ is no longer a closed form. In the presence of a non-trivial B-field it satisfies

$$
\pm d\omega^{(\pm)}\big(J^{(\pm)}X, J^{(\pm)}Y, J^{(\pm)}Z\big) = dB(X, Y, Z). \tag{12}
$$

Then from (11) it follows that

$$
d\mu_\pm = \omega^{(\pm)} \cdot \xi \mp J^{(\pm)T} \cdot u, \tag{13}
$$

where $i_\xi H = du$ and $H = 3dB$. When $\mu_\pm$ can be defined globally they are called moment maps. Since the isometry generator ξ preserves the complex structures, it also respects the natural decomposition of the tangent space induced by the chiral ϕ^i and twisted chiral $\psi^{i'}$ coordinates. For $\xi = \xi^i\partial_i + \xi^{\bar{i}}\partial_{\bar{i}}$, the gauging of the sigma model is done by coupling with an ordinary $(2, 2)$ vector multiplet. For $\tilde{\xi} = \tilde{\xi}^{i'}\partial_{i'} + \tilde{\xi}^{\bar{i}'}\partial_{\bar{i}'}$, the gauging is done by coupling with a twisted $(2, 2)$ vector multiplet [15].[3] Following an off-shell $(2, 2)$ supersymmetric sigma model analysis, Hull et al. [15] showed that the moment maps are identified with the Killing potentials $i\xi^{\bar{i}}\partial_{\bar{i}}K$ and respectively $i\tilde{\xi}^{\bar{i}'}\partial_{\bar{i}'}K$. In terms of the significance of the moment maps for the generalized Kähler geometry, it can be shown that it is either the sum or the difference of the two moment maps μ_+, μ_- which defines an eigenvector of the generalized complex structure $\mathcal{J}_{1/2}$ [6], i.e. $(\xi \pm \frac{i}{2}(d\mu_+ \pm d\mu_-)) \in T \oplus T^*$ lies in the eigenbundle of $\mathcal{J}_{1/2}$.

Lastly, we turn to the generic case of bi-hermitean geometry with non-commuting complex structure, which is realized by a semichiral superfield sigma-model [12]. In [6] it was found by studying a certain example of generalized Kähler geometry, the $SU(2) \times U(1)$ WZNW sigma model, that the two a priori distinct moment maps are indeed distinct. This point deserves a further investigation since the on-shell $(2, 2)$ supersymmetric sigma model analysis in [16] points out to the existence of a unique moment map, with μ_+ and μ_- being identified.

---

[3] The large vector multiplet introduced in [8] can be used to gauge an isometry which mixes the chiral and twisted chiral directions.

The connection between the moment maps and the gauged sigma model action was previously discussed in [15]. The idea is to start from the reduced $(1, 1)$ supersymmetric sigma-model action, and require that it is invariant under the additional, non-manifest $(1, 1)$ supersymmetries generated by $\tilde{\nabla}_\pm$. These act on the $(1, 1)$ sigma-model superfields as

$$\delta\Phi = \frac{i}{\sqrt{2}}\big[\epsilon^+(\nabla_+ - \bar{\nabla}_+) + \epsilon^-(\nabla_- - \bar{\nabla}_-)\big]\Phi = \frac{1}{\sqrt{2}}\big(\epsilon^+ J^{(+)}\cdot\hat{\nabla}_+\Phi + \epsilon^- J^{(-)}\cdot\hat{\nabla}_-\Phi\big), \tag{14}$$

where Φ stands for the sigma-model superfields $\varphi^I, \chi^{I'}$ [12].

The action of the non-manifest supersymmetries on the gauge superconnections is inferred from:

$$\delta\hat{\nabla}_\pm\Phi^m = \pm 2i\lambda\epsilon^\mp(S \pm 2T_2)\xi^m - \epsilon^+\hat{\nabla}_\pm\big(J^{(\pm)m}{}_n\hat{\nabla}_\pm\Phi^n\big) - \epsilon^-\hat{\nabla}_\pm\big(J^{(\mp)m}{}_n\hat{\nabla}_\mp\Phi^n\big). \tag{15}$$

