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Integrable systems on the lattice and orthogonal polynomials of discrete variable

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Abstract

Some particular examples of classical and quantum systems on the lattice are solved with the help of orthogonal polynomials and its connection to continuous models are explored.

Keywords: Integrable systems; Discrete models; Difference equations; Orthogonal polynomials; Raising and lowering operators

1. Introduction

Recently, there is an increasing interest on discrete models in classical and quantum physics. Numerical calculations use computational technics to solve differential equations [3]. Lattice field theories have become a very powerful tool to avoid infinities in perturbative methods, and to obtain exact solutions of the field equations [8]. Ponzano–Regge calculus introduced in gravitational field is equivalent to Penrose spin network that discretizes riemannian manifolds [1]. Statistical mechanics has been working from the beginning with lattice approximation.

The orthogonal polynomials of discrete variable [9] offers a new approach to these models. They are exact solutions of difference equations from which raising and lowering operators, eigenvalue equation, symmetries and constant of motion can be calculated. If a physical problem is not given in a discrete form, this can be guessed if we know the differential equation of some orthogonal polynomial which is the continuous limit of the difference equation.

We present some simple examples of discrete models in classical and quantum systems, that can be solved by the above method. In the last case, special care has to be taken in the construction of Hilbert space when the basis are the orthogonal polynomials of discrete variables.

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2. Orthogonal polynomials of discrete variable

A polynomial of hypergeometric type $P_n(x)$ of discrete variable x satisfies two fundamental relations from which one derives raising and lowering operators [7]:

(i) Difference equation:

$$\sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0, \tag{1}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of, at most, second and first degree, respectively, $\Delta(\nabla)$ are the forward (backward) operators and λ_n is the eigenvalue corresponding to the eigenfunction $P_n(x)$.

(ii) Three term recurrence relations:

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x). \tag{2}$$

(iii) Raising operator

$$\sigma(x)\nabla P_n(x) = \frac{\lambda_n}{n\tau_n'}\tau_n(x)P_n(x) - \frac{\lambda_{2n}}{2n}P_{n+1}(x), \tag{3}$$

where

$$\tau_n(x) = \tau(x+n) + \sigma(x+n) - \sigma(x).$$

(iv) Lowering operator

$$(\sigma(x)+\tau(x))\Delta P_n(x) = \left[-\frac{\lambda_n}{n}\frac{2n+1}{\lambda_{2n+1}}\tau(x) - \lambda_n - \frac{\lambda_{2n}}{2n}(x-\beta_n)\right]P_n(x) + \frac{\lambda_{2n}}{2n}\gamma_n P_{n-1}(x). \tag{4}$$

These polynomials are orthogonal with respect to the weight function $\rho(x)$, particular examples of which are the Kravchuk, Meixner, Charlier, and Hahn polynomials.

From the orthogonal polynomials of discrete variable we can construct the corresponding orthonormal functions of discrete variable, by the definition

$$\phi_n(x) = d_n^{-1}\sqrt{\rho(x)}P_n(x), \tag{5}$$

where d_n is some normalization constant. They satisfy

(i) Difference equation

$$\sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}\phi_n(x+1) + \sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}\phi_n(x-1) - (2\sigma(x)+\tau(x))\phi_n(x) + \lambda_n\phi_n(x) = 0. \tag{6}$$

(ii) Three term recurrence relation:

$$\frac{\lambda_{2n}}{2n}\alpha_n\frac{d_{n+1}}{d_n}\phi_{n+1}(x) + \frac{\lambda_{2n}}{2n}\gamma_n\frac{d_{n-1}}{d_n}\phi_{n-1}(x) + \frac{\lambda_{2n}}{2n}(\beta_n - x)\phi_n(x) = 0. \tag{7}$$

(iii) Raising operator

$$L^{+}(x,n) \equiv \left[\frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'} - \sigma(x)\right]\phi_n(x) + \sqrt{(\sigma(x-1)+\tau(x-1))\sigma(x)}\phi_n(x-1)$$
$$= \frac{\lambda_{2n}}{2n}\alpha_n\frac{d_{n+1}}{d_n}\phi_{n+1}(x). \quad (8)$$

