Convergence of Cascade Algorithms in Sobolev Spaces Associated with Multivariate Refinement Equations¹

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This paper is concerned with multivariate inhomogeneous refinement equations written in the form

\[ \phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(Mx - \alpha) + g(x), \quad x \in \mathbb{R}^s, \]

where \( \phi \) is the unknown function defined on the \( s \)-dimensional Euclidean space \( \mathbb{R}^s \), \( g \) is a given compactly supported function on \( \mathbb{R}^s \), \( a \) is a finitely supported sequence on \( \mathbb{Z}^s \), and \( M \) is an integer matrix such that \( \lim_{n \to \infty} M^{-n} = 0 \). Equation (1.1) is called an inhomogeneous refinement equation, \( M \) is called a dilation matrix, and the sequence \( a \) is called a refinement mask. Any function satisfying a refinement equation (1.1) is called a

1. INTRODUCTION

We are interested in functional equations of the form

\[ \phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(Mx - \alpha) + g(x), \quad x \in \mathbb{R}^s, \]

where \( \phi \) is the unknown function defined on the \( s \)-dimensional Euclidean space \( \mathbb{R}^s \), \( g \) is a given compactly supported function on \( \mathbb{R}^s \), \( a \) is a finitely supported sequence on \( \mathbb{Z}^s \), and \( M \) is an \( s \times s \) integer matrix such that \( \lim_{n \to \infty} M^{-n} = 0 \). Any function satisfying a refinement equation (1.1) is called a

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inhomogeneous refinement equations

Refinable function. The refinement equation plays an important role in computer graphics and wavelet analysis. Associated with the inhomogeneous refinement equation (1.1) is the homogeneous refinement equation

$$\varphi(x) = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha) \varphi(Mx - \alpha), \quad x \in \mathbb{R}^d.$$ (1.2)

It is well known that the refinement equations for the scaling functions $\varphi$ are the fundamental equations in wavelet theory.

The inhomogeneous refinement equation appeared in construction of multi-wavelets and construction of wavelets on a finite interval [13]. Strang and Zhou [14] gave a systematic study of distributional solutions for the case $s = 1$. In the multivariate case $s > 1$, Jia et al. [10] gave a study of distributional solutions of the refinement equation (1.1).

Choose an initial function $\varphi_0 \in W_k^2(\mathbb{R}^d)$ with compact support. Let

$$\varphi_n(x) = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha) \varphi_{n-1}(Mx - \alpha) + g(x), \quad x \in \mathbb{R}^d, \ n \in \mathbb{N}. \quad (1.3)$$

The algorithm (1.3) is called the cascade algorithm with $a$, $g$, dilation $M$, and $\varphi_0$. If there exists a function $\varphi \in W_k^2(\mathbb{R}^d)$ such that

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_{W_k^2(\mathbb{R}^d)} = 0,$$

then we say that the cascade algorithm associated with $a$, $g$, dilation $M$, and $\varphi_0$ is convergent in $W_k^2(\mathbb{R}^d)$. If this is the case, then the limit $\varphi$ is a solution of the inhomogeneous refinement equation (1.1) in $W_k^2(\mathbb{R}^d)$.

The convergence of the cascade algorithms is fundamental to wavelet theory and subdivision. It has been studied in connection with solutions of refinement equations and the description of curves and surfaces in computer aided geometric design (see [1, 2, 5, 15, 16]). The $L_2$-convergence and $L_p$-convergence of cascade algorithms associated with homogeneous refinement equations were investigated in many papers such as [4, 5, 11, 12, 15]. When $s = 1, 1 \leq p \leq \infty$, with dilation 2, Strang and Zhou [14] gave a complete characterization for $L_p$-convergence of the cascade algorithm associated with inhomogeneous refinement equations (1.1). These results were further extended to the multivariate case for $p = 2$ [8].

The purpose of this paper is to investigate the strong convergence in Sobolev space $W_2^k(\mathbb{R}^d)$ of cascade algorithms associated with inhomogeneous refinement equations (1.1). In one dimension ($s = 1$) with dilation 2 and $p = 2$, characterization of weak and strong convergence in the Sobolev space $W_2^k(\mathbb{R}^d)$ of cascade algorithms associated with the homogeneous refinement equation (1.2) has been studied in [3]. For the multivariate case, Jia et al. [7] investigated strong convergence in the Sobolev space $W_2^k(\mathbb{R}^d)$ of cascade algorithms associated with homogeneous refinement equations (1.2) when $M$ is isotropic. Similar characterizations were
established in [17] for strong convergence of the cascade algorithms (1.3) in the Sobolev space $W^k_2(\mathbb{R}^s)$ with dilation matrix $M = 2I$. Our main object here is to give a characterization of strong convergence of the cascade algorithms (1.3) in the Sobolev space $W^k_2(\mathbb{R}^s)$ for the case in which the dilation matrix $M$ is isotropic, i.e., $M$ is similar to a diagonal matrix $\text{diag}(\sigma_1, \ldots, \sigma_s)$ with $|\sigma_1| = \cdots |\sigma_s|$.