Further using that $S - iP$ is a twisted chiral superfield and that T is chiral, we find the non-manifest supersymmetry variation of the field strength superfields:

$$\delta(S - iP) = i\big(-\epsilon^+\hat{\nabla}_+ + \epsilon^-\hat{\nabla}_-\big)(S - iP), \qquad \delta T = i\big(\epsilon^+\hat{\nabla}_+ + \epsilon^-\hat{\nabla}_-\big)T. \tag{16}$$

Let us now concentrate on the invariance of the manifestly $(1, 1)$ supersymmetric gauged sigma model action

$$\mathcal{S} = \int d^2x\, d^2\hat{\theta}\big(2i\hat{\nabla}_+\Phi\cdot(g + B)\cdot\hat{\nabla}_-\Phi + 4\lambda S\mu_1 - 8\lambda T_2\mu_2 + 2\lambda\sigma(2T_1 + P)\big) \tag{17}$$

under the additional (14)–(16) supersymmetries. In the case we are investigating, we have assumed that the Kähler potential is strictly invariant under the action of the $U(1)$ isometry generator ξ. Because of this assumption, the first term in the gauged action is actually obtained by minimal coupling. In other words, since $\mathcal{L}_\xi g = \mathcal{L}_\xi B = 0$, then the kinetic terms and the B-field dependent terms in the sigma-model are gauged in the same way, by minimal coupling. We have introduced the notation μ_1, μ_2 for the terms which multiply the superfield strengths S, T_2 in (17), even though we have their concrete expression in terms of derivatives of the Kähler potential from (6). The reason for our feigned ignorance is that we want to be able to show the rapport between μ_1, μ_2 and the moment maps. This will become transparent once we require that (17) has the additional $(1, 1)$ supersymmetries.

The invariance of (17) is conditioned, among other things, by the cancellation of the terms in $\delta\mathcal{S}$ which are proportional to the superfield strengths S, P, T_1, T_2. Those terms which are proportional to S are

$$4\lambda\epsilon^+\left(-\xi^m(g + B)_{nm} + \partial_m\mu_1 J^{(+)m}{}_n - \frac{1}{2}\partial_n\sigma\right)\hat{\nabla}_+\Phi^n + 4\lambda\epsilon^-\left(-\xi^m(g + B)_{mn} + \partial_m\mu_1 J^{(-)m}{}_n + \frac{1}{2}\partial_n\sigma\right)\hat{\nabla}_-\Phi^n. \tag{18}$$

Therefore we find that

$$d\mu_1 = -\xi\cdot(g - B)\cdot J^{(+)} - \frac{1}{2}d\sigma\cdot J^{(+)}, \qquad d\mu_1 = -\xi\cdot(g + B)\cdot J^{(-)} + \frac{1}{2}d\sigma\cdot J^{(-)}. \tag{19}$$

Similarly, the terms which are proportional to T_2 are

$$8\lambda\epsilon^+\left(-\xi^m(g + B)_{nm} - \partial_m\mu_2 J^{(+)m}{}_n + \frac{1}{2}\partial_n\sigma\right)\hat{\nabla}_+\Phi^n + 8\lambda\epsilon^-\left(\xi^m(g + B)_{mn} - \partial_m\mu_2 J^{(-)m}{}_n + \frac{1}{2}\partial_n\sigma\right)\hat{\nabla}_-\Phi^n, \tag{20}$$

which implies that the action is invariant provided that

$$d\mu_2 = \xi\cdot(g - B)\cdot J^{(+)} - \frac{1}{2}d\sigma\cdot J^{(+)}, \qquad d\mu_2 = -\xi\cdot(g + B)\cdot J^{(-)} - \frac{1}{2}d\sigma\cdot J^{(-)}. \tag{21}$$

In order for these two sets of equations to be satisfied, σ must be such that

$$d\sigma = (d\mu_1 + d\mu_2)\cdot J^{(+)} = -(d\mu_1 - d\mu_2)\cdot J^{(-)}. \tag{22}$$

To complete our investigation of the relationship between the Killing potentials μ_1, μ_2, σ and the moment maps μ_+, μ_-, we recall that we have worked under the assumption that the Kähler potential is invariant under the action of the isometry generator $\xi K = 0$. With the metric and B-field determined by the invariant Kähler potential, then $\mathcal{L}_\xi g = \mathcal{L}_\xi B = 0$. As a consequence, the one form u defined in (3) can be explicitly solved

$$\mathcal{L}_\xi B = d(i_\xi B) + i_\xi dB = 0 \implies u = -\xi\cdot B + d\tilde{\sigma}, \tag{23}$$

where $d\tilde{\sigma}$ is an exact one-form, invariant under ξ. What (19) and (21) show is that

$$u = -\xi\cdot B + \frac{1}{2}d\sigma \tag{24}$$

and that $2\mu_1$ and $2\mu_2$ are equal to the sum and respectively the difference of the moment maps $\mu_\pm$.