(iv) Lowering operator

$$L^{-}(x,n) \equiv \left[-\frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'} + \lambda_n + \frac{\lambda_{2n}}{2n}(x-\beta_n) - \sigma(x) - \tau(x)\right]\phi_n(x)$$
$$+ \sqrt{(\sigma(x)+\tau(x))\sigma(x+1)}\phi_n(x+1) = \frac{\lambda_{2n}}{2n}\gamma_n\frac{d_{n-1}}{d_n}\phi_{n-1}(x). \quad (9)$$

The difference equation (i) written in the form $H(x,n)\phi_n(x)=0$, can be factorized with the help of the raising and lowering operators:

$$L^{-}(x,n+1)L^{+}(x,n) = \mu(n) + u(x+1,n)H(x,n), \quad (10)$$

$$L^{+}(x,n)L^{-}(x,n+1) = \mu(n) + u(x,n-1)H(x,n+1), \quad (11)$$

where

$$\mu(n) = \frac{\lambda_{2n}}{2n}\frac{\lambda_{2n+2}}{2n+2}\alpha_n\gamma_{n+1}, \quad u(x,n) = \frac{\lambda_n\tau_n(x)}{n\tau_n'} - \sigma(x).$$

3. The quantum harmonic oscillator of discrete variable

We start from the orthogonal polynomials of a discrete variable, the Kravchuk polynomials $k_n^{(p)}(x)$ and the corresponding orthonormal Kravchuk functions [6]

$$K_n^{(p)}(x) = d_n^{-1}\sqrt{\rho(x)}k_n^{(p)}(x), \quad (12)$$

where $d_n^2 = [N!/n!(N-n)!](pq)^n$ is a normalization constant, $\rho(x) = [N!\,p^x q^{N-x}/x!(N-x)!](pq)^n$ is the weight function, with $p>0,\ q>0,\ p+q=1,\ x=0,1,\ldots,N-1$.

From the difference equation

$$\sqrt{pq(N-x)(x+1)}K_n^{(p)}(x+1)$$
$$+ \sqrt{pq(N-x+1)x}K_n^{(p)}(x-1) + [x(p-q)-Np+n]K_n^{(p)}(x) = 0 \quad (13)$$

and the recurrence relation

$$\sqrt{pq(N-n)(x+1)}K_{n+1}^{(p)}(x)$$
$$+ \sqrt{pq(N-n+1)n}K_{n-1}^{(p)}(x) + [n(q-p)+Np-x]K_n^{(p)}(x) = 0, \quad (14)$$

we construct raising and lowering operators

$$L^+(x,n)K_n^{(p)}(x) = pq(x+n-N)K_n^{(p)}(x) + \sqrt{pq(N-x+1)x}K_n^{(p)}(x-1) = \sqrt{pq(N-n)(n+1)}K_{n+1}^{(p)}(x), \tag{15}$$

$$L^-(x,n)K_n^{(p)}(x) = pq(x+n-N)K_n^{(p)}(x) + \sqrt{pq(N-x)(x+1)}K_n^{(p)}(x+1) = \sqrt{pq(N-n+1)n}K_{n-1}^{(p)}(x). \tag{16}$$

As in the general case, these operators factorize the difference equation

$$L^+(x,n-1)L^-(x,n) = pq(N-n+1)n + pq(x+n-1-N)H(x,n), \tag{17}$$

$$L^-(x,n+1)L^+(x,n) = pq(N-n)(n+1) + pq(x+n+1-N)H(x,n). \tag{18}$$

In order to justify the name of quantum oscillator of discrete variable we substitute $x = Np + \sqrt{2Npq}s$, and take the limit $N \to \infty$ in the former expressions. We get

$$K_n^{(p)}(x) \to \sqrt{\frac{1}{2^n n!\pi}}\,\mathrm{e}^{-s^2/2}H_n(s) \equiv \psi(s), \tag{19}$$

where $H_n(s)$ is the Hermite polynomial of continuous variable. The difference and recurrence relations for the Kravchuk functions becomes in the limit the differential and recurrence relations for the normalized Hermite functions $\psi(s)$ which are the solution of the quantum harmonic oscillator.

