CHANGES OF SIGN IN CUMULATIVE SUMS [1]). I

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1. *Introduction.* Let x denote a fixed non-negative real number. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables and $S_n = X_1 + \ldots + X_n$. Then N_n will denote the number of indices $m = 1, \ldots, n$ for which

$$\text{either } S_m > x \geqslant S_{m-1} \quad \text{or} \quad S_{m-1} > x \geqslant S_m.$$

Further, K_n will denote the number of indices $m = 1, \ldots, n$ with $S_m > x$. We shall be interested in the generating functions

$$C_k = \sum_{n=0}^{\infty} t^n \Pr(N_n = k)\, E(\varrho^{K_n} e^{s(S_n - x)} | N_n = k),$$

where $|t| < 1$, $|\varrho| \leqslant 1$, $\mathrm{Re}(s) = 0$. For the case $x = 0$, $\varrho = 1$, the problem of determining the C_k was first considered by BAXTER [2], who reduced it to the problem of finding a certain factorization, see section 5.

In this paper, a method is presented which in many important special cases yields an explicit formula for the C_k. More specifically, such formulae can be found when, for $y > 0$, $\Pr(X_n > y)$ is an exponential polynomial in y and also when the X_n are integral valued and, for $j \geqslant j_0$, $\Pr(X_n = j)$ is an exponential polynomial in j, (j an integer, j_0 sufficiently large). In fact, a little more is proved by allowing $\{z_n = S_n - x\}$ to be a Markov chain satisfying $z_0 = -x$ and (3.1).

In working out the details, we have concentrated our attention on the random variable N_n (and its limit N_∞), the random variable K_n having been studied already in [1], [4] and [6]. Further results on N_n may be found in [7], [8] and [9].

The method employed has a definite interest of its own and can easily be modified so as to apply to a large number of important problems. For some such applications, see [4], [6] and [10].

2. *Preliminaries.* Let $\mathfrak{M}$ denote the commutative Banach algebra of all the complex-valued *finite* regular Borel measures on the real line R, the product $\mu\nu$ being defined as the convolution of μ and ν. The norm $||\mu||$ of $\mu \in \mathfrak{M}$ is defined as the total variation of the measure μ; in partic-

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ular, $||\mu||=1$ if μ is a probability measure, that is, a non-negative measure with $\mu(R)=1$.

The Fourier transform $\hat{\mu}$ of $\mu \in \mathfrak{M}$ is defined as

$$\hat{\mu}(s) = \int_{-\infty}^{\infty} e^{sy}\, \mu(dy), \qquad (\mathrm{Re}(s)=0),$$

thus, $|\hat{\mu}(s)| \leqslant ||\mu||$; we shall define $||\hat{\mu}||=||\mu||$. In fact, the mapping $\mu \to \hat{\mu}$ is a 1 : 1 linear norm-preserving mapping from $\mathfrak{M}$ unto a commutative Banach algebra, denoted by B, of bounded and continuous functions such that

$$(\mu\nu)^{\wedge} = \hat{\mu}\,\hat{\nu}.$$

By $\mathfrak{M}^-$ and $\mathfrak{M}^+$ we shall denote the class of measures $\mu \in \mathfrak{M}$ having their support contained in the interval $(-\infty, 0]$ or $(0, \infty)$, respectively. The corresponding classes of Fourier transforms will be denoted as B^- and B^+. Each of B^- and B^+ is a closed linear subspace of B which is also closed under multiplication. Thus, if $\alpha \in B^+$ then $e^{\alpha}-1 = \sum_{1}^{\infty} \alpha^n/n! \in B^+$.

Each $\alpha \in B$ has a unique decomposition as $\alpha=\alpha^-+\alpha^+$ with $\alpha^- \in B^-$ and $\alpha^+ \in B^+$. In fact, if $\alpha=\hat{\mu}$ then

$$\alpha^- = \int_{-\infty}^{0+} e^{sy}\, \mu(dy), \qquad \alpha^+ = \int_{0+}^{\infty} e^{sy}\, \mu(dy),$$

(s purely imaginary).

If $\alpha=\hat{\mu}$ with μ as the probability distribution of a random variable Z one can write $\alpha=E(e^{sZ})$ and

$$\alpha^-=E(\{Z\leqslant 0\}\, e^{sZ}), \qquad \alpha^+=E(\{Z>0\}\, e^{sZ}). \tag{2.1}$$

Here, and furtheron, if Π denotes a possible event then $\{\Pi\}=1$ or 0 according to whether the event Π does or does not occur.

