Analytic aspects of Sobolev orthogonal polynomials revisited

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Abstract

This paper surveys some recent achievements in the analytic theory of polynomials orthogonal with respect to inner products involving derivatives. Asymptotic properties, zero location, approximation and moment theory are some of the topic considered. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

During the VIII Symposium on Orthogonal Polynomials and Applications, held in September 1997 in Sevilla (Spain), a survey on some new results and tools in the study of the analytic properties of Sobolev orthogonal polynomials was presented, which was published later in [22]. This is a modest attempt to update that survey, including some topics where progress has been made in the two intervening years.

In general terms, we refer to Sobolev orthogonal polynomials when the underlying inner product involves derivatives (in the classical or distributional sense). Here we restrict ourselves to the following setups, which are general enough to exhibit the main features of the subject (for a more general definition, see e.g. [4]):

- **Diagonal case**: let \( \{ \mu_k \}_{k=0}^m \), with \( m \in \mathbb{Z}_+ \), be a set of \( m + 1 \) finite positive Borel measures such that at least one of the measures, say \( \mu_j \), has infinitely many points of increase, in which case \( \mu_k \)
has at least $k + 1$ points of increase, when $0 \leq k < j$. If $f^{(k)}$ is the $k$th derivative of the function $f$, then we denote

$$
(f, g)_S = \sum_{k=0}^{m} \int f^{(k)}(z)g^{(k)}(z) \, d\mu_k.
$$

(1)

- **Nondiagonal case:** for a function $f$ put $f = (f, f', \ldots, f^{(m)})$. Given a finite positive Borel measure $\mu$ on $\mathbb{C}$ with infinitely many points of increase and an $(m+1) \times (m+1)$ Hermitian positive-definite matrix $A$, set

$$
(f, g)_S = \int f A(g^T) \, d\mu_k.
$$

(2)

Then either (1) or (2) defines an inner product in the linear space $\mathbb{P}$ of polynomials with complex coefficients. The Gram–Schmidt process applied to the canonical basis of $\mathbb{P}$ generates the orthonormal sequence of polynomials $\{q_n\}, n = 0, 1, \ldots, \deg q_n = n$; we denote the corresponding monic polynomials by $Q_n(x) = x^n + \text{lower degree terms}$, so that

$$
q_n(x) = \frac{Q_n(x)}{||Q_n||_S}, \quad n = 0, 1, \ldots,
$$

(3)

where $||f||_S = \sqrt{(f, f)_S}$. As usual, we will call these polynomials **Sobolev orthogonal polynomials**.

In addition, we will use the following notation. If $\mu$ is a measure, then $\text{supp}(\mu)$ is its support and $P_n(\cdot; \mu)$ is the corresponding $n$th monic polynomial (if it exists) orthogonal with respect to the inner product

$$
\langle f, g \rangle_\mu = \int_{\text{supp}(\mu)} f(z)g(z) \, d\mu(z).
$$

We have

$$
P_n(z; \mu) = z^n + \sum_{k=0}^{n-1} c_{n,k} z^k, \quad \langle P_n(z; \mu), z^k \rangle_\mu = 0, \quad k = 0, \ldots, n - 1.
$$

If $\Gamma$ is a compact set in the complex plane, we denote by $C(\Gamma)$ its logarithmic capacity and by $\Omega$ the unbounded component of $\mathbb{C} \setminus \Gamma$; for simplicity we assume in what follows that $\Omega$ is regular with respect to the Dirichlet problem. Also, $\varphi$ is the conformal mapping of $\Omega$ onto the exterior of a disc $|z| = r$, normalized by

$$
\lim_{z \to \infty} \varphi(z)/z = 1,
$$

so that the radius $r = C(\Gamma)$. Finally, $\omega_\Gamma$ stands for the equilibrium measure of the compact set $\Gamma$. For details, see e.g. [35] or [36].