2. CONVERGENCE OF THE CASCADE ALGORITHM IN SOBOLEV SPACE $W^k_2(\mathbb{R}^s)$

Let $Q_a$ be the cascade operator defined by

$$Q_a \varphi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \varphi(Mx - \alpha), \quad \varphi \in L_2(\mathbb{R}^s).$$

(2.1)

As usual, we use $L_2(\mathbb{R}^s)$ to denote the space of square integral functions on $\mathbb{R}^s$. The norm on $L_p(\mathbb{R}^s)$ is given by

$$\|f\|_p = \left( \int_{\mathbb{R}^s} |f|^p \, ds \right)^{1/p}, \quad f \in L_p(\mathbb{R}^s).$$

The Fourier transform of an integrable function $f$ on $\mathbb{R}^s$ is defined to be

$$\hat{f}(\xi) = \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\xi} \, dx, \quad \xi \in \mathbb{R}^s,$$

where $x \cdot \xi$ denotes the inner product of two vectors $x$ and $\xi$ in $\mathbb{R}^s$.

Let $\ell_p(\mathbb{Z}^s)$ denote the linear space of all finitely supported sequences on $\mathbb{Z}^s$. Furthermore, we denote by $\ell_\infty(\mathbb{Z}^s)$ the linear space of all bounded sequences. The norm on $\ell_\infty(\mathbb{Z}^s)$ is given by

$$\|v\|_\infty = \sup \{|v(\alpha)| : \alpha \in \mathbb{Z}^s\}, \quad v \in \ell_\infty(\mathbb{Z}^s).$$

We use $\mathbb{C}^r$ to denote the linear space of all $r \times 1$ complex vectors. The norm of a vector $\xi = (\xi_1, \ldots, \xi_r)^T \in \mathbb{C}^r$ is defined by $|\xi| = \left( \sum_{j=1}^r |\xi_j|^2 \right)^{1/2}$.

For a positive integer $k \in \mathbb{N}$, we use $W^k_2(\mathbb{R}^s)$ to denote the Sobolev space that consists of all distributions $f$ such that $D^\mu f \in L_2(\mathbb{R}^s)$ for all multi-indices $\mu = (\mu_1, \ldots, \mu_s)$ with $|\mu| \leq k$, equipped with the norm given by

$$\|f\|_{W^k_2(\mathbb{R}^s)} := \sum_{|\mu| \leq k} \|D^\mu f\|_2,$$

(2.2)

where $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, $|\mu| = \mu_1 + \cdots + \mu_s$, $D^\mu$ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$, and $D_j f$ is the partial derivative of the differentiable function $f$ with respect to $j$th coordinate.
It is easily seen that the norm defined by (2.2) is equivalent to the norm given by
\[ ||f||^*_{W^2_2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}} \left( \int_{\mathbb{R}^d} (1 + |\xi|^{2k})|\hat{f}(\xi)|^2 d\xi \right)^{1/2}. \]  
(2.3)

Given a finitely supported sequence \( c \in l_0(\mathbb{Z}^d) \), we use \( \tilde{c}(z) \) to denote its symbol
\[ \tilde{c}(z) = \sum_{a \in \mathbb{Z}^d} c(a)z^a, \]
where \( z^n = z_1^{n_1} \cdots z_d^{n_d} \) for \( z = (z_1, \ldots, z_d) \in (\mathbb{C}\setminus\{0\})^d \), and \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d \). The sequence \( c \) can be recovered from \( \tilde{c} \) through the inversion formula;
\[ c(\alpha) = \frac{1}{2\pi} \int_{[0,1)^d} \tilde{c}(e^{i\xi e}) e^{-i\alpha \cdot \xi} d\xi, \quad \alpha \in \mathbb{Z}^d. \]  
(2.4)

For \( c, d \in l_0(\mathbb{Z}^d) \), the discrete convolution of \( c \) and \( d \), denoted \( c \ast d \), is given by
\[ c \ast d(\alpha) = \sum_{\beta \in \mathbb{Z}^d} c(\alpha - \beta)d(\beta), \quad \alpha \in \mathbb{Z}^d. \]

It is easily seen that
\[ \tilde{c} \ast d(z) = \tilde{c}(z)\tilde{d}(z), \quad z = (z_1, \ldots, z_d) \in (\mathbb{C}\setminus\{0\})^d. \]  
(2.5)

For \( z \in \mathbb{C} \), we use \( \bar{z} \) to denote the complex conjugate of \( z \), and for \( a \in \ell_0(\mathbb{Z}^d) \), we denote by \( a^* \) the sequence given by \( a^*(\alpha) := \overline{a(-\alpha)}, \alpha \in \mathbb{Z}^d \). If \( b = a \ast a^* \), then we have
\[ \tilde{b}(e^{-i\xi}) = \overline{\tilde{a}(e^{-i\xi})}\tilde{a}^*(e^{-i\xi}) = |\tilde{a}(e^{-i\xi})|^2, \quad \text{for } \xi \in \mathbb{R}^d. \]  
(2.6)

Let \( a \) be an element in \( \ell_0(\mathbb{Z}^d) \). We define the transition operator \( T_a \) to be the linear mapping from \( \ell_0(\mathbb{Z}^d) \) to \( \ell_0(\mathbb{Z}^d) \) given by
\[ T_av(\alpha) := \sum_{\beta \in \mathbb{Z}^d} a(M\alpha - \beta)v(\beta), \quad \alpha \in \mathbb{Z}^d, v \in \ell_0(\mathbb{Z}^d). \]  
(2.7)

Following the ideas of [4, 9], it is easy to know that the minimal invariant subspace \( W \) of \( T_a \) generated by each \( w \in \ell_0(\mathbb{Z}^d) \) is finite dimensional. We use \( \rho(T_a\vert_W) \) to denote the spectral radius of \( T_a\vert_W \).