The supersymmetry variations which are proportional to the superfield strengths P and T_1 give rise to a set of constraints which is equivalent to the one we have just discussed. There are three terms which are proportional to each of these superfield strengths. Two of these terms are obvious, coming from supersymmetry variations of $(\delta S)\mu_1$ and $P(\delta\sigma)$, and similarly for the terms proportional to T_1. The third term will arise from the supersymmetry variations of $\hat{\nabla}_-\Phi\cdot(g + B)\cdot\hat{\nabla}_+\Phi$, where we keep the contributions coming from the second and third term in (15). After partial integration, these terms combine by using the anti-commutator $\{\hat{\nabla}_+, \hat{\nabla}_-\}$.

Of course, in addition to these constraints, in order to ensure the invariance of the action under the non-manifest $(1, 1)$ supersymmetries, the metric and B field must satisfy the usual requirements which define the bi-hermitean geometry. A perhaps unexpected requirement emerging from our supersymmetry analysis is that $E = g + B$ ought to be bi-hermitean. This is, however, in complete agreement with the manifestly $(2, 2)$ supersymmetric origin of the $(1, 1)$ action (17). From a $(2, 2)$ superspace perspective, the complex

structures, the metric and the B field arise from second order derivatives of the Kähler potential [12]. These explicit expressions enable the check that indeed $E = g + B$ is bi-hermitean. These expressions should also allow a demonstration that the constraints, Eqs. (19), (21), are also satisfied. While we were unable to show this in general, we have observed that they hold in the flat space and $SU(2) \otimes U(1)$ examples.

3.1. Generalized moment maps

In [6,16] an effort was made to check whether the moment maps obtained from the sigma model correspond to the moment maps used in [9,17] as part of the definition of generalized moment maps. The equations derived at the end of Section 3 allow us to extend these previous attempts to $(2,2)$ supersymmetric sigma models with semichiral superfields, i.e. sigma models with three Killing potentials: the two moment maps and the function σ. More explicitly, Eqs. (19) and (21) can be rewritten as

$$2\,d\mu_1 = (\omega^{(+)} + \omega^{(-)})\xi - (J^{(+)T} - J^{(-)T})\left(-\xi B + \frac{1}{2}d\sigma\right),$$

$$2\,d\mu_2 = (\omega^{(+)} - \omega^{(-)})\xi + (J^{(+)T} + J^{(-)T})\left(-\xi B - \frac{1}{2}d\sigma\right),$$

$$2\,d\mu_1 = -(J^{(+)T} - J^{(-)T})\,d\sigma, \qquad 2\,d\mu_2 = -(J^{(+)T} + J^{(-)T})\,d\sigma. \tag{25}$$

From these equations it follows

$$0 = (J^{(+)} - J^{(-)})\xi - (\omega^{(+)-1} + \omega^{(-)-1})u,$$

$$2\,d\mu_1 = (\omega^{(+)} + \omega^{(-)})\xi - (J^{(+)T} - J^{(-)T})u, \tag{26}$$

where we have used that $u = -\xi B + \frac{1}{2}d\sigma$. As in [16], (26) can be written in terms of $\mathcal{J}_2$, one of the generalized complex structures given in [4], to show that $\xi + u - i\,d\mu_1$ is an eigenvector of $\mathcal{J}_2$. This corresponds to the definition of a generalized moment map for twisted generalized Kähler geometry [9]. The proof is given by noting that since

$$\mathcal{J}_2 = \frac{1}{2}\begin{pmatrix} J_+ - J_- & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & -(J_+^t - J_-^t) \end{pmatrix}, \tag{27}$$

then (26) can be written as $\mathcal{J}_2(\xi + u) = d\mu_1$. This is equivalent to the equation $\mathcal{J}_2(\xi + u - i\,d\mu_1) = i(\xi + u - i\,d\mu_1)$ which verifies the claim. Similarly, $d\mu_2$ is used in the construction of a second twisted generalized moment map, eigenvector of $\mathcal{J}_1$.