During the 1990s very active research on Sobolev orthogonal polynomials was in progress. Nevertheless, most of the results are connected with the algebraic aspects of the theory and classical measures in the inner product. For a historical review of this period the reader is referred to [22] (see also [26]). Here we are mainly interested in the analytic theory: asymptotics, Fourier series, approximation properties, etc.
2. Strong and comparative asymptotics

To study the asymptotic properties of the sequence \( \{Q_n\} \) as \( n \to \infty \), a natural approach is to compare it with the corresponding behavior of the sequence \( \{P_n(\cdot, \mu)\} \), where \( \mu \) is one of the measures involved in the Sobolev inner product (1) or (2). If \( \mu \) is “good”, the asymptotic properties of the standard orthogonal polynomials are well known, so we can arrive at conclusions about \( Q_n \).

Probably, the simplest case of a nontrivial Sobolev inner product is

\[
(f, g)_s = \int f(x)g(x) \, d\mu_0(x) + \lambda f'(\xi)g'(\xi).
\]

It corresponds to (1) with \( m = 1 \) and \( \mu_1 = \lambda \delta_\xi \), where as usual \( \delta_\xi \) is the Dirac delta (unit mass) at \( \xi \). The case when the measures corresponding to derivatives are finite collections of Dirac deltas is known as discrete, although it is implicitly assumed that \( \mu_0 \) has a nonzero absolutely continuous component. The asymptotic properties of \( \{Q_n\} \) in such a situation have been thoroughly studied in [16]. As it was shown there, a discrete measure \( \mu_1 \) cannot “outweigh” the absolutely continuous \( \mu_0 \), and the asymptotic behavior of the polynomials \( Q_n \) is identical to the standard orthogonal polynomials corresponding to a mass-point modification of \( \mu_0 \).

For example, assume that \( \mu_0 \) is supported on an interval \([a, b]\) \( \subset \mathbb{R} \) and belongs to the Nevai class \( M(a, b) \) (see [29]), and that \( \lambda > 0 \) and \( \xi \in \mathbb{R} \). Then,

\[
\lim_{n \to \infty} \frac{Q_n(z)}{P_n(z; \mu_0 + \mu_1)} = 1
\]

holds uniformly on compact subsets of \( \mathbb{C} \setminus ([a, b] \cup \{\xi\}) \) (see [16]).

We also may try to construct an analogue of the classical theory when we have derivatives in the inner product, considering Szegő or Nevai classes of measures or weights.

Assume that all the measures \( \mu_k \) in the inner product (1) are supported on the same Jordan curve or arc \( \Gamma \subset \mathbb{C} \). If we recall that one of the motivations for introducing Sobolev orthogonal polynomials is a least-squares fitting of differentiable functions, this seems to be the most natural situation in practice. On the support \( \Gamma \) we impose a restriction: the natural (arclength) parametrization of \( \Gamma \) belongs to the class \( C^{2+} \), which is the subclass of functions of \( C^2 \) whose second derivatives satisfy a Lipschitz condition.

In order to extend Szegő’s theory to Sobolev orthogonality, we assume first that all the measures \( \mu_k \) belong to the Szegő class on \( \Gamma \).

The experience accumulated so far shows that the right approach to the asymptotics of \( \{Q_n\} \) consists in a “decoupling” of terms in (1). Observe that after taking derivatives the polynomials involved are no longer monic. The factor \( O(n^k) \) which multiples \( Q_n^{(k)} \) plays a crucial role as \( n \to \infty \). Decoupling here means that we can restrict our attention to the last term of (1) and show that only \( \{Q_n^{(m)}\} \) “matters”. This can be done by comparing the Sobolev norm \( \|Q_n^{(m)}\|_S \) with the standard \( L^2(\mu_n) \) norm of \( Q_n^{(m)} \) and using the extremality of the \( L^2(\mu_n) \) norm for \( P_{n-m}(\cdot; \mu_m) \). This allows us to find the asymptotics of \( \{Q_n^{(m)}\} \) described by the Szegő theory. The second step is to “recover” the behavior of the sequences \( \{Q_n^{(k)}\} \) for \( k = 0, \ldots, m-1 \).