Let us investigate the cascade algorithm as given in (1.3). For \( n = 1, 2, \ldots, \) by (2.1) and (1.3) we have
\[ \varphi_n = g + Q_ag + \cdots + Q_{a}^{n-1}g + Q_{a}^{n}\varphi_0. \]
It follows that
\[ \varphi_{n+1} - \varphi_{n} = Q_{a}^{n}g + Q_{a}^{n+1}\varphi_0 - Q_{a}^{n}\varphi_0 = Q_{a}^{n}g_0, \]  
(2.8)

where \( g_0 := g + Q_{a}^{n}\varphi_0 - \varphi_0 \). The following is the main theorem which gives a characterization for the strong convergence of cascade algorithms given by (1.3) in the Sobolev space \( W^2_2(\mathbb{R}^d) \).
Theorem. Suppose $M$ is an $s \times s$ isotropic dilation matrix and $m := |\det M|$. Let $k \in \mathbb{N}$, $g$, and $\varphi_0$ be compactly supported functions in $W^2_s(\mathbb{R}^s)$, and let $b := a * a^* / m$. Then the cascade algorithm associated with $a, g, M$, and $\varphi_0$ is convergent in the Sobolev space $W^2_s(\mathbb{R}^s)$ if and only if

$$\lim_{n \to \infty} m^{2nk/s} \|T^n_b u_1\|_{\infty} = 0,$$

and

$$\lim_{n \to \infty} \|T^n_b w_2\|_{\infty} = 0,$$

where $w_1, w_2 \in l_0(\mathbb{Z}^s)$ are given respectively by

$$\tilde{w}_1(e^{-i\xi}) = \sum_{j=1, \alpha \in \mathbb{Z}^s} |\xi_j + 2\pi \alpha_j|^{2k} \hat{g}_0(\xi + 2\pi \alpha)\hat{g}_0(\xi + 2\pi \alpha),$$

and

$$\tilde{w}_2(e^{-i\xi}) = \sum_{\alpha \in \mathbb{Z}^s} \hat{g}_0(\xi + 2\pi \alpha)\hat{g}_0(\xi + 2\pi \alpha),$$

and $g_0 := g + Q_a \varphi_0 - \varphi_0$, or, equivalently

$$\rho(T_b|_{W_1}) < m^{-2k/s},$$

and

$$\rho(T_b|_{W_2}) < 1,$$

where $W_1$ and $W_2$ are the minimal invariant subspaces of $T_b$ generated respectively by $w_1$ and $w_2$, $\xi = (\xi_1, \xi_2, \ldots, \xi_s)^T$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$.

3. PROOF OF THE THEOREM

From (2.3) and (2.8) we have

$$\|\varphi_{n+1} - \varphi_n\|_{W_2^s(\mathbb{R}^s)}^2 = \frac{1}{(2\pi)^{2s}} \int_{\mathbb{R}^s} (1 + |\xi|^{2k})|\hat{Q}_a \varphi_0(\xi)|^2d\xi.$$

Let $a_n(n = 1, 2, \ldots)$ be the sequences defined by $a_1 = a$ and

$$a_n(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_{n-1}(\beta) a(\alpha - M^\beta), \quad \alpha \in \mathbb{Z}^s, n = 2, 3, \ldots$$

It can be easily seen by induction that

$$Q^n_a \varphi_0 = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) g_0(M^n \cdot -\alpha).$$
Taking the Fourier transform on both sides, we obtain

$$\hat{Q}_n(g_0)(\xi) = \frac{1}{m^n} \hat{a}_n(e^{-i(M^T)^{-n}}) \hat{g}_0((M^T)^{-n} \xi), \quad \xi \in \mathbb{R}^s.$$  

It follows that

$$\int_{\mathbb{R}^s} (1 + |\xi|^2)^2 |\hat{Q}_n(g_0)(\xi)|^2 d\xi$$

$$= \frac{1}{m^{2n}} \int_{\mathbb{R}^s} (1 + |\xi|^2)^2 |\hat{a}_n(e^{-i(M^T)^{-n}}) \hat{g}_0((M^T)^{-n} \xi)|^2 d\xi$$

$$= \frac{1}{m^n} \int_{\mathbb{R}^s} (1 + |(M^T)^n \xi|^2)^2 |\hat{a}_n(e^{-i\xi}) \hat{g}_0(\xi)|^2 d\xi.$$  

By periodization, this equals

$$\frac{1}{m^n} \int_{[0,2\pi]^s} (1 + |(M^T)^n(\xi + 2\pi\alpha)|^2) |\hat{a}_n(e^{-i\xi}) \hat{g}_0(\xi + 2\pi\alpha)|^2 d\xi.$$  