In the limit the raising and lowering operators becomes the creation and annihilation operators

$$\frac{1}{\sqrt{Npq}}L^+(x,n)K_n^{(p)}(x) \underset{N\to\infty}{\to} \frac{1}{2}\left\{s - \frac{\mathrm{d}}{\mathrm{d}s}\right\}\psi_n(s) \equiv a^+\psi(s), \tag{20}$$

$$\frac{1}{\sqrt{Npq}}L^-(x,n)K_n^{(p)}(x) \underset{N\to\infty}{\to} \frac{1}{2}\left\{s + \frac{\mathrm{d}}{\mathrm{d}s}\right\}\psi_n(s) \equiv a\psi(s). \tag{21}$$

The commutator of the raising and lowering operators are closed under the SO(3) algebra.

$$\frac{1}{Npq}[L,L^+]K_n^{(p)}(x) = \left(1 - \frac{n}{j}\right)K_n^{(p)}(x) \equiv L^3K_n^{(p)}(x), \tag{22}$$

which in the limit becomes $[a,a^+]\psi(s) = \psi(s)$.

Similarly, the anticommutation relation of the raising and lowering operators becomes in the limit the Hamiltonian of the quantum oscillator

$$\frac{1}{Npq}[L,L^+]K_n^{(p)}(x) = \frac{1}{j}(j(j+1)-(j-n)^2)K_n^{(p)}(x) \to (aa^+ + a^+a)\psi_n(s) = (2n+1)\psi_n(s). \tag{23}$$

4. The hydrogen atom of discrete variable

We define the orthonormal Meixner functions [6]

$$M_n^{(\gamma,\mu)}(x) \equiv d_n^{-1}\sqrt{\rho_1(x)}m_n^{(\gamma,\mu)}(x), \tag{24}$$

where $m_n^{(\gamma,\mu)}(x)$ are the Meixner polynomials,

$$d_n = \frac{n!\Gamma(n+\gamma)}{\mu^n(1-\mu)^\gamma\Gamma(\gamma)}, \quad \rho_1(x) = \frac{\mu^x\Gamma(x+\gamma+1)}{\Gamma(x+1)\Gamma(\gamma)}$$

and γ, μ are real constants $0 < \mu < 1$, $\gamma > 0$.

They satisfy the orthogonality condition

$$\sum M_n^{(\gamma,\mu)}(x)M_{n'}^{(\gamma,\mu)}(x)\frac{1}{\mu(x+\gamma)} = \delta_{nn'}$$

and the following properties:

(i) Difference equation

$$\sqrt{\frac{\mu(x+\gamma)(x+1)(x+\gamma)}{x+\gamma+1}}M_n(x+1) + \sqrt{\mu(x+\gamma)x}M_n(x-1) - [\mu(x+\gamma)+x-n(1-\mu)]M_n(x) = 0. \tag{25}$$

(ii) Recurrence relation

$$-\sqrt{\mu(n+\gamma)(n+1)}M_{n+1}(x) - \sqrt{\mu(n+\gamma-1)n}M_{n-1}(x) + (\mu x + \mu n + \mu\gamma + n - x)M_n(x) = 0. \tag{26}$$

(iii) Raising operator

$$L^+(x,n)M_n(x) = -\mu(x+\gamma+n)M_n(x) + \sqrt{\mu(x+\gamma)x}M_n(x-1) = \sqrt{\mu(n+\gamma)(n+1)}M_{n+1}(x). \tag{27}$$

(iv) Lowering operator

$$L^-(x,n)M_n(x) = -\mu(x+\gamma+n)M_n(x) + \sqrt{\frac{\mu(x+\gamma)(x+1)(x+\gamma)}{x+\gamma+1}}M_n(x+1) = -\sqrt{\mu(n+\gamma-1)n}M_{n-1}(x). \tag{28}$$

Notice that we have omitted, for the sake of brevity, the superindices (γ,μ) in $M_n(x)$.