By means of this scheme, Bernstein–Szegő type theorems were established in [23] (case \( m = 1 \)) and in [25] (for \( m > 1 \)). A combination of some of these results can be stated in the form of comparative asymptotics:
Theorem 1. With the above-mentioned assumptions on the measures $\mu_k$, $k = 0, \ldots, m$, we have

$$\lim_{n \to \infty} \frac{Q_n^{(k)}(z)}{n^k P_{n-k}(z; \mu_m)} = \frac{1}{[\varphi'(z)]^{m-k}}$$

for $k = 0, 1, \ldots, m$, uniformly on compact subsets of $\Omega$.

By considering a slightly different extremal problem, we can extend this result to the case when $\mu_m$ has an absolutely continuous part from the Szegö class on $\Gamma$ plus a finite number of mass points in $C$. The condition that the measures $\mu_k$, $k = 0, \ldots, m - 1$, belong to the Szegö class, is introduced because of some technicalities in the proof: at some stage we must derive strong asymptotics from the $L^2$ one. Indeed, this condition is clearly not necessary for (4). An easy example is produced in [23]. Later, in [3], a case of Sobolev orthogonality (1) on the unit circle with $m = 1$ was studied, where, assuming more restrictive conditions on $\mu_1$, the same asymptotics was established for a wider class of measures $\mu_0$.

Thus, a necessary condition for (4) or, on the contrary, any nontrivial examples when this asymptotics does not hold, is still an open problem.

The case of noncoincident supports of the measures $\mu_k$ is very interesting and, for the time being, practically unexplored. An insight into the difficulties inherent in this situation is given in the paper [14].

3. Recent extensions or the importance to be coherent

One approach to the study of asymptotics has not yet been mentioned, namely the coherence of measures $\mu_k$. Although its scope is limited, it has played an important role during the last few years.

Historically, the coherence of measures was introduced in connection with Sobolev orthogonality and was essential in establishing first results on asymptotics in the nondiscrete case. Briefly, we say that two measures, $\mu$ and $\nu$, form a coherent pair if there exists a fixed constant $k \in \mathbb{N}_0$ such that for each $n \in \mathbb{N}$ the monic orthogonal polynomial $P_n(\cdot; \nu)$ can be expressed as a linear combination of the set

$$P_{n+1}(\cdot; \mu), \ldots, P_{n-k}(\cdot; \mu).$$

Coherence is then classified in terms of $k$. Coherent pairs of measures on $\mathbb{R}$ have been known for several years, but the complete classification (for $k = 0$) was given only in [27]. This work was a basis for [24], where the first more or less general result on nondiscrete asymptotics was obtained by means of a very simple but successful technique: establishing an algebraic relation between the sequence $\{Q_n\}$ and the corresponding sequence $\{P_n(\cdot; \mu)\}$, having a fixed number of terms, and studying the asymptotic behavior of the parameters involved. This leads easily to the comparative outer asymptotics.

As recent results show, this idea can be exploited in a variety of contexts. For instance, the case of a coherent pair with $m = 1$ and unbounded support of the measures was studied in [28] (where a result of [20] was extended). According to [27], in this case, either one of the measures $\mu_0$ or $\mu_1$ in (1) is given by the classical Laguerre weight. Following the path described above one can show
that for a suitable parameter $\alpha$ of the Laguerre polynomials $L_n^{(\alpha)}$, the ratio $Q_n(x)/L_n^{(\alpha)}(x)$ tends to a constant for $x \in \mathbb{C} \setminus [0, +\infty)$, which in turn shows that the zeros of $Q_n$ accumulate on $[0, +\infty)$.

