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Upper bounds for the condition numbers of the GCD and the reciprocal GCD matrices in spectral norm

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ABSTRACT

Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. The $n \times n$ matrix having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry is called the greatest common divisor (GCD) matrix defined on S, denoted by $((x_i, x_j))$, or abbreviated as (S). The $n \times n$ matrix $(S^{-1}) = (g_{ij})$, where $g_{ij} = \frac{1}{(x_i, x_j)}$, is called the reciprocal greatest common divisor (GCD) matrix on S. In this paper, we present upper bounds for the spectral condition numbers of the reciprocal GCD matrix (S^{-1}) and the GCD matrix (S) defined on $S = \{1, 2, \ldots, n\}$, with $n \geq 2$, as a function of Euler's ϕ function and n.

1. Introduction

Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. The $n \times n$ matrix having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry is called the greatest common divisor (GCD) matrix defined on S, denoted by $((x_i, x_j))$, or abbreviated as (S). The matrix having the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry is called the least common multiple (LCM) matrix, denoted by $([x_i, x_j])$, or abbreviated by $[S]$. Smith was the first mathematician who studied the GCD and LCM matrices. Later on, many authors considered the generalizations of Smith's determinant. In 1875, Smith [1] showed that the determinant of the GCD matrix (S) defined on a factor-closed $S = \{x_1, x_2, \ldots, x_n\}$ is the product

$$\det(S) = \phi(x_1)\phi(x_2)\ldots\phi(x_n),$$

and the determinant of the GCD matrix (S) defined on $S = \{1, 2, \ldots, n\}$ is

$$\det(S) = \phi(1)\phi(2)\ldots\phi(n) \tag{1}$$

where ϕ is Euler's totient function. Since then many generalizations of Smith's determinant and related results have been published, see, for example, [2–10] and the references listed there. From Smith's result one can see that the GCD matrix (S) is invertible when S is a factor-closed set. In 1989, Beslin and Ligh [11] initiated the study of the GCD matrix (S) on any set S in the direction of structure, determinant and inverse. In particular, they proved that the GCD matrix (S) on any set S is positive definite.

Turkmen and Bozkurt [12] have shown that the Euclidean norm of the GCD matrix (S) on $S = \{1, 2, \ldots, n\}$ possesses the upper bound

$$\|(S)\|_E \leq \frac{n(n+1)}{2}. \tag{2}$$

where $\|.\|_E$ is the Euclidean norm. Beslin [13] defined the $n \times n$ matrix $(S^{-1}) = (g_{ij})$, where $g_{ij} = \frac{1}{(x_i, x_j)}$, and it is the reciprocal GCD matrix on $S = \{x_1, x_2, \ldots, x_n\}$.

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Throughout this paper, (S) and (S^{-1}) denote the GCD matrix and the reciprocal GCD matrix defined on $S = \{1, 2, \dots, n\}$, with $n \geq 2$, respectively.

This paper is organized as follows. Section 2 collects definitions and some preliminary results. We present upper bounds for the spectral condition numbers of the reciprocal GCD (S^{-1}) and the GCD (S) matrices as a function of Euler's ϕ function and n in Section 3.

2. Preliminaries

In this section, we review the basic results on matrices needed in this paper. For more comprehensive treatments on matrices we refer to [14]. Let A be any $n \times n$ matrix. The Euclidean norms of the matrix A and jth column of the matrix A are defined as

$$\|A\|_E = \left(\sum_{i,j=1}^{n} \left| a_{ij} \right|^2 \right)^{1/2} \tag{3}$$

and

$$c_j(A) = \left(\sum_{i=1}^{n} \left| a_{ij} \right|^2 \right)^{1/2}, \quad 1 \leq j \leq n, \tag{4}$$

respectively. Also, the spectral norm of the matrix A is

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n} \lambda_i},$$

where λ_i is eigenvalue of $A^H A$ and A^H is the conjugate transpose of the matrix A. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A, then

$$\det A = \lambda_1 \lambda_2 \dots \lambda_n. \tag{5}$$

The square roots of the n eigenvalues of $A^H A$ are the singular values A. Since $A^H A$ is Hermitian and positive semidefinite, the singular values of A are real and nonnegative. This lets us write them in sorted order

$$\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0.$$

If $\sigma_1, \sigma_2, \dots, \sigma_n$ are the singular values of the matrix A, then

$$\|A\|_E^2 = \sum_{i=1}^{n} \sigma_i^2(A). \tag{6}$$

Throughout this note, we denote the smallest singular value of A by $\sigma_n(A)$, and its largest singular value by $\sigma_1(A)$.