4. T-duality

Next, in discussing T-duality, we follow the basic procedure outlined in [18]. First we gauge the $U(1)$ isometry of the sigma model using the prepotentials of the gauge multiplet. Then we add the Lagrange multipliers which will force the field strength of the gauge multiplet to vanish. In the last step leading to the duality functional, we use the gauge freedom to gauge away the appropriate superfields. By solving the Lagrange multiplier constraints and substituting back into the duality functional we return to the original action. The dual action is obtained by imposing the prepotential equations of motion. We will work out one concrete example, T^4, and observe the characteristic interchange of the S^1 radius $R \leftrightarrow 1/R$ in the T-dual actions.

In our discussion of T-duality we will maintain manifest the $(2,2)$ supersymmetry. The gauging of the $(2,2)$ supersymmetric sigma model action [7] is done at the level of the Kähler potential by replacing the left and right semichiral superfields X and Y by:

$$X \to \tilde{X} = e^{V_1 \xi}X, \qquad \bar{X} \to \tilde{\bar{X}} = e^{\bar{V}_1 \xi}\bar{X}, \qquad Y \to \tilde{Y} = e^{V_2 \xi}Y, \qquad \bar{Y} \to \tilde{\bar{Y}} = e^{\bar{V}_2 \xi}\bar{Y}, \tag{28}$$

where V_1 and V_2 are the prepotentials of the semichiral vector multiplet and shift respectively by left and right semichiral superfields under gauge transformations. The replacement in (28) ensures the invariance of the gauged Kähler potential

$$K_g = K(\tilde{\bar{X}}, \tilde{X}, \tilde{\bar{X}}, \tilde{Y}). \tag{29}$$

For concreteness, let us assume that the $U(1)$ generator is

$$\xi = \partial_X - \partial_{\bar{X}} - \partial_Y + \partial_{\bar{Y}} \tag{30}$$

and accordingly,

$$K \equiv K(X + Y, X + \bar{X}, Y + \bar{Y}) = K(X + Y - \bar{X} - \bar{Y}, X + \bar{X}, Y + \bar{Y}). \tag{31}$$

On the tangent bundle T we can define three other vectors, which together with ξ form a vector basis:

$$\xi_1 = \partial_X + \partial_{\bar{X}}, \qquad \xi_2 = \partial_Y + \partial_{\bar{Y}}, \qquad \xi_3 = \partial_X + \partial_Y - \partial_{\bar{X}} - \partial_{\bar{Y}}. \tag{32}$$

The gauged Kähler potential can then be rewritten as

$$K_g = K + \frac{e^{2i\operatorname{Im}V_1\xi_1} - 1}{\operatorname{Im}V_1\xi_1}\operatorname{Im}V_1\xi_1 K + \frac{e^{2i\operatorname{Im}V_2\xi_2} - 1}{\operatorname{Im}V_2\xi_2}\operatorname{Im}V_2\xi_2 K + \frac{e^{2(\operatorname{Re}V_1 - \operatorname{Re}V_2)\xi_3} - 1}{(\operatorname{Re}V_1 - \operatorname{Re}V_2)\xi_3}(\operatorname{Re}V_1 - \operatorname{Re}V_2)\xi_3 K$$

$$= K + \frac{e^{2i\operatorname{Im}V_1\xi_1} - 1}{\operatorname{Im}V_1\xi_1}\operatorname{Im}V_1(\mu_1 + \mu_2) + \frac{e^{2i\operatorname{Im}V_2\xi_2} - 1}{\operatorname{Im}V_2\xi_2}\operatorname{Im}V_2(\mu_1 - \mu_2) + \frac{e^{2(\operatorname{Re}V_1 - \operatorname{Re}V_2)\xi_3} - 1}{(\operatorname{Re}V_1 - \operatorname{Re}V_2)\xi_3}(\operatorname{Re}V_1 - \operatorname{Re}V_2)\sigma, \tag{33}$$

which emphasizes the role of μ_1, μ_2 and σ as Killing potentials. In addition, this expression of the gauged Kähler potential makes manifest the dependence on only three of the four prepotentials.