Since $M$ is isotropic, there exists an invertible matrix $A$ such that

$$M = A \text{diag}(\sigma_1, \ldots, \sigma_s) A^{-1},$$

with $|\sigma_1| = \cdots = |\sigma_s| = m^{1/s}$. Therefore, there exist two positive constants $M_1$ and $M_2$ independent of $n$ such that

$$M_1 m^{n/s} |\xi + 2\pi\alpha| \leq |(M^T)^n(\xi + 2\pi\alpha)| \leq M_2 m^{n/s} |\xi + 2\pi\alpha|. \quad (3.3)$$

By definitions of $w_1$ and $w_2$ we have

$$\sum_{\alpha \in \mathbb{Z}^s} \hat{g}_0(\xi + 2\pi\alpha) \hat{g}_0(\xi + 2\pi\alpha) = \hat{w}_2(e^{-i\xi}),$$

and

$$\sum_{j=1}^s \sum_{\alpha \in \mathbb{Z}^s} |\xi_j + 2\pi\alpha_j|^2 \hat{g}_0(\xi + 2\pi\alpha) \hat{g}_0(\xi + 2\pi\alpha) = \hat{w}_1(e^{-i\xi}).$$

From [6], we know that for $\alpha \in \mathbb{Z}^s$, $w_2(\alpha) = \int_{\mathbb{R}^s} \hat{g}_0(x) \overline{\hat{g}_0(x + \alpha)} dx$, and $w_1(\alpha) = \sum_{j=1}^s \int_{\mathbb{R}^s} D_j^k \hat{g}_0(x) \overline{D_j^k \hat{g}_0(x + \alpha)} dx$. This shows that $w_1, w_2 \in \ell_0(\mathbb{Z}^s)$.

Let $b_n(n = 1, 2, \ldots)$ be the sequences defined in a similar way to (3.2). Then one can easily show that

$$T_n^\alpha v(\alpha) = \sum_{\beta \in \mathbb{Z}^s} b_n(M^n \alpha - \beta) v(\beta). \quad (3.4)$$

From the definitions of $b_n$ and $a_n$, we can obtain

$$b_n := \frac{1}{m^n} a_n * a_n^*. \quad (3.5)$$
In fact, we can prove (3.5) by induction on \( n \). By the definitions of \( a_1 \) and \( b_1 \), (3.5) is true for \( n = 1 \). Suppose \( n > 1 \) and (3.5) has been verified for \( n - 1 \). By the induction hypothesis, for \( \alpha \in \mathbb{Z}^2 \) we have

\[
 b_n(\alpha) = \sum_{n \in \mathbb{Z}^2} b_{n-1}(\eta) b(\alpha - M\eta)
 = m^{-n} \sum_{\beta \in \mathbb{Z}^2} \sum_{\eta \in \mathbb{Z}^2} \sum_{\tau \in \mathbb{Z}^2} (a_n(\tau) a_{n-1}(\eta + \tau)) (\alpha(\beta - M\eta + \beta))
 = m^{-n} \sum_{\beta \in \mathbb{Z}^2} \sum_{\eta \in \mathbb{Z}^2} \sum_{\tau \in \mathbb{Z}^2} (a_n(\tau) a(\beta - M\tau)) (a_{n-1}(\eta) a(\alpha - M\eta + \beta))
 = m^{-n} \sum_{\beta \in \mathbb{Z}^2} a_n(\alpha + \beta) a_n(\beta),
\]

which implies that (3.5) is true for all \( n \).

Following (2.5), (2.6), (3.3), and above discussions, we know that

\[
\frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} (b_n \ast \tilde{w}_2(e^{-i\xi})) + M_1 m^{2n/k/s} b_n \ast \tilde{w}_1(e^{-i\xi})) d\xi
\leq \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} (1 + |\xi|^{2k}) |\hat{Q}_d g_0(\xi)|^2 d\xi
\leq \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} (b_n \ast \tilde{w}_2(e^{-i\xi})) + M_2 m^{2n/k/s} b_n \ast \tilde{w}_1(e^{-i\xi})) d\xi.
\]

By (2.4) and (3.4), it follows that

\[
T_b^n w_2(0) + M_1 m^{2n/k/s} T_b^n w_1(0) \leq \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} (1 + |\xi|^{2k}) |\hat{Q}_d g_0(\xi)|^2 d\xi
\leq T_b^n w_2(0) + M_2 m^{2n/k/s} T_b^n w_1(0). \tag{3.6}
\]

If \( \rho(T_b|W) < m^{-2k/s} \) and \( (\rho T_b|W_1) < 1 \), then we can find \( \eta, 0 < \eta < 1 \), such that \( \|T_b^n w_1\|_1^{1/n} < \eta m^{-2k/s} \) and \( \|T_b^n w_2\|_\infty^{1/n} < \eta \) are valid for sufficiently large \( n \). Consequently, there exists a positive constant \( C \) independent of \( n \), such that for all \( n \in \mathbb{N} \)

\[
\|T_b^n w_1\|_\infty \leq C(m^{-2k/s} \eta)^n,
\]

and

\[
\|T_b^n w_2\|_\infty \leq C \eta^n.
\]

Therefore, we have

\[
\|\varphi_{n+1} - \varphi_n\|_{W^k_1(R)}^2 \leq C_1 \eta^n,
\]

where \( C_1 \) is a positive constant independent of \( n \).
Since the supports of $\varphi_n$ are uniformly bounded, this shows that the sequences $\{\varphi_n\}_{n \in \mathbb{N}}$ converges to a function $\varphi$ in $W^{k,2}_2(\mathbb{R}^s)$. The sufficiently part of the theorem is proved.