In order to make connection between the Meixner functions of discrete variable and Laguerre functions of continuous variable we substitute $\gamma = \alpha + 1$, $\mu = 1 - h$, $x = s/h$ in the former and take the limit $h \to 0$, $x \to \infty$, $hx \to s$;

$$M_n^{(\gamma,\mu)}(x) \to \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}\,\mathrm{e}^{-s}s^{\alpha+1}L_n^\alpha(s) \equiv \psi_n^\alpha(s). \tag{29}$$

We also take also the limit of the following expressions:

(i) Differential equation

$$\psi_n^{\alpha''}(s) + \left[\frac{v}{s} - \frac{1}{4} - \frac{\alpha^2-1}{s^2}\right]\psi_n^{\alpha}(s) = 0, \quad v = n + \frac{1}{2}(\alpha+1). \tag{30}$$

(ii) Recurrence relations

$$-\sqrt{(n+\alpha+1)(n+1)}\psi_{n+1}^{\alpha}(s) - \sqrt{(n+\alpha)n}\psi_{n-1}^{\alpha}(s) + (2n+\alpha+1-s)\psi_n^{\alpha}(s) = 0. \tag{31}$$

(iii) Raising operator

$$L^{+}(s,n)\psi_n^{\alpha}(s) = -\frac{1}{2}(2n+\alpha+1-s)\psi_n^{\alpha}(s) - s\frac{\mathrm{d}}{\mathrm{d}s}\psi_n^{\alpha}(s) = -\sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}^{\alpha}(s). \tag{32}$$

(iv) Lowering operator

$$L^{-}(s,n)\psi_n^{\alpha}(s) = -\frac{1}{2}(2n+\alpha+1-s)\psi_n^{\alpha}(s) + s\frac{\mathrm{d}}{\mathrm{d}s}\psi_n^{\alpha}(s) = -\sqrt{n(n+\alpha)}\psi_{n-1}^{\alpha}(s). \tag{33}$$

If we substitute in the differential equation (30) $\alpha = 2l+1$, $v = n+l+1$, we obtain the reduced radial equation for the hydrogen atom,

$$\frac{\mathrm{d}^2u}{\mathrm{d}s^2} + \left[\frac{v}{s} - \frac{1}{4} - \frac{l(l+1)}{s^2}\right]u(s) = 0, \tag{34}$$

the solutions of which are given by the generalized Laguerre functions

$$u_{vl}(s) = \left\{\frac{(v-l-1)!}{(v+l)!}\right\}^{1/2} s^{l+1}\mathrm{e}^{s/2}L_{v-l-1}^{2l+1}(s). \tag{35}$$

This correspondence shows that we can use the difference equation of Meixner function as quantum model of hydrogen atom of discrete variable.

5. Calogero–Sutherland model on the lattice

We start with the difference equation for the Hahn polynomials of discrete variable, in the particular case $\alpha = \beta = \lambda - \frac{1}{2}$, namely,

$$\begin{aligned}&[x(N-x-\lambda-\tfrac{3}{2}) + (\lambda+\tfrac{1}{2})(N-1)]h_n^{(\lambda-1/2,\lambda-1/2)}(x+1)\\ &\quad + x(N+\lambda-\tfrac{1}{2}-x)h_n^{(\lambda-1/2,\lambda-1/2)}(x-1)\\ &\quad - [2x(N-\lambda-1) + (\lambda+\tfrac{1}{2})(N-1)]h_n^{(\lambda-1/2,\lambda-1/2)}(x)\\ &\quad + n(n+2\lambda)h_n^{(\lambda-1/2,\lambda-1/2)}(x) = 0.\end{aligned} \tag{36}$$

In the continuous limit $N \to \infty$, $x/N \to s$, the Hahn polynomials become the Jacobi polynomials of continuous variable, that in the particular case $\alpha = \beta = \lambda - \frac{1}{2}$ are proportional to the Gegenbauer polynomials, namely,

$$h_n^{(\lambda-1/2,\lambda-1/2)}(x) \to P_n^{(\lambda-1/2,\lambda-1/2)}(s) = \frac{(\lambda+1/2)_n}{(2\lambda)_n} C_n^{\lambda}(s). \tag{37}$$