In order to select the appropriate supersymmetry representation of the Lagrange multipliers, we first solve the gauge superfield strengths in terms of the prepotentials

$$T = \frac{1}{4}\bar{D}^2(V_2 - V_1), \qquad \bar{T} = \frac{1}{4}D^2(\bar{V}_2 - \bar{V}_1),$$
$$S + iP = \frac{1}{2}D_-\bar{D}_+(\bar{V}_2 - V_1), \qquad S - iP = \frac{1}{2}D_+\bar{D}_-(V_2 - \bar{V}_1). \tag{34}$$

As mentioned in the previous section, T is a chiral superfield and $S - iP$ is twisted chiral. Next we add the Lagrange multipliers which enforce the condition that the gauge field is pure gauge

$$K_L = K_g + Z_1 T + \bar{Z}_1 \bar{T} + Z_2(S + iP) + \bar{Z}_2(S - iP)$$
$$= K_g + \phi(V_2 - V_1) + \bar{\phi}(\bar{V}_2 - \bar{V}_1) + \psi(V_2 - \bar{V}_1) + \bar{\psi}(\bar{V}_2 - V_1), \tag{35}$$

where in the second step we substituted the superfield strengths in terms of the prepotentials, and integrated by parts twice. Therefore, ϕ is chiral and ψ is twisted chiral. Lastly, because the prepotential gauge transformation is a shift by a semichiral superfield we can choose the gauge in which $X = 0$ and $Y = 0$. This yields the duality functional

$$K_D = K\big(i(V_1 - V_2), i(\bar{V}_2 - \bar{V}_1), i(V_1 - \bar{V}_1), -i(V_2 - \bar{V}_2)\big)$$
$$+ \phi(V_2 - V_1) + \bar{\phi}(\bar{V}_2 - \bar{V}_1) + \psi(V_2 - \bar{V}_1) + \bar{\psi}(\bar{V}_2 - V_1). \tag{36}$$

To see how we recover the original Kähler potential we study the constraints imposed by the Lagrange multipliers. The ϕ and ψ equations of motion require

$$V_2 - V_1 = iX + iY, \qquad \bar{V}_2 - V_1 = iX + i\bar{Y}. \tag{37}$$

Plugging this back into (36) we obtain the original potential.

If on the other hand, we impose the equations of motion of the prepotentials, solve for V_1 and V_2 and substitute back into (36), we obtain the T-dual Kähler potential. This duality replaces a pair of left and right semichiral superfields with a pair of chiral and twisted chiral superfields.

We would like to mention that the duality functional obtained before appears to be related to the Legendre transforms described in [19]. The authors of [19] began by writing the Kähler potential as $K = K(V, \bar{V}, W, \bar{W}) - (XV + YW + \text{c.c.})$, where X, Y are left, right semichiral superfields and V, W are unrestricted. If the Kähler potential has an isometry, resulting in a dependence of only three real independent linear combinations of the unconstrained complex V and W, then by integrating out the semichiral superfields, one is left with a Kähler potential expressed in terms of chiral and twisted chiral superfields.

As a concrete example of the T-duality map we consider the torus T^4. Its $(2, 2)$ supersymmetric sigma model action is derived from the Kähler potential

$$K = R(\bar{X} + Y)(X + \bar{Y}) - \frac{R}{4}(\bar{Y} + Y)^2. \tag{38}$$

The duality functional is

$$K_D = R(V_2 - \bar{V}_1)(\bar{V}_2 - V_1) + \frac{R}{4}(V_2 - \bar{V}_2)^2 + \phi(V_2 - V_1) + \bar{\phi}(\bar{V}_2 - \bar{V}_1) + \psi(V_2 - \bar{V}_1) + \bar{\psi}(\bar{V}_2 - V_1). \tag{39}$$

The dual potential, up to generalized Kähler gauge transformations,

$$\tilde{K} = \frac{1}{R}(\bar{\phi}\phi - \bar{\psi}\psi) \tag{40}$$

is indeed is the potential for the T-dual T^4, this time written in terms of chiral and twisted chiral superfields. As expected, the radius R of the dualized S^1 is mapped into $1/R$.