Next, we establish the necessity part of the theorem.

If one of (2.9) and (2.10) does not hold, then we have

$$\inf_{n \geq 1} \| T^n_b \|_{W^s_{1,1}}^{1/n} = \lim_{n \to \infty} \| T^n_b \|_{W^s_{1,1}}^{1/n} \geq m^{-2k/s},$$

or

$$\inf_{n \geq 1} \| T^n_b \|_{W^s_{2,1}}^{1/n} = \lim_{n \to \infty} \| T^n_b \|_{W^s_{2,1}}^{1/n} \geq 1.$$ 

It follows that

$$\| T^n_b \|_{W^s_{1,1}} \geq 1, \quad n \in \mathbb{N},$$

or

$$m^{2k/s} \| T^n_b \|_{W^s_{2,1}} \geq 1, \quad n \in \mathbb{N}.$$ 

From the proof of Lemma 2.4 in [4], we see that there exists a positive constant $M_3$ independent of $n$ such that

$$\| T^n_b \|_{W^s_{1,1}} \geq M_3, \quad n \in \mathbb{N},$$

or

$$m^{2k/s} \| T^n_b \|_{W^s_{2,1}} \geq M_3, \quad n \in \mathbb{N}.$$ 

In light of (2.4), (2.5), (2.6), and (3.4), we have

$$| T^n_b \varphi_1(0) | = | b_n \ast w_1(M^n \alpha) |$$

$$\leq \frac{1}{(2\pi)^{r'}} \int_{[0,2\pi)^{r'}} | b_n \ast \tilde{w}_1(e^{-i\xi}) e^{-i(M^n \alpha) \cdot \xi} | d\xi$$

$$\leq \frac{1}{(2\pi)^{r'}} \int_{[0,2\pi)^{r'}} | b_n \ast \tilde{w}_1(e^{i\xi}) | d\xi$$

$$= b_n \ast w_1(0)$$

$$= T^n_b \varphi_1(0).$$

Similarly, we obtain $| T^n_b \varphi_2(\alpha) | \leq T^n_b \varphi_2(0)$.

This in connection with (3.6) gives

$$\| \varphi_{n+1} - \varphi_n \|_{W^s_{2}(\mathbb{R}^s)} \geq M_4, \quad n \in \mathbb{N},$$

where $M_4$ is a positive constant independent of $n$. This contradicts the fact that

$$\lim_{n \to \infty} \| \varphi_{n+1} - \varphi_n \|_{W^s_{2}(\mathbb{R}^s)} = 0.$$ 

The necessity part of the theorem is also proved.
4. EXAMPLE

In this section we give a few examples to illustrate our theory.

**Example 4.1.** Consider the inhomogeneous multivariate refinement equation of the form

\[ \phi(x, y) = t \phi(2x, 2y) + t \phi(2x - 1, 2y - 1) + g(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (4.1) \]

where \( t \) is a nonzero complex number and \( g \) is a function in \( W^2_2(\mathbb{R}^2) \) supported in \([0, 2] \times [0, 2]\). Let \( \phi_0 \) be a function in \( W^k_2(\mathbb{R}^n) \) supported in \([0, 2] \times [0, 2]\) and \( g_0(x, y) = g(x, y) + t \phi_0(2x, 2y) + t \phi_0(2x - 1, 2y - 1) - \phi_0(x, y) \). The corresponding cascade algorithm is given by

\[ \phi_{n+1}(x, y) = t \phi_n(2x, 2y) + t \phi_n(2x - 1, 2y - 1) + g(x, y), \quad n = 1, 2, \ldots. \]

In this case, \( a(0, 0) = t, a(1, 1) = t, \) and \( a(\alpha) = 0 \) for \( \alpha \notin \{(0, 0), (1, 1)\} \). We observe that \( g_0 \) is a function in \( W^k_2(\mathbb{R}^2) \) supported in \([0, 2] \times [0, 2]\).

Let \( b \) be the sequence given by \( b = (a * a^*)/4 \). Then \( b(0, 0) = |t|^2/2, b(1, 1) = |t|^2/4, b(-1, -1) = |t|^2/4, \) and \( b(\alpha) = 0 \) for \( \alpha \notin \{(-1, -1), (0, 0), (1, 1)\} \). By the definitions of \( b_n \), it can be easily seen by induction that

\[ b_n(0, 0) = \left( \frac{1}{2} \right)^n |t|^{2n}, \quad b_n(-1, -1) = \left[ \left( \frac{1}{2} \right)^n - \left( \frac{1}{2} \right)^{2n} \right] |t|^{2n} \]

and

\[ b_n(1, 1) = \left[ \left( \frac{1}{2} \right)^n - \left( \frac{1}{2} \right)^{2n} \right] |t|^{2n}. \]