For invertible $A \in \mathbb{C}^{n\times n}$, the condition number $\kappa(A)$ is defined by

$$\kappa(A) = \|A\| \left\| A^{-1} \right\|.$$

The condition number depends on the choice of norm $\|.\|$. The condition number of a given nonsingular matrix $A \in \mathbb{C}^{n\times n}$ in the spectral norm is $\sigma_1(A)/\sigma_n(A)$, and is also known as the spectral condition number. The condition number of the matrix A plays an important role in the numerical solution of linear systems since it measures the sensitivity of the solution of linear systems $Ax = b$ to the perturbations on A and b.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of the same size, not necessarily square. Then their Hadamard product (also called the Schur product) $A \circ B$ is defined by entrywise multiplication: $A \circ B = (a_{ij} b_{ij})$. The Hadamard unit matrix is the matrix U all of whose entries are 1 (the size of U being understood). A matrix A is Hadamard invertible if all its entries are non-zero, and $A^{\circ -1} = (a_{ij}^{-1})$ is then called the Hadamard inverse of A.

The arithmetic–geometric-mean inequality, or briefly the AGM inequality is the most important inequality in the classical analysis. It simply states that if $x_1, x_2, \dots, x_n$ are nonnegative real numbers and $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, then

$$\prod_{i=1}^{n} x_i^{\lambda_i} \leq \sum_{i=1}^{n} \lambda_i x_i.$$

Moreover, the equality holds if and only if $x_1 = x_2 = \cdots = x_n = 1$. The important unweighted case occurs if we put $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \frac{1}{n}$:

$$\sqrt[n]{x_1 x_2, \dots, x_n} \leq \frac{x_1 + x_2, \dots + x_n}{n}. \tag{7}$$

Lynn [15] have given the following result: if both $A = (a_{ij})$ and $B = (b_{ij})$ are any two matrices of order n, then

$$\det(A \circ B) + \det(A)\det(B) \geq \det(A) \prod_{i=1}^{n} b_{ii} + \det(B) \prod_{i=1}^{n} a_{ii} \tag{8}$$

is valid where the operation "∘" is a Hadamard product.

3. Main results

First, we give the following lemma that will be needed.

Lemma 1. *Suppose $A \in \mathbb{C}^{n\times n}$ $(n \geq 3)$ is nonsingular, with $\|A\|_E = \sqrt{n}$. Then,*

$$\sigma_n(A) > \left(\frac{n-1}{n}\right)^{(n-1)/2} |\det A| \left[1 + \frac{1}{2}\left(\frac{n-1}{n}\right)^n |\det A|^2\right]$$

[16].

By using his solution [17] to the Bourque–Ligh conjecture [18], Hong [19] proved that if $n \leq 7$, then the reciprocal GCD matrix (S^{-1}) defined on any GCD-closed set S (i.e. one has $(x_i, x_j) \in S$ for all $1 \leq i, j \leq n$) with $|S| = n$ and for any $n \geq 8$, there is a GCD-closed set S with $|S| = n$ such that the reciprocal GCD matrix (S^{-1}) defined on S is singular. In what follows we investigate the spectral condition number of the reciprocal GCD matrix.

Theorem 2. *If $\kappa\left(\left(S^{-1}\right)\right)$ is the spectral condition number of the reciprocal GCD matrix $\left(S^{-1}\right)$, with $n \geq 3$, then*

$$\kappa\left(\left(S^{-1}\right)\right) \leq n! \sqrt{\frac{n^{2n+1}}{(n-1)^{n-1}} \frac{\left|\prod_{i=1}^{n} \phi(i) - n!\right|}{\prod_{i=1}^{n} \phi(i)}}$$

is valid where ϕ is Euler's totient function and $|.|$ is the absolute value.