5. Quotients

Lastly we address the quotient construction for the semichiral sigma models. The quotient manifold is obtained by extremizing the gauged Kähler potential with respect to the three real linear independent combinations of the prepotentials for each of the isometry group generators. The dimension of the quotient manifold, which remains bi-hermitean, is $\dim \mathcal{M} - 4 \dim G$.

After adding the FI terms[4]

$$\text{FI terms} = \Delta K_g = r_0 \tilde{V} + r_1 \tilde{V}_1 + r_2 \tilde{V}_2 \tag{41}$$

where

$$\tilde{V} = \text{Re}(V_2) - \text{Re}(V_1), \qquad \tilde{V}_1 = \text{Im}(V_1), \qquad \tilde{V}_2 = \text{Im}(V_2), \tag{42}$$

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[4] The quotient construction relies on the same duality functional as used for the T-duality map. The FI terms correspond to the Lagrange multiplier terms, where the Lagrange multipliers are taken to be constant.

to the gauged Kähler potential given in (33). Extremizing with respect to $\tilde{V}$, $\tilde{V}_1$ and $\tilde{V}_2$, one finds

$$e^{2i\tilde{V}_1\xi_1}(\mu_1 + \mu_2) = r_1, \qquad e^{2i\tilde{V}_2\xi_2}(\mu_1 - \mu_2) = r_2, \qquad e^{2\tilde{V}\xi_3}\sigma = r_0. \tag{43}$$

These are the equations which define the quotient. In practical terms, one solves them for the prepotentials, and substitutes back into the gauged Kähler potential to arrive at the quotient manifold potential. We recall that a similar constraint (involving the only Killing potential) defines the quotient on a Kähler manifold [14].

Let us consider the flat space quotient as an example. We take two copies of R^4 and gauge the fields for both copies with the same charge (which we have set to 1). The local $U(1)$ acts on the fields as

$$X_{1/2} \to X'_{1/2} = e^{i\Lambda} X_{1/2}, \qquad Y_{1/2} \to Y'_{1/2} = e^{-iU} Y_{1/2} \tag{44}$$

and, correspondingly, the gauged Kähler potential is

$$K_g = e^{i(V_1 - \bar{V}_1)}(\bar{X}_1 X_1 + \bar{X}_2 X_2) + e^{i(V_1 - V_2)}(Y_1 X_1 + Y_2 X_2)$$
$$+ e^{i(\bar{V}_1 - V_2)}(\bar{Y}_1 \bar{X}_1 + \bar{Y}_2 \bar{X}_2) + \frac{1}{2}e^{i(\bar{V}_2 - V_2)}(\bar{Y}_1 Y_1 + \bar{Y}_2 Y_2) + \Delta K_g. \tag{45}$$

To write the equations of motion for the prepotentials in a compact form we introduce the notation: $x = (\bar{X}_1 X_1 + \bar{X}_2 X_2)$, $y = (\bar{Y}_1 Y_1 + \bar{Y}_2 Y_2)$, $z = (Y_1 X_1 + Y_2 X_2)$, $A = (Y_1 X_1 + Y_2 X_2)e^{-i\tilde{V}}e^{-\tilde{V}_1}e^{\tilde{V}_2}$. Then we can rewrite them as

$$2\,\mathrm{Im}(A) = r_0, \qquad ye^{2\tilde{V}_2} + 2\,\mathrm{Re}\,A = r_1, \qquad -2xe^{-2\tilde{V}_1} + ye^{2\tilde{V}_2} = r_1 + r_2. \tag{46}$$

Next, solving for $|A|$ we find

$$|A|^2 = z\bar{z}e^{-2\tilde{V}_1 + 2\tilde{V}_2} = \frac{r_0^2}{4} + \frac{1}{4}(r_2 - ye^{2\tilde{V}_2})^2. \tag{47}$$

Further substituting $\tilde{V}_1$ in terms of $\tilde{V}_2$ yields

$$z\bar{z}e^{2\tilde{V}_2}(ye^{2\tilde{V}_2} - r_1 - r_2) = \frac{x}{2}(r_0^2 + (r_2 - ye^{2\tilde{V}_2})^2) \tag{48}$$

which may be solved directly for $\tilde{V}_2$, giving

$$e^{2\tilde{V}_2} = \frac{(r_1 + r_2)\bar{z}z - r_2 xy \pm \sqrt{((r_1 + r_2)\bar{z}z - r_2 xy)^2 + 2(r_2^2 + r_0^2)(\bar{z}z - \frac{1}{2}xy)xy}}{2y(\bar{z}z - \frac{1}{2}xy)}. \tag{49}$$