By simple computation and the definitions of \( w_1 \) and \( w_2 \), we obtain

\[ T^*_h w_2(0) = \sum_{\beta \in \mathbb{Z}^2} b_n(-\beta) w_2(\beta) \]

\[ = \left[ \left( \frac{1}{2} \right)^n - \left( \frac{1}{2} \right)^{2n} \right] |t|^{2n} \int_{\mathbb{R}^2} g_0(x, y) g_0(x + 1, y + 1) dxdy \]

\[ + \left( \frac{1}{2} \right)^n |t|^{2n} \int_{\mathbb{R}^2} |g_0(x, y)|^2 dxdy + \left[ \left( \frac{1}{2} \right)^n - \left( \frac{1}{2} \right)^{2n} \right] |t|^{2n} \]

\[ \times \int_{\mathbb{R}^2} g_0(x, y) g_0(x - 1, y - 1) dxdy. \]
Similarly, we have
\[ T_n^0 w_1(0) = \sum_{\beta \in \mathbb{Z}^2} b_n(-\beta) w_1(\beta) \]
\[ = \left[ \left( \frac{1}{2} \right)^n - \left( \frac{1}{2} \right)^{2n} \right] |t|^{2n} \int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k g_0(x, y) \left( \frac{\partial}{\partial x} \right)^k g_0(x+1, y+1) \right. \]
\[ + \left( \frac{\partial}{\partial y} \right)^k g_0(x, y) \left( \frac{\partial}{\partial y} \right)^k g_0(x+1, y+1) \] \[
\left. + \left( \frac{\partial}{\partial y} \right)^k g_0(x, y) \left( \frac{\partial}{\partial y} \right)^k g_0(x-1, y-1) \right] dxdy \]
\[ + \left[ \left( \frac{1}{2} \right)^n - \left( \frac{1}{2} \right)^{2n} \right] |t|^{2n} \int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k g_0(x, y) \left( \frac{\partial}{\partial x} \right)^k g_0(x-1, y-1) \right. \]
\[ + \left( \frac{\partial}{\partial y} \right)^k g_0(x, y) \left( \frac{\partial}{\partial y} \right)^k g_0(x-1, y-1) \] \[
\left. + \left( \frac{\partial}{\partial y} \right)^k g_0(x, y) \left( \frac{\partial}{\partial x} \right)^k g_0(x+1, y+1) \right] dxdy. \]

By our theorem, we conclude that if \(|t| < 2^{1/2-k}\), the cascade algorithm associated with \(a, g\), and any \(\varphi_0\) is convergent in \(W^k_2(\mathbb{R}^2)\); if \(2^{1/2-k} \leq |t| < 2^{1-k}\), the cascade algorithm associated with \(a, g\), and \(\varphi_0\) is convergent in \(W^k_2(\mathbb{R}^2)\) if and only if
\[ \int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k g_0(x, y) \left( \frac{\partial}{\partial x} \right)^k g_0(x+1, y+1) + \left( \frac{\partial}{\partial x} \right)^k g_0(x-1, y-1) \right. \]
\[ + \left. \left( \frac{\partial}{\partial y} \right)^k g_0(x, y) \left( \frac{\partial}{\partial y} \right)^k g_0(x+1, y+1) \right] \]
\[ + \left. \left( \frac{\partial}{\partial y} \right)^k g_0(x, y) \left( \frac{\partial}{\partial y} \right)^k g_0(x-1, y-1) \right] dxdy = 0; \]
if \(|t| \geq 2^{1-k}\), the cascade algorithm associated with \(a, g\), and \(\varphi_0\) is convergent in \(W^k_2(\mathbb{R}^2)\) if and only if \(\varphi_0\) is a solution of Eq. (4.1).

**Example 4.2.** Consider the inhomogeneous multivariate refinement equation of the form
\[ \varphi(x, y) = t \varphi(x-y, x+y) + g(x, y), \quad (x, y) \in \mathbb{R}^2, \]
(4.2)
where \(t\) is a nonzero complex number and \(g\) is a function in \(W^k_2(\mathbb{R}^2)\) supported in \([0, 2] \times [0, 2]\). Let \(\varphi_0\) be a function in \(W^k_2(\mathbb{R}^3)\) supported
in $[0, 2] \times [0, 2]$. The corresponding cascade algorithm is given by

$$
\varphi_{n+1}(x, y) = t\varphi_n(x - y, x + y) + g(x, y), \quad n = 1, 2, \ldots.
$$

We have $a(0, 0) = t$, and $a(\alpha) = 0$ for $\alpha \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Let $b$ be the sequences given by $b = a \ast a^*/2$. Then $b(0, 0) = |t|^2/2$ and $b(\alpha) = 0$ for $\alpha \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Let $g_0(x, y) = g(x, y) + t\varphi_0(x - y, x + y) - \varphi_0(x, y)$. It is easily seen that $g_0$ is a function in $W^k_2(\mathbb{R}^2)$ with compact support. By iteration relation (3.2) and (3.5) we can obtain

$$
b_n(0, 0) = (|t|^2/2)^n, \quad b_n(\alpha) = 0, \quad \text{for } \alpha \notin \{(0, 0)\}.
$$