The nondiagonal Sobolev inner product can also be dealt with if we have coherence. In [21], the authors take (2) with a $2 \times 2$ matrix $A$ and the measure $d\mu(x) = x^2e^{-x}dx$ on $[0, +\infty)$. Observe that this case can be reduced to the diagonal one (1) but with a sign-varying weight in the first integral. Thus, the results in [21] are not immediate consequences of those in [20]. Exploiting the algebraic relation between the two families (classical and Sobolev), once again one can derive the outer comparative asymptotics in $[0, +\infty)$, which gives in the limit a constant. Perhaps more informative is the scaled asymptotics $Q_n(nx)/L_n^{(\alpha)}(nx)$ holding uniformly in $C_\alpha[0, 4]$, which leads directly to an analogue of the Plancherel–Rotach asymptotics for $Q_n$. Finally, we can use the well-known Mehler–Heine asymptotic formula for $L_n^{(-1)}(nx)$ in order to relate it (and its zeros) to the Bessel functions. Analogous research for $\mu$ given by the Hermite weight on $\mathbb{R}$ was carried out in [2].

More complicated coherent pairs have been studied in [19], which yield similar asymptotic results. At any rate, after the work [27] it became apparent that coherence cannot lead us very far from the classical weights on $\mathbb{R}$. Some attempts to extend this notion to supports in the complex plane (say, on the unit circle) have not given any important results so far.

From the discussion above, it becomes clear that a discretization of the measures $\mu_k$ in (1) for $k \geq 1$ changes the asymptotic behavior of the corresponding Sobolev polynomials. An alternative approach could be to get rid of derivatives in the inner product by replacing them with a suitable finite-difference scheme.

One of the first problems in this direction was considered in [9] for

$$(f, g)_S = \int_{\mathbb{R}} fg \, d\mu_0 + \lambda \Delta f(c)\Delta g(c), \quad \lambda > 0,$$

where $\Delta f(x) = f(x+1) - f(x)$ and the support of $\mu_0$ is disjoint with $(c, c+1)$. The paper is devoted mainly to algebraic properties and zero location of polynomials $Q_n$ orthogonal with respect to this inner product.

Thus, we have the following problem: given two measures on $\mathbb{R}$, $\mu_0$ and $\mu_1$, describe the properties of the sequence of polynomials $Q_n$ orthogonal with respect to the inner product

$$(f, g)_S = (f, g)_{\mu_0} + \langle \Delta f, \Delta g \rangle_{\mu_1}. \quad (5)$$

As far as I know, only the cases of discrete measures $\mu_1$ have so far been studied, both from algebraic and analytic points of view. For instance, in [15] the construction of the corresponding Sobolev space is discussed.

We can generalize in some sense the classical families of orthogonal polynomials of a discrete variable if in (5) we take both measures $\mu_k$ discrete. In a series of papers [5,7], the so-called Meixner–Sobolev polynomials are studied, which are orthogonal with respect to (5) and

$$\mu_0 = \sum_{i=0}^{\infty} \left( \gamma + i - 1 \choose i \right) t^i \delta_i, \quad 0 < t < 1, \quad \gamma > 0, \quad \mu_1 = \lambda \mu_0, \quad \lambda > 0.$$

The Sobolev inner product $(\cdot, \cdot)_S$ obtained in this way fits in both schemes (1) and (2).

The asymptotic properties of the corresponding Sobolev orthogonal polynomials $Q_n$ are studied in [7]. Once again we observe the use of coherence in the proof: the authors show that $Q_n$, $Q_{n+1}$, $P_n(\cdot; \mu_0)$
and $P_{n+1}(\cdot; \mu_0)$ are linearly dependent and find expressions for the (nonzero) coefficients of a vanishing linear combination. This yields bounds or recurrence relations for the Sobolev norms of $Q_n$, which in turn allows us to establish comparative asymptotics of the coefficients above. As usual, an analogue of Poincaré’s theorem does the rest.

As the support of the Meixner discrete measure is unbounded, it is more interesting to study a contracted asymptotics obtained by a scaling of the variable. The authors of [7] find the behavior of the ratio $Q_n(nz)/P_n(nz; \mu_0)$ for $z \notin [0, (1 + \sqrt{t})/(1 - t)]$ and show that the zeros accumulate on the complement of this interval.