3. *Stating the problem.* In this paper $\{z_n;\, n=0, 1, 2, \ldots\}$ denotes a Markov chain defined by

$$z_0 = -x$$

and

$$\begin{aligned} \Pr(z_n \in A | z_{n-1}=y) &= \mu(A-y) && \text{if } y\leqslant 0, \\ &= \nu(A-y) && \text{if } y>0. \end{aligned}$$

Here, x denotes a *fixed* non-negative number, while μ and ν denote probability measures. Hence, letting

$$X_n=z_n-z_{n-1}$$

and $\hat{\mu}=\varphi(s)$, $\hat{\nu}=\Phi(s)$, we have

$$\begin{cases} E(e^{sX_n}|z_{n-1}=y)=\varphi(s) & \text{if } y\leqslant 0, \\ \qquad\qquad\qquad\quad\;\; =\Phi(s) & \text{if } y>0, \end{cases} \tag{3.1}$$

(s purely imaginary).

By N_n we denote the (random) number of indices $m=1, 2, \ldots, n$ for which either $z_m > 0 \geqslant z_{m-1}$ or $z_{m-1} > 0 \geqslant z_m$. We denote by K_n the number of indices $m=1, 2, \ldots, n$ for which $z_m > 0$; in particular, $N_0 = K_0 = 0$. Further, t and ϱ denote *fixed* real or complex constants with

$$0 < |t| < 1, \qquad |\varrho| \leqslant 1;$$

the dependence of a quantity on t or ϱ will usually not be exhibited by the notation used.

We now introduce

$$C_{nk}(s) = E(\{N_n = k\}\, \varrho^{K_n}\, e^{sz_n}), \tag{3.2}$$

(s purely imaginary), in particular, from $z_0 = -x$,

$$\begin{cases} C_{0k}(s) = 0 & \text{if } k \neq 0, \\ \qquad\;\; = e^{-sx} & \text{if } k = 0. \end{cases} \tag{3.3}$$

Finally, let

$$C_k = C_k(s) = \sum_{n=0}^{\infty} t^n C_{nk}(s). \tag{3.4}$$

Our aim is to present a general method which in many important special cases enables us to obtain explicit formulae for the generating functions C_k. These in turn often lead to useful explicit formulae for

$$\Pr(N_n = n | z_0 = -x) = [C_{nk}(0)]_{\varrho=1}.$$

Of special importance is the particular case $\varphi = \Phi$ where the increments X_n are independent and identically distributed.

4. *Basic relations.* Because $z_0 = -x \leqslant 0$ we have that N_n is even if and only if $z_n \leqslant 0$, thus, N_n is odd if and only if $z_n > 0$. Therefore, we have the identities

$$\{N_n = 2k+1\}\, \varrho^{K_n} = \{z_n > 0\}\, [\{N_{n-1} = 2k\} + \{N_{n-1} = 2k+1\}]\, \varrho^{1+K_{n-1}},$$

and

$$\{N_n = 2k\}\, \varrho^{K_n} = \{z_n \leqslant 0\}\, [\{N_{n-1} = 2k\} + \{N_{n-1} = 2k-1\}]\, \varrho^{K_{n-1}},$$

($k = 0, 1, 2, \ldots$). Hence, from (3.1) and (3.2), (cf. (2.1)),

$$C_{n,2k+1}(s) = \varrho[C_{n-1,2k}(s)\, \varphi(s) + C_{n-1,2k+1}(s)\, \Phi(s)]^+ \tag{4.1}$$

and

$$C_{n,2k}(s) = [C_{n-1,2k}(s)\, \varphi(s) + C_{n-1,2k-1}(s)\, \Phi(s)]^-, \tag{4.2}$$

($n \geqslant 1$, $k \geqslant 0$), where $C_{0,-1}(s) \equiv 0$. Using (3.3) and (3.4), we obtain

$$C_{2k+1} = \varrho t[C_{2k}\varphi + C_{2k+1}\Phi]^+,$$
$$C_{2k+2} = t[C_{2k+2}\varphi + C_{2k+1}\Phi]^-,$$

($k \geqslant 0$), and

$$C_0 = e^{-sx} + t[C_0\varphi]^-.$$

Consequently,

$$C_{2k} \in B^-, \quad C_{2k+1} \in B^+, \qquad (k \geqslant 0), \tag{4.3}$$

$$(1-\varrho t\Phi)C_{2k+1} - \varrho t\varphi C_{2k} \in B^-, \qquad (k \geqslant 0), \tag{4.4}$$

$$(1-t\varphi)C_{2k+2} - t\Phi C_{2k+1} \in B^+, \qquad (k \geqslant 0), \tag{4.5}$$

and

$$(1-t\varphi)C_0 - e^{-sx} \in B^+. \tag{4.6}$$

These relations form the starting point of our method, see section 6.