By the definitions of $w_1$ and $w_2$, we have

$$
T_0^n w_2(0) = (|t|^2/2)^n \int_{\mathbb{R}^2} |g_0(x, y)|^2 \, dx \, dy,
$$

and

$$
T_0^n w_1(0) = (|t|^2/2)^n \int_{\mathbb{R}^2} \left|\left(\frac{\partial}{\partial x}\right)^k g_0(x, y)\right|^2 + \left|\left(\frac{\partial}{\partial y}\right)^k g_0(x, y)\right|^2 \, dx \, dy.
$$

Then our theorem tells us that if $|t| < 2^{1/2-k/2}$, the cascade algorithm associated with $a, g, \left(\begin{smallmatrix} 1 \\ 1 \\ -1 \\ 1 \end{smallmatrix}\right)$, and any $\varphi_0$ is convergent in $W^k_2(\mathbb{R}^2)$; if $|t| \geq 2^{1/2-k/2}$, the cascade algorithm associated with $a, g, \left(\begin{smallmatrix} 1 \\ 1 \\ -1 \\ 1 \end{smallmatrix}\right)$, and $\varphi_0$ is convergent in $W^k_2(\mathbb{R}^2)$ if and only if $\varphi_0$ is a solution of Eq. (4.2).

**Example 4.3.** Consider the inhomogeneous multivariate refinement equation of the form

$$
\varphi(x, y) = t\varphi(x - y, x + y) + t\varphi(x - y - 1, x + y - 1)
+
g(x, y), \quad (x, y) \in \mathbb{R}^2,
$$

where $t$ is a nonzero complex number and $g$ is a function in $W^k_2(\mathbb{R}^2)$ supported in $[0, 1] \times [0, 1]$. Let $\varphi_0$ be a function in $W^k_2(\mathbb{R}^2)$ supported in $[0, 1] \times [0, 1]$ and $g_0(x, y) = g(x, y) + t\varphi_0(x - y, x + y) + t\varphi_0(x - y - 1, x + y - 1) - \varphi_0(x, y)$. The corresponding cascade algorithm is given by

$$
\varphi_{n+1}(x, y) = t\varphi_n(x - y, x + y) + t\varphi_n(x - y - 1, x + y - 1) + g(x, y),
$$

$$
n = 1, 2, \ldots.
$$

In this case, $a(0, 0) = t$, $a(1, 1) = t$ and $a(\alpha) = 0$ for $\alpha \notin \{(0, 0), (1, 1)\}$. We observe that $g_0$ is a function in $W^k_2(\mathbb{R}^2)$ supported in $[0, 2] \times [-1, 1]$.

Let $b$ be the sequence given by $b = a \ast a^*/2$. By the definitions of $b_n$, it can be easily seen by induction that

$$
b_n(0, 0) = |t|^{2n}, \quad b_n(-1, -1) = b_n(1, 1) = |t|^{2n}/2
$$

and

$$
b_n(-1, 1) = b_n(1, -1) = (1/2 - 1/2^n)|t|^{2n}.
$$
Similarly, we have
\[ T^n w_2(0) = \sum_{\beta \in \mathbb{Z}^2} b_n(-\beta)w_2(\beta) = |t|^{2n} \int_{\mathbb{R}^2} |g_0(x, y)|^2 dx dy \]
\[ + |t|^{2n/2} \int_{\mathbb{R}^2} g_0(x, y)(g(x + 1, y + 1) + g(x - 1, y - 1)) dx dy \]
\[ + (1/2 - 1/2^n)|t|^{2n} \int_{\mathbb{R}^2} g_0(x, y)(g(x + 1, y - 1) \]
\[ + g_0(x - 1, y + 1)) dx dy. \]

By simple computation and the definitions of \( w_1 \) and \( w_2 \), we obtain
\[ T^n w_1(0) = \sum_{\beta \in \mathbb{Z}^2} b_n(-\beta)w_1(\beta) \]
\[ = |t|^{2n} \int_{\mathbb{R}^2} \left( \left| \left( \frac{\partial}{\partial x} \right)^k g_0(x, y) \right|^2 + \left| \left( \frac{\partial}{\partial y} \right)^k g_0(x, y) \right|^2 \right) dx dy \]
\[ + |t|^{2n/2} \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial x} \right)^k g_0(x, y)\left( \frac{\partial}{\partial x} \right)^k g_0(x + 1, y + 1) \]
\[ + \left( \frac{\partial}{\partial x} \right)^k g_0(x - 1, y - 1)) dx dy \]
\[ + |t|^{2n/2} \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial y} \right)^k g_0(x, y)\left( \frac{\partial}{\partial y} \right)^k g_0(x + 1, y + 1) \]
\[ + \left( \frac{\partial}{\partial y} \right)^k g_0(x - 1, y - 1)) dx dy \]
\[ + (1/2 - 1/2^n)|t|^{2n} \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial x} \right)^k g_0(x, y)\left( \frac{\partial}{\partial x} \right)^k g_0(x + 1, y - 1) \]
\[ + \left( \frac{\partial}{\partial x} \right)^k g_0(x - 1, y + 1)) dx dy \]
\[ + (1/2 - 1/2^n)|t|^{2n} \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial y} \right)^k g_0(x, y)\left( \frac{\partial}{\partial y} \right)^k g_0(x - 1, y + 1) \]
\[ + \left( \frac{\partial}{\partial y} \right)^k g_0(x + 1, y - 1)) dx dy. \]