One step further in the direction of discretizing of derivatives in (1) is to consider a nonuniform mesh on $\mathbb{R}$. In particular, we could take discretization knots of the form $q^k$ and substitute the differential operator $D$ in (1) by the $q$-difference operator $D_q$:

$$D_q f(x) = \frac{f(qx) - f(x)}{(q - 1)x}, \quad x \neq 0, \quad q \neq 1, \quad D_q f(0) = f'(0).$$

In [6], the little $q$-Laguerre measure was considered,

$$\mu_0 = \sum_{k \geq 0} \frac{(aq)^k (aq; q)_\infty}{(q; q)_k} \delta_{q^k}, \quad 0 < aq, \quad q < 1,$$

where as usual,

$$(b; q)_0 = 1, \quad (b; q)_k = \prod_{j=1}^{k} (1 - bq^{j-1}), \quad 0 < k \leq \infty.$$  

Then (1) becomes

$$(f, g)_S = (f, g)_{\mu_0} + (D_q f, D_q g)_{\mu_1};$$

for $\mu_1 = \lambda \mu_0, \lambda > 0$, the corresponding $Q_n$ are called little $q$-Laguerre–Sobolev polynomials.

Once again, the “coherent” scheme works perfectly. In particular, the properties of Laguerre–Sobolev polynomials [20] can be recovered by taking appropriate limits in the little $q$-Laguerre–Sobolev family when $q \uparrow 1$.

4. Balanced Sobolev orthogonal polynomials

The role of the derivatives in the last term in (1), which introduce large factors as $n \to \infty$, was discussed above along with the idea of “decoupling” the study of each derivative $Q_n^{(k)}$. These considerations motivate the idea of balancing the terms of the Sobolev inner product by considering only monic polynomials. In other words, we can look for monic polynomials $Q_n$ of degree $n$, which minimize the norm

$$||Q_n||^2 = (Q_n, Q_n)_{\mu_0} + (Q_n'/n, Q_n'/n)_{\mu_1}$$

in the class of all monic polynomials of degree $n$. In a more general setting, we can study orthogonality with respect to (1), where each term is multiplied by a parameter which depends on the degree of the polynomial.
In [1], we made use once again of coherence of measures supported on \([-1, 1]\) in order to explore the following situation: let \(\{\lambda_n\}\) be a decreasing sequence of real positive numbers such that
\[
\lim_{n \to \infty} n^2 \lambda_n = L, \quad 0 < L < +\infty,
\]
and let \(Q_n\) now stand for the monic polynomial of degree \(n\) orthogonal to all polynomials of degree \(< n\) with respect to the inner product
\[
(p, q)_{S,n} = \langle p, q \rangle_{\mu_0} + \lambda_n \langle p', q' \rangle_{\mu_1}.
\]
Observe that the inner product varies with the degree \(n\); thus we could speak also of varying Sobolev orthogonality.

If we introduce, in addition, the measure \(\mu^*\) on \([-1, 1]\),
\[
d\mu^*(x) = \left\{ \mu_0'(x) + 4L|\varphi'(x)|^2 \mu_1'(x) \right\} dx, \quad x \in [-1, 1],
\]
we have the following theorem.

**Theorem 2** (Alfaro et al. [1]). Let \((\mu_0, \mu_1)\) be a coherent pair of measures satisfying Szegö’s condition on \([-1, 1]\), and let the sequence \(\{\lambda_n\}\) be as in (6). Then,
\[
\lim_{n \to \infty} \frac{Q_n(z)}{P_n(z; \mu^*)} = 1,
\]
locally uniformly in \(C \setminus [-1, 1]\).

In other words, the sequence \(\{Q_n\}\) asymptotically behaves like the monic orthogonal polynomial sequence corresponding to the measure (8).

The study of polynomials orthogonal with respect to a varying inner product (7) under assumption (6) should be extended to a wider class of measures \(\mu_k\). Coherence still can give something new for unbounded support, but the general case of bounded \(\text{supp}(\mu_k)\) probably must be attacked with the help of the Szegö type theory as described above.