The reality of $\tilde{V}_2$ will require that $\bar{z}z \geqslant \frac{1}{2}xy$ indicating the presence of a boundary in the quotient target space. The solutions for $\tilde{V}_1$ and $\tilde{V}$ follow:

$$e^{-2\tilde{V}_1} = \frac{1}{2x}(ye^{2\tilde{V}_2} - r_1 - r_2), \qquad e^{-i\tilde{V}} = \frac{1}{2z}e^{\tilde{V}_1}e^{-\tilde{V}_2}(r_2 + ir_0 - ye^{2\tilde{V}_2}). \tag{50}$$

To complete the discussion of the quotient, we have to choose a gauge. Considering (44), we will pick the gauge where $X'_1 = 1$, $X'_2 = \frac{X_2}{X_1}$, $Y'_1 = 1$ and $Y'_2 = \frac{Y_2}{Y_1}$. Despite the complexity of the final answer, the gauge fixing step demonstrates that the dimension of the quotient manifold is smaller by 4, as expected. The quotient geometry remains bi-hermitean since $(2, 2)$ supersymmetry has been preserved. An interesting point is that the quotient target space has non-trivial H flux.

Acknowledgements

References

[1] S.J. Gates, C.M. Hull, M. Rocek, Nucl. Phys. B 248 (1984) 157.
[2] U. Lindstrom, M. Rocek, R. von Unge, M. Zabzine, hep-th/0512164.
[3] N. Hitchin, Quart. J. Math. Oxford Ser. 54 (2003) 281, math.DG/0209099.
[4] M. Gualtieri, math.DG/0401221.
[5] U. Lindstrom, R. Minasian, A. Tomasiello, M. Zabzine, Commun. Math. Phys. 257 (2005) 235, hep-th/0405085;
  A. Kapustin, Y. Li, hep-th/0407249;
  R. Zucchini, JHEP 0503 (2005) 022, hep-th/0501062;
  A. Bredthauer, U. Lindstrom, J. Persson, M. Zabzine, Lett. Math. Phys. 77 (2006) 291, hep-th/0603130;
  M. Zabzine, hep-th/0605148;
  R. Zucchini, JHEP 0612 (2006) 039, hep-th/0608145;
  I.T. Ellwood, hep-th/0612100;
  U. Lindstrom, M. Rocek, R. von Unge, M. Zabzine, hep-th/0703111.
[6] W. Merrell, L.A.P. Zayas, D. Vaman, hep-th/0610116.
[7] S.J. Gates, W. Merrell, arXiv: 0705.3207 [hep-th].
[8] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge, M. Zabzine, arXiv: 0705.3201 [hep-th].

[9] Y. Lin, S. Tolman, math.DG/0510010.
[10] C.M. Hull, A. Karlhede, U. Lindstrom, M. Rocek, Nucl. Phys. B 266 (1986) 1.
[11] A. Sevrin, J. Troost, Nucl. Phys. B 492 (1997) 623, hep-th/9610102;
     A. Sevrin, J. Troost, hep-th/9610103.
[12] U. Lindstrom, M. Rocek, R. von Unge, M. Zabzine, JHEP 0507 (2005) 067, hep-th/0411186.
[13] B. Zumino, Phys. Lett. B 87 (1979) 203;
     L. Alvarez-Gaume, D.Z. Freedman, Phys. Lett. B 94 (1980) 171.
[14] N.J. Hitchin, A. Karlhede, U. Lindstrom, M. Rocek, Commun. Math. Phys. 108 (1987) 535.
[15] C.M. Hull, G. Papadopoulos, B.J. Spence, Nucl. Phys. B 363 (1991) 593.
[16] A. Kapustin, A. Tomasiello, hep-th/0610210.
[17] Y. Lin, S. Tolman, math.DG/0509069.
[18] M. Rocek, E.P. Verlinde, Nucl. Phys. B 373 (1992) 630, hep-th/9110053.
[19] M.T. Grisaru, M. Massar, A. Sevrin, J. Troost, Fortschr. Phys. 47 (1999) 301, hep-th/9801080.