The difference equation for the Hahn polynomials becomes in the continuous limit the differential equation for the Gegenbauer polynomials

$$(s^2-1)\frac{d^2}{ds^2} C_n^{\lambda}(s) + (2\lambda+1)s\frac{d}{ds} C_n^{\lambda}(s) = n(n+2\lambda)C_n^{\lambda}(s). \tag{38}$$

Using polar coordinates $s = \cos q$, this equation becomes,

$$-\frac{d^2}{dq^2} C_n^{\lambda}(q) - 2\lambda \cot q \frac{d}{dq} C_n^{\lambda}(q) = \varepsilon_n(\lambda) C_n^{\lambda}(q)$$

with $\varepsilon_n(\lambda) = E_n(\lambda) - E_0 = (n+\lambda)^2 - \lambda^2 = n(n+2\lambda)$.

If we normalize the solution by the weight function $\rho(q) = (\sin q)^{\lambda}$, that is,

$$\psi_n^{\lambda}(q) = d_n (\sin q)^{\lambda} C_n^{\lambda}(q), \tag{39}$$

we get the standard differential equation for the Calogero–Sutherland model [10] in one dimension

$$H\psi_n^{\lambda}(q) = E_n(\lambda)\psi_n^{\lambda}(q), \quad E_n(\lambda) = (n+\lambda)^n$$

$$H = -\frac{d^2}{dq^2} - \lambda(\lambda-1)\frac{1}{\sin^2 q}. \tag{40}$$

6. Discrete time quantum mechanical systems

Let $H(q, p)$ be the time-independent Hamiltonian of some quantum mechanical system in one dimension [4]. The Heisenberg equation of motion for the position and momentum operators as functions of discrete time can be written as follows:

$$\frac{i}{\varepsilon}(q_{n+1} - q_n) = \frac{1}{2}[q_{n+1} + q_n, H], \tag{41}$$

$$\frac{i}{\varepsilon}(p_{n+1} - p_n) = \frac{1}{2}[p_{n+1} + p_n, H], \tag{42}$$

where the position operator $q_n \equiv q(n\varepsilon)$, and momentum operator $p_n \equiv p(n\varepsilon)$ must satisfy $[q_n, p_n] = i \ \forall n$.

Let $P_k(x)$ be an polynomial of the variable $x \equiv qp + pq$. It is easy to prove that

$$[q, P_k(x)] = (P_k(x+2i) - P_k(x))q,$$

$$[p, P_k(x) = (P_k(x-2i) - P_k(x))p.$$

When H is a polynomial of the type $P_k(x)$ we have

$$\frac{\mathrm{i}}{\varepsilon}(q_{n+1}-q_n)=[P_k(x+2\mathrm{i})-P_k(x)]\frac{1}{2}(q_{n+1}+q_n)$$

hence

$$q_{n+1}=\frac{1-(1/n)\mathrm{i}\varepsilon(P_k(x+2\mathrm{i})-P_k(x))}{1+(1/2)\mathrm{i}\varepsilon(P_k(x+2\mathrm{i})-P_k(x))}\,q_n. \tag{43}$$

By iteration we can calculate q_n in terms of the initial condition q_0. Similarly

$$p_{n+1}=\frac{1-(1/n)\mathrm{i}\varepsilon(P_k(x+2\mathrm{i})-P_k(x))}{1+(1/2)\mathrm{i}\varepsilon(P_k(x+2\mathrm{i})-P_k(x))}\,p_n. \tag{44}$$

Since $P_k(x)$ is Hermitian we have $[q_{n+1}, p_{n+1}]=[q_n, p_n]$.