5. Moments and approximation properties of Sobolev polynomials

It is clear that the moment theory plays an essential role in the study of the properties of standard orthogonal polynomials. At the same time, first works in this direction for the Sobolev orthogonality are very recent. In [8] (see also [30]) the diagonal case (1) is considered, when all the measures \(\mu_k\) are supported on \(\mathbb{R}\). As usual, the moment problem associated with (1) looks for the inversion of the mapping
\[
\mu = (\mu_0, \ldots, \mu_m) \to \mathcal{M} = (s_{i,j})_{i,j=0}^\infty, \quad s_{i,j} = (x^i, x^j)_{\mathbb{R}}.
\]
To be more precise, the Sobolev moment problem (or the S-moment problem) is the following: given an infinite matrix \(\mathcal{M} = (s_{i,j})_{i,j=0}^\infty\) and \(m + 1\) subsets of \(\mathbb{R}\), \(\Gamma_k, k = 0, \ldots, m\), find a set of measures \(\mu_0, \ldots, \mu_m\) \((\mu_m \neq 0)\) such that
\[
\text{supp} \mu_k \subset \Gamma_k, \quad k = 0, \ldots, m \quad \text{and} \quad s_{i,j} = (x^i, x^j)_S \quad \text{for} \; i, j = 0, \ldots.
\]
As usual, the problem is considered “definite” if it has a solution, and “determinate” if this solution is unique.
In this setting we can observe once again the phenomenon of “decoupling” mentioned above. Indeed, using (1), we can see that if $\mathcal{M}$ is given by the right-hand side of (10), then

$$\mathcal{M} = \sum_{k=0}^{m} D_k \mathcal{M}_k D_k^T, \quad \mathcal{M}_m \neq 0,$$

(11)

where

$$D_k = (d_{i,j}^k)_{i,j=0}^{\infty}, \quad d_{i,j}^k = \frac{i!}{(i-k)!} \delta_{k,i-j}, \ k = 0,\ldots,m,$$

and $\mathcal{M}_k$ are infinite Hankel matrices. Thus, questions about S-moment problem can be tackled by means of the classical tools of moment theory. In fact, in [8] it was shown that if $\mathcal{M}$ has a decomposition (11) then definiteness (determinacy) of the S-moment problem is equivalent of that for the classical moment problem for each measure $\mu_k, k = 0,\ldots,m$. A characterization of all the matrices $\mathcal{M}$ which for a fixed $m$ admit (11) can be found in [30, Theorem 2.1], where they are called Hankel–Sobolev matrices.

As moments and recurrence go hand in hand in the theory of orthogonal polynomials, it is natural to explore this path here. It is immediate to see that the Sobolev inner products (1) and (2) lack of an essential feature of the standard inner product, namely

$$(xp(x),q(x))_S \neq (p(x),xq(x))_S.$$

As a consequence, we cannot expect a three-term recurrence relation for $Q_n$ (neither any recurrence relation with a fixed number of terms, except in the case that the measures corresponding to derivatives are discrete, see [10]). Nevertheless, expanding polynomials $x^n q_n(x)$ in the basis $q_0,\ldots,q_n$ we obtain the Hessenberg matrix of coefficients,

$$R = (r_{i,j})_{i,j=0}^{\infty}, \quad r_{i,j} = (xq_i(x),q_j(x))_S.$$

(12)

As in the standard case, the zeros of the Sobolev orthogonal polynomial $q_n$ are the eigenvalues of the $n \times n$ principal minor of $R$. This shows that the zeros are connected with the operator of multiplication by the variable (or shift operator).

Now we can try to use the tools of operator theory. But here special care is needed in defining the appropriate function space. For the time being we can work it out formally: since (1) or (2) define an inner product in the space $\mathbb{P}$, we can take its completion identifying all the Cauchy sequences of polynomials whose difference tends to zero in the norm $||\cdot||_S$. Let us denote the resulting Hilbert space by $\mathbb{P}_S = \mathbb{P}_S(\mu), \mu = (\mu_0,\ldots,\mu_m)$.

Considerations above show that by means of the matrix (12) we can define in $\mathbb{P}$ a linear operator $R$ such that

$$Rp(x) = xp(x).$$

(13)

By continuity, it can be extended to the multiplication operator in $\mathbb{P}_S$.