By our theorem, we conclude that if \(|t| < 2^{-k/2}\), the cascade algorithm associated with \( a, g, \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \), and any \( \varphi_0 \) is convergent in \( W^2_k(\mathbb{R}^2) \);
if $2^{-k/2} \leq |t| < 2^{1/2-k/2}$, the cascade algorithm associated with $a, g, \left(\frac{1}{t}, -\frac{1}{t}\right)$, and $\varphi_0$ is convergent in $W_2^k(\mathbb{R}^2)$ if and only if

$$\int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k g_0(x, y) \right] \frac{dxdy}{(x^2 + y^2)^k} = 0;$$

if $2^{1/2-k/2} \leq |t| < 1(k > 1)$, the cascade algorithm associated with $a, g, \left(\frac{1}{t}, -\frac{1}{t}\right)$, and $\varphi_0$ is convergent in $W_2^k(\mathbb{R}^2)$ if and only if

$$\int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k g_0(x, y) \right] \frac{dxdy}{(x^2 + y^2)^k} = 0;$$

and

$$\int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k g_0(x, y) \right] \frac{dxdy}{(x^2 + y^2)^k} = 0;$$
INHOMOGENEOUS REFINEMENT EQUATIONS

if $1 \leq t < 2^{1/2}$, the cascade algorithm associated with $a, g, \{l_{i}^{1} \}_{i}^{-1}$, and $\varphi_0$ is convergent in $W^k_{2}([0, 1]^2)$ if and only if

$$\int_{\mathbb{R}^2} \left[ |g_0(x, y)|^2 + g_0(x, y)/2(g_0(x + 1, y + 1) + g_0(x - 1, y - 1))
\right. $$

$$\left. + g_0(x, y)/2(g_0(x + 1, y - 1) + g_0(x - 1, y + 1)) \right] dxdy = 0;$$

$$\int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k g_0(x, y)/2 \left( \left( \frac{\partial}{\partial x} \right)^k g_0(x + 1, y + 1) + \left( \frac{\partial}{\partial x} \right)^k g_0(x - 1, y - 1) \right)
\right. $$

$$\left. + \left( \frac{\partial}{\partial y} \right)^k g_0(x, y)/2 \left( \left( \frac{\partial}{\partial y} \right)^k g_0(x + 1, y + 1) + \left( \frac{\partial}{\partial y} \right)^k g_0(x - 1, y - 1) \right) \right] dxdy = 0$$

and

$$\int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k g_0(x, y) \left( \frac{\partial}{\partial x} \right)^k g_0(x + 1, y - 1) + \left( \frac{\partial}{\partial x} \right)^k g_0(x - 1, y + 1) \right.
\right. $$

$$\left. + \left( \frac{\partial}{\partial y} \right)^k g_0(x, y) \left( \frac{\partial}{\partial y} \right)^k g_0(x + 1, y - 1) + \left( \frac{\partial}{\partial y} \right)^k g_0(x - 1, y + 1) \right] dxdy = 0;$$

if $|t| \geq 2^{1/2}$, the cascade algorithm associated with $a, g, \{l_{i}^{1} \}_{i}^{-1}$, and $\varphi_0$ is convergent in $W^k_{2}([0, 1]^2)$ if and only if

$$\int_{\mathbb{R}^2} (|g_0(x, y)|^2 + g_0(x, y)/2(g_0(x + 1, y + 1) + g_0(x - 1, y - 1)) dxdy = 0;$$

$$\int_{\mathbb{R}^2} g_0(x, y)(g_0(x + 1, y - 1) + g_0(x - 1, y + 1)) dxdy = 0;$$

$$\int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k g_0(x, y)/2 \left( \left( \frac{\partial}{\partial x} \right)^k g_0(x + 1, y + 1) + \left( \frac{\partial}{\partial x} \right)^k g_0(x - 1, y - 1) \right)
\right. $$

$$\left. + \left( \frac{\partial}{\partial x} \right)^k g_0(x, y)/2 \left( \left( \frac{\partial}{\partial y} \right)^k g_0(x + 1, y + 1) + \left( \frac{\partial}{\partial y} \right)^k g_0(x - 1, y - 1) \right) \right] dxdy = 0.$$
\[
+ \left( \frac{\partial}{\partial y} \right)^k \frac{g_0(x, y)}{2} \left( \left( \frac{\partial}{\partial y} \right)^k \frac{g_0(x + 1, y + 1)}{g_0(x - 1, y - 1)} \right) \right] dxdy = 0
\]

and
\[
\int_{\mathbb{R}^2} \left[ \left( \frac{\partial}{\partial x} \right)^k \frac{g_0(x, y)}{2} \left( \left( \frac{\partial}{\partial x} \right)^k \frac{g_0(x + 1, y - 1)}{g_0(x - 1, y + 1)} \right) \right. \\
\left. + \left( \frac{\partial}{\partial y} \right)^k \frac{g_0(x, y)}{2} \left( \left( \frac{\partial}{\partial y} \right)^k \frac{g_0(x + 1, y - 1)}{g_0(x - 1, y + 1)} \right) \right] dxdy = 0.
\]

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