Proof. To obtain a lower bound for the smallest singular value of the matrix $\left(S^{-1}\right)$, we define

$$U = \left(S^{-1}\right) \operatorname{diag}\left(\frac{1}{c_1\left(\left(S^{-1}\right)\right)}, \ldots, \frac{1}{c_n\left(\left(S^{-1}\right)\right)}\right).$$

Thus, the matrix U has the following basic properties, which are verified by direct computations: U is nonsingular, the Frobenius norm of the matrix U is equal to $\sqrt{n}$. Since U satisfies the conditions of Lemma 1, it follows that

$$\begin{aligned} \sigma_n^2\left(\left(S^{-1}\right)\right) &\geq \min_i c_i^2\left(\left(S^{-1}\right)\right)\sigma_n^2(U) \\ &\geq \frac{1}{n}\sigma_n^2(U). \end{aligned} \tag{9}$$

Multiplying $\left[\sum_{i=1}^{n-1} \sigma_i^2(U)/(n-1)\right]^{(n-1)}$ on both sides of Eq. (9), we get

$$\begin{aligned} \sigma_n^2\left(\left(S^{-1}\right)\right) \left[\frac{\sum_{i=1}^{n-1} \sigma_i^2(U)}{n-1}\right]^{(n-1)} &\geq \frac{1}{n}\sigma_n^2(U) \left[\frac{\sum_{i=1}^{n-1} \sigma_i^2(U)}{n-1}\right]^{(n-1)} \\ &\geq \frac{1}{n}\prod_{i=1}^{n} \sigma_i^2(U) \\ &= \frac{1}{n}(\det U)^2 \\ &= \frac{1}{n}\left(\det\left(S^{-1}\right)\right)^2 \prod_{i=1}^{n} \frac{1}{c_i^2\left(\left(S^{-1}\right)\right)}, \end{aligned} \tag{10}$$

where we have used (5) and (7). We shall now obtain some inequalities which will be used in the rest of the proof of this theorem such as upper bounds for $\det(S^{-1})$ and $\left\|\left(S^{-1}\right)\right\|_E$, where $\det(S^{-1})$ and $\left\|\left(S^{-1}\right)\right\|_E$ are the determinant and the Euclidean norm of the reciprocal GCD matrix $\left(S^{-1}\right)$, respectively.

Firstly, if we consider (3), then by a simple computation we say that

$$\left\|\left(S^{-1}\right)\right\|_F^2 \leq n^2. \tag{11}$$

Also we know that the matrix $(S) \circ \left(S^{-1}\right)$ is the matrix U, where the operation "$\circ$" is the Hadamard product and the matrix U is the Hadamard unit matrix. In this case we deduce that determinant of the matrix $(S) \circ \left(S^{-1}\right)$ is to be zero. From [20], it is known that one of the Euler's totient function's fundamental properties is if

$$n = p_1^{k_1} p_2^{k2} \dots p_r^{k_r},$$

where the p_j are distinct primes and $k_i > 0$ are integers, then

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{12}$$

Hence, since $n \geq 2$, by virtue of (12), we can immediate write the inequality

$$\prod_{i=1}^{n} \phi(i) - n! < 0. \tag{13}$$

Using the information obtained above, we now will present a lower bound determinant of the matrix $\left(S^{-1}\right)$.

Now let us consider the fact that

$$\det(A \circ B) + \det(A)\det(B) \geq \det(A) \prod_{i=1}^{n} b_{ii} + \det(B) \prod_{i=1}^{n} a_{ii}$$

and then apply this inequality to (S) and $\left(S^{-1}\right)$ matrices. Thus, from (1) we have the inequality

$$\begin{aligned} \det\left[(S) \circ \left(S^{-1}\right)\right] + \det\left(S^{-1}\right) \prod_{i=1}^{n} \phi(i) &\geq \prod_{i=1}^{n} \phi(i) \frac{1}{(i,i)} + \det\left(S^{-1}\right) \prod_{i=1}^{n} (i,i) \\ &= \frac{1}{n!} \prod_{i=1}^{n} \phi(i) + \det\left(S^{-1}\right) n!, \end{aligned} \tag{14}$$

where ϕ is Euler's totient function. Therefore, from (13) and (14) we see that

$$\left[\det\left(S^{-1}\right)\right]^2 \geq \left[\frac{\prod_{i=1}^{n} \phi(i)}{n!\left(\prod_{i=1}^{n} \phi(i) - n!\right)}\right]^2. \tag{15}$$

Also, from (3) and (7) we have directly the inequality

$$\prod_{i=1}^{n} \frac{1}{c_i^2(\left(S^{-1}\right))} \geq \left(\frac{n}{\left\|\left(S^{-1}\right)\right\|_E^2}\right)^n. \tag{16}$$

We now turn our attention to the inequality (10).