Recall that the location of zeros of Sobolev orthogonal polynomials is not a trivial problem. Simple examples show that they do not necessarily remain in the convex hull of the union of the supports of the measures $\mu_k$ and can be complex even when all the $\mu_k$ are supported on $\mathbb{R}$. Some accurate numerical results in this regard can be found in [14]. In particular, the following question is open: is it true that whenever the measures $\mu_0,\ldots,\mu_m$ are compactly supported in $\mathbb{C}$, the zeros of $Q_n$ are uniformly bounded?
The first benefit from the interpretation of the recurrence for $q_n$ in terms of operator theory was obtained in [17;30, Theorem 3.6] and this result was improved in [18]: if $R$ is bounded and $\|R\|$ is its operator norm, then all the zeros of the Sobolev orthogonal polynomials $Q_n$ are contained in the disc
\[ \{ z \in \mathbb{C}: |z| \leq \|R\| \}. \]
Indeed, if $x_0$ is a zero of $Q_n$ then $xp(x) = x_0p(x) + Q_n(x)$ for a $p \in \mathbb{P}_{n-1}$. Since $p$ and $Q_n$ are orthogonal,
\[ |x_0|^2 \|p\|^2 = \|xp(x)\|^2 - \|Q_n(x)\|^2 \leq \|xp(x)\|^2 = \|R\| \|p\|^2 \leq \|R\|^2 \|p\|^2 \]
which yields the result.
Thus, the question whether or not the multiplication operator $R$ is bounded turns out to be a key for the location of zeros and, as it was shown in [14], to asymptotic results for the $n$th root of $Q_n$.
Clearly enough, without a thorough knowledge of the space $\mathbb{P}_{\mu}$ this condition ($R$ is bounded) lacks of any practical application.
But we can have a simple and verifiable sufficient condition for $\|R\| < \infty$, introduced also in [17]: the sequential domination of the Sobolev inner product (1). It means that for $k = 1, \ldots, m$,
\[ \text{supp } \mu_k \subseteq \text{supp } \mu_{k-1}, \quad \mu_k \ll \mu_{k-1}, \quad \frac{d\mu_k}{d\mu_{k-1}} \in L^\infty(\mu_{k-1}), \tag{14} \]
where $\mu \ll \nu$ means that $\mu$ is absolutely continuous with respect to $\nu$ and $d\mu/d\nu$ stands for the Radon–Nikodym derivative. A bound for $\|R\|$ in terms of $\max_{x \in \text{supp}(\mu)} |x|$ and the sup-norm of the derivatives above can be obtained (see [18]).
At a second look, the assumption of sequential domination seems more natural. Indeed, we have seen above that in part owing to derivatives in the integrals defining (1), the last measure, $\mu_m$, plays the leading role and determines the behavior of $Q_n^{(m)}$, while the other measures are bound to “control” the proper asymptotic behavior of $Q_n^{(k)}$, for $k = 0, \ldots, m-1$. This can be achieved by assigning more weight to measures with smaller index, like in (14).
It turns out that the condition of sequential domination is in some sense not far from being also necessary. This comes as a result of a series of works [31–34], aiming in particular at a full understanding of the structure and properties of the space $\mathbb{P}_\mu$ defined above. We are talking here about a general theory of Sobolev spaces.
Weighted Sobolev spaces are studied from several points of view, motivated mainly by the analysis of differential equations. Their extension to general measures is less explored. Some examples have been considered in [11–13], but the beginnings of a systematic study can be found in the papers mentioned above. In particular, two key questions are discussed in [32,33]: what is a reasonable extension of the definition of a Sobolev space of functions with respect to a vectorial measure $\mu = (\mu_0, \ldots, \mu_m)$? For example, we could define it as the largest space where the Sobolev norm $\| \cdot \|_S$ has sense and is finite. Then, the second question arises: what is the relation of this space to $\mathbb{P}_\mu$? In other words, we should study the possibility of approximation of a class of functions by polynomials in the norm $\| \cdot \|_S$.
A good description of $\mathbb{P}_\mu$ led in [31,34] to a proof of both necessary and sufficient conditions for the multiplication operator $R$ to be bounded in $\mathbb{P}_\mu$. Rather remarkable is the result that the sufficient condition of sequential domination is not far from being necessary. Roughly speaking, Theorem 4.1
in [31] says that if all the measures \( \mu_k \) are compactly supported on \( \mathbb{R} \) and \( R \) is bounded, then there exists a vectorial measure \( \nu \), also compactly supported on \( \mathbb{R} \), whose components are sequentially dominated, and such that the Sobolev norms induced by \( \mu \) and \( \nu \) are equivalent.