When H is a function of x we can expand it in terms of some orthonormal polynomials of the variable x. In particular, if we take the continuous Hahn polynomials $S_k(x)$ defined by the two terms recursion relation

$$kS_k(x)=xS_{k-1}(x)-(k-1)S_{k-2}(x),$$

we can express the totally symmetric polynomial $T_{k,k}(q,p)$ of all possible monomials containing k factors of q and k factors of p as this formula was proved rigorously by Koornwinder [2]

$$T_{k,k}(q,p)=\frac{(2k)!}{k!2^k}\,S_k(qp+pq).$$

In all these cases we have

$$q_n=\left[\frac{1-1/2\mathrm{i}\varepsilon(H(x+2\mathrm{i})-H(x))}{1+1/2\mathrm{i}\varepsilon(H(x+2\mathrm{i})-H(x))}\right]^n q_0, \tag{45}$$

$$p_n=\left[\frac{1-1/2\mathrm{i}\varepsilon(H(x+2\mathrm{i})-H(x))}{1+1/2\mathrm{i}\varepsilon(H(x+2\mathrm{i})-H(x))}\right]^n p_0, \tag{46}$$

which in the limit $n\to\infty$, $\varepsilon\to 0$, $n\varepsilon\to t$, become

$$q_n\to q(t)=\exp(\mathrm{i}tH(x))q(0)\exp(-\mathrm{i}tH(x)),$$

$$p_n\to p(t)=\exp(\mathrm{i}tH(x))p(0)\exp(-\mathrm{i}tH(x)).$$

7. Dirac equation on the lattice

Given a function $\psi(n_\mu)$ defined on the grid points of a Minkowski lattice with elementary lengths ε_μ, difference operators $\Delta(\nabla)$, average operators $\tilde{\Delta}(\tilde{\nabla})$, we construct the Hamiltonian for the Dirac fields $\psi_\alpha(n_\mu)$ on the lattice [5]

$$H = \varepsilon_1\varepsilon_2\varepsilon_3 \sum_{n_1n_2n_3=0}^{N-1} \tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\psi^+(n_\mu)$$

$$\times\left\{\gamma_0\gamma_1\frac{1}{\varepsilon_1}\Delta_1\tilde{\Delta}_2\tilde{\Delta}_3 + \gamma_0\gamma_2\tilde{\Delta}_1\frac{1}{\varepsilon_2}\Delta_2\tilde{\Delta}_3 + \gamma_0\gamma_3\tilde{\Delta}_1\tilde{\Delta}_2\frac{1}{\varepsilon_3}\Delta_3 + m_0c\gamma_0\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\right\}\psi(n_\mu) \tag{47}$$

from which we obtain (by the Hamilton equation of motion) the Dirac equation

$$\left(\mathrm{i}\gamma_\mu\frac{1}{\varepsilon_\mu}\Delta_\mu\prod_{v\neq\mu}\tilde{\Delta}_v - m_0c\tilde{\Delta}_0\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3\right)\psi(n_\mu) = 0, \tag{48}$$

the solution of which can be expressed in terms of the plane waves on the lattice, namely, the orthogonal functions:

$$f(n_\mu) = \prod_{\mu=0}^{3}\exp\left(-\mathrm{i}\frac{2\pi}{N}m_\mu n_\mu\right), \qquad m_\mu, n_\mu = 0, 1\ldots N-1$$

provided the dispersion relations are satisfied

$$k_\mu k^\mu = m_0^2c^2, \quad k_\mu \equiv \frac{2}{\varepsilon_\mu}\, tg\frac{\pi m_\mu}{N}. \tag{49}$$

The transfer matrix, which carries the Dirac field from one time to the next time step can be obtain from the evolution operator

$$U = \frac{1 + 1/2\mathrm{i}\varepsilon_0 H}{1 - 1/2\mathrm{i}\varepsilon_0 H}, \quad \psi(n_0+1) = U\psi(n_0)U^+. \tag{50}$$

Our model for the fermion field on the lattice satisfies the following conditions in order to escape the no-go theorem of Nielsen–Ninomiya.

(i) the Hamiltonian is translational invariant,
(ii) the Hamiltonian is Hermitian,
(iii) for $m_0 = 0$, the wave equation is invariant under global chiral transformation,
(iv) there is no fermion doubling,
(v) the Hamiltonian is non-local (its Fourier transform has a singularity in the Brillouin zone) but the evolution operator, due to the Stone theorem, is unitary.

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