If we first multiply by $\left[(n-1)/\sum_{i=1}^{n-1} \sigma_i^2(U)\right]^{(n-1)}$ both sides of Eq. (10), then by (11), (15) and (16), we obtain

$$\begin{aligned} \sigma_n^2(\left(S^{-1}\right)) &\geq \frac{1}{n}(\det\left(S^{-1}\right))^2 \prod_{i=1}^{n} \frac{1}{c_i^2(\left(S^{-1}\right))} \left[\frac{n-1}{\sum_{i=1}^{n-1} \sigma_i^2(U)}\right]^{(n-1)} \\ &\geq \frac{1}{n}\left(\frac{n}{\left\|\left(S^{-1}\right)\right\|_E^2}\right)^n \left[\frac{\prod_{i=1}^{n} \phi(i)}{n!\left(\prod_{i=1}^{n} \phi(i) - n!\right)}\right]^2 \left[\frac{n-1}{\sum_{i=1}^{n} \sigma_i^2(U) - \sigma_n^2(U)}\right]^{(n-1)} \\ &\geq \frac{1}{n^{n+1}} \left[\frac{\prod_{i=1}^{n} \phi(i)}{n!\left(\prod_{i=1}^{n} \phi(i) - n!\right)}\right]^2 \left[\frac{n-1}{\sum_{i=1}^{n} \sigma_i^2(U)}\right]^{(n-1)} \\ &= \frac{1}{n^{n+1}} \left(\frac{n-1}{n}\right)^{n-1} \left[\frac{\prod_{i=1}^{n} \phi(i)}{n!\left(\prod_{i=1}^{n} \phi(i) - n!\right)}\right]^2 \end{aligned} \tag{17}$$

$$\sigma_n^2((S^{-1})) \geq \frac{1}{n^{2n}} (n-1)^{n-1} \left[\frac{\prod_{i=1}^{n} \phi(i)}{n! \left(\prod_{i=1}^{n} \phi(i) - n! \right)} \right]^2,$$

or

$$\sigma_n((S^{-1})) \geq \frac{1}{n^n} (n-1)^{(n-1)/2} \frac{\prod_{i=1}^{n} \phi(i)}{n! \left| \prod_{i=1}^{n} \phi(i) - n! \right|}, \tag{18}$$

where $|.|$ is the absolute value.

Now, we will present an upper bound for the largest singular value $\sigma_1((S^{-1}))$ of the reciprocal GCD matrix (S^{-1}). For this, firstly from [14] we recall that an upper bound for $\sigma_1((S^{-1}))$ is given by

$$\sigma_1((S^{-1})) \leq \left[\|(S^{-1})\|_1 \, \|(S^{-1})\|_\infty \right]^{1/2},$$

where $\|.\|_1$ is the maximum column sum matrix norm and $\|.\|_\infty$ is the maximum row sum matrix norm. Thus, we get an upper bound, which is an immediate consequence of Eq. (9), such that

$$\sigma_1((S^{-1})) \leq \left[\left(\sum_{j=1}^{n} \frac{1}{(1,j)^2} \right)^{1/2} \left(\sum_{i=1}^{n} \frac{1}{(i,1)^2} \right)^{1/2} \right]^{1/2} = \sqrt{n}. \tag{19}$$

Consequently, from (18) and (19) we get

$$\kappa\left((S^{-1})\right) \leq \frac{\beta}{\alpha},$$

where

$$\beta = \sqrt{n}$$

and

$$\alpha = \frac{1}{n^n} (n-1)^{(n-1)/2} \frac{\prod_{i=1}^{n} \phi(i)}{n! \left| \prod_{i=1}^{n} \phi(i) - n! \right|}.$$

This completes the proof. □

Theorem 3. *If $\kappa((S))$ is the spectral condition number of the GCD matrix (S), with $n \geq 3$, then*

$$\kappa((S)) \leq \frac{\gamma}{\delta},$$

where

$$\gamma = (4n-4)^{\frac{1-n}{2n-4}} \left(\frac{n(n+1)}{2} \right)^{\frac{2n-2}{n-2}} \left(\prod_{i=1}^{n} \phi(i) \right)^{\frac{1}{2-n}}$$

and

$$\delta = \frac{n}{n-1} \left(\frac{2(n-1)}{n(n+1)} \right)^n \prod_{i=1}^{n} \phi(i)$$

and ϕ is Euler's totient function.

Proof. If we consider the matrix U defined in the proof of Theorem 2, similarly, for the GCD matrix (S) we define the matrix W such that

$$W = (S)\mathrm{diag}\left(\frac{1}{c_1((S))}, \dots, \frac{1}{c_n((S))}\right).$$

Thus, W is a nonsingular matrix and the Euclidean norm of the matrix W is equal to $\sqrt{n}$.