Let us go back to asymptotics. If we have located the zeros of \( \{ Q_n \} \), which turn out to be uniformly bounded, we can apply the ideas of [14] to obtain the zero distribution along with the \( n \)th root asymptotics of the Sobolev polynomials. This is the second part of [17,18].

For any polynomial \( Q \) of exact degree \( n \) we denote by

\[
v(Q) = \frac{1}{n} \sum_{Q(z) = 0} \delta_z,
\]

the normalized zero counting measure associated with \( Q \). The (weak) zero distribution of the polynomials \( Q_n \) studies the convergence of the sequence \( v(Q_n) \) in the weak-* topology. The class of regular measures \( \mu \in \text{Reg} \), compactly supported on \( \mathbb{R} \), has been introduced in [36] and is characterized by the fact that

\[
\lim_{n \to \infty} \left\| P_n (: ; \mu) \right\|_{L_2(\mu)}^{1/n} = C(\text{supp}(\mu)).
\]

Consider again the Sobolev inner product (1). Assume that there exists an \( l \in \{0, \ldots, m\} \) such that

\[
\bigcup_{k=0}^l \text{supp}(\mu_k) = \bigcup_{k=0}^m \text{supp}(\mu_k)
\]

and \( \mu_0, \ldots, \mu_l \in \text{Reg} \), with their supports regular with respect to the Dirichlet problem. Following [17], we call this inner product \( l \)-regular. For example, if it is sequentially dominated and the support of \( \mu_0 \) is regular with respect to the Dirichlet problem, then the condition \( \mu_0 \in \text{Reg} \) is equivalent to \( 0 \)-regularity.

If (1) is \( l \)-regular, then the derivatives \( Q_n^{(k)} \) for \( l \leq k \leq m \) exhibit regular behavior. Indeed, let \( \Gamma = \bigcup_{k=0}^m \text{supp}(\mu_k) \); then we have

**Theorem 3** (Lopez et al. [18]). If (1) is \( l \)-regular, then for \( l \leq k \leq m \),

\[
\lim_{n \to \infty} \left( \max_{z \in \Gamma} |Q_n^{(k)}(z)| \right)^{1/n} = C(\Gamma)
\]

and, if the interior of \( \Gamma \) is empty and \( \mathbb{C} \setminus \Gamma \) is connected,

\[
\lim_{n \to \infty} v(Q_n^{(k)}) = \omega_\Gamma,
\]

the equilibrium measure of \( \Gamma \).

In particular, if (1) is sequentially dominated and \( 0 \)-regular, then for \( k = 0, \ldots, m \),

\[
\lim_{n \to \infty} |Q_n^{(k)}(z)|^{1/n} = |\varphi(z)|
\]

holds locally uniformly in the intersection of \( \{ z \in \mathbb{C}: |z| > |R| \} \) with the unbounded connected component of \( \mathbb{C} \setminus \Gamma \). As before, \( R \) is the multiplication operator (13) in \( \mathbb{P}_S(\mu) \).

This exposition shows that the analytic theory of Sobolev orthogonal polynomials, though very abundant in results and conjectures, is still in its beginning. New approaches and fresh nonstandard
ideas are needed. Moreover, in spite of its “numerical” motivation, the development of the theory up to now has obeyed more its own internal logic than the needs of the practitioner. Thus, a good stimulus outside to this field would be more than welcome and could help to state the right questions leading to beautiful answers.

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References