Since W satisfies the conditions of Lemma 1, it follows that

$$\begin{aligned}\sigma_n^2(S) &\geq \min_i c_i^2(S)\sigma_n^2(W) \\ &= n\sigma_n^2(W).\end{aligned} \tag{20}$$

If we multiply by $\left[\sum_{i=1}^{n-1}\sigma_i^2(W)/(n-1)\right]^{(n-1)}$ both sides of Eq. (20), we obtain

$$\sigma_n^2(S)\left[\frac{\sum_{i=1}^{n-1}\sigma_i^2(W)}{n-1}\right]^{(n-1)} \geq n\sigma_n^2(W)\left[\frac{\sum_{i=1}^{n-1}\sigma_i^2(W)}{n-1}\right]^{(n-1)} \tag{21}$$

$$\geq n(\det(S))^2\prod_{i=1}^{n}\frac{1}{c_i^2((S))}. \tag{22}$$

Also, from (3), (4) and (7) it follows that

$$\prod_{i=1}^{n}\frac{1}{c_i^2((S))} \geq \left(\frac{n}{\|(S)\|_E^2}\right)^n. \tag{23}$$

Thus, by (1), (21) and (22) we obtain

$$\begin{aligned}\sigma_n^2(S) &\geq n\left(\frac{n}{\|S\|_E^2}\right)^n\prod_{i=1}^{n}\phi^2(i)\left[\frac{n-1}{\sum_{i=1}^{n-1}\sigma_i^2(S)}\right]^{(n-1)} \\ &\geq n\left(\frac{n-1}{n}\right)^{(n-1)}\left(\frac{n}{\|S\|_E^2}\right)^n\prod_{i=1}^{n}\phi^2(i),\end{aligned} \tag{24}$$

where ϕ is Euler's totient function. By combining (2) and (24) we easily obtain the following inequality

$$\sigma_n^2(S) \geq \frac{2^{2n}}{n^{2(n-1)}}\frac{1}{(n+1)^{2n}}(n-1)^{n-1}\prod_{i=1}^{n}\phi^2(i)$$

or

$$\sigma_n(S) \geq \frac{n}{n-1}\left(\frac{2(n-1)}{n(n+1)}\right)^n\prod_{i=1}^{n}\phi(i). \tag{25}$$

From (5)–(7), the following inequality is easily seen to hold for the largest singular value $\sigma_1(S)$ of the GCD matrix S,

$$\begin{aligned}\sigma_1^{2n-4}(S)(\det S)^2 &= \sigma_1^{2n-2}(S)\prod_{i=2}^{n}\sigma_i^2(S) \\ &\leq \sigma_1^{2n-2}(S)\left[\frac{\sum_{i=2}^{n}\sigma_i^2(S)}{n-1}\right]^{n-1} \\ &\leq \left[\frac{\sigma_1^2(S)\sum_{i=2}^{n}\sigma_i^2(S)}{n-1}\right]^{n-1}\end{aligned}$$

$$\leq \left[\frac{1}{n-1} \left(\frac{\sum_{i=1}^{n} \sigma_i^2(S)}{2} \right)^2 \right]^{n-1}$$

$$= (4n-4)^{1-n} \, \|(S)\|_E^{4n-4} . \tag{26}$$

Hence we apply (1), (2) to (26) to get

$$\sigma_1(S) \leq (4n-4)^{\frac{1-n}{2n-4}} \left(\frac{n(n+1)}{2} \right)^{\frac{2n-2}{n-2}} \left(\prod_{i=1}^{n} \phi(i) \right)^{\frac{1}{2-n}} , \tag{27}$$

where ϕ is Euler's totient function. Finally,

$$\kappa\left((S)\right) \leq \frac{\gamma}{\delta},$$

where

$$\gamma = (4n-4)^{\frac{1-n}{2n-4}} \left(\frac{n(n+1)}{2} \right)^{\frac{2n-2}{n-2}} \left(\prod_{i=1}^{n} \phi(i) \right)^{\frac{1}{2-n}}$$

and

$$\delta = \frac{n}{n-1} \left(\frac{2(n-1)}{n(n+1)} \right)^n \prod_{i=1}^{n} \phi(i),$$

which proves Theorem 3. □

4. Discussion

Bounds for the spectral condition numbers of the reciprocal GCD matrix (S^{-1}) and the GCD matrix (S) defined on $S = \{1, 2, \ldots, n\}$ have not hitherto been studied in the literature. We initiated the study of bounds for the spectral condition numbers of these matrices in a different sense. Furthermore, these bounds have been obtained as a function of the Euler's ϕ function and n.

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