5. *Baxter's method.* Though the results of the present section will not be needed later on, it might be helpful to explain how BAXTER's [2] method would work for the problem on hand. For this purpose, we introduce the generating functions

$$H_z(s) = \sum_{k=0}^{\infty} u^k \sum_{n=0}^{\infty} t^n E(\{N_n = k\}\, \varrho^{K_n} e^{sz_n} | z_0 = z) \tag{5.1}$$

and

$$\begin{cases} E_z = (1-t\varphi)H_z^- - tu\Phi H_z^+, \\ F_z = -\varrho tu\varphi H_z^- + (1-\varrho t\Phi)H_z^+. \end{cases} \tag{5.2}$$

Here, u denotes a fixed parameter, $|u| < 1$. If $z_0 = -x \leqslant 0$ then C_k as defined by (3.2) and (3.4) satisfies (4.3)–(4.6), consequently,

$$E_z^- = e^{sz}, \quad F_z^+ = 0 \qquad \text{if } z \leqslant 0. \tag{5.3}$$

Vice versa, if $z \leqslant 0$ is fixed then the functions H_z^- and H_z^+ are uniquely determined by (5.2), (5.3) and the condition that each can be expanded as a power series in t and u with coefficients in B^- and B^+, respectively. A similar statement holds for $z > 0$, where (5.3) is to be replaced by

$$E_z^- = 0, \quad F_z^+ = e^{sz} \qquad \text{if } z > 0. \tag{5.4}$$

Now, consider the pair of matrices

$$P = \begin{pmatrix} E_{z_1} & E_{z_2} \\ H_{z_1}^+ & H_{z_2}^+ \end{pmatrix}, \quad Q = \begin{pmatrix} H_{z_1}^- & H_{z_2}^- \\ F_{z_1} & F_{z_2} \end{pmatrix}, \tag{5.5}$$

where z_1, z_2 denote fixed real numbers. Then (5.2) for $z = z_1$, $z = z_2$ is equivalent to

$$(1-\varrho t\Phi) \begin{pmatrix} 1 & tu\Phi \\ 0 & 1-t\varphi \end{pmatrix} P = (1-t\varphi) \begin{pmatrix} 1-\varrho t\Phi & 0 \\ \varrho tu\varphi & 1 \end{pmatrix} Q,$$

in other words,

$$P = SQ, \tag{5.6}$$

where

$$S = (1-\varrho t\Phi)^{-1} \begin{pmatrix} (1-t\varphi)(1-\varrho t\Phi) - (tu\varphi)(\varrho tu\Phi) & -tu\Phi \\ \varrho tu\varphi & 1 \end{pmatrix}.$$

In following Baxter's method, one would try to find a pair of matrices (5.5) satisfying (5.6) such that the E_{z_i}, F_{z_i} have the properties (5.3), (5.4), while $H^-_{z_i}$ and $H^+_{z_i}$ are all power series in t and u with coefficients in B^- and B^+, respectively. Somewhat by trial, Baxter [2] succeeded in doing so when $\varrho=1, z_1=0, z_2=0+$ and either $\varphi=\Phi=(1-s^2)^{-1}$ or $\varphi=\Phi=pe^s+qe^{-s}$.

6. *Recursion formulae.* We shall now convert the basic relations (4.3)–(4.6) into recursion formulae. Namely, let

$$(6.1)\qquad \begin{cases} l^-(s) = \sum_{n=1}^{\infty} \frac{t^n}{n} (\varphi^n)^-, \\ l^+(s) = \sum_{n=1}^{\infty} \frac{t^n}{n} (\varphi^n)^+, \end{cases}$$

and

$$(6.2)\qquad \begin{cases} L^-(s) = \sum_{n=1}^{\infty} \frac{(\varrho t)^n}{n} (\Phi^n)^-, \\ L^+(s) = \sum_{n=9}^{\infty} \frac{(\varrho t)^n}{n} (\Phi^n)^+. \end{cases}$$

Consider further the linear operators on B defined by

$$(6.3)\qquad T_-\chi = t\,[\Phi e^{l^+ + L^+}\chi]^-$$

and

$$(6.4)\qquad T_+\chi = \varrho t\,[\varphi e^{l^- + L^-}\chi]^+.$$

Then the quantities

$$(6.5)\qquad \begin{cases} \Gamma^-_{2k} = e^{-l^-} C_{2k}, & (k\geqslant 0) \\ \Gamma^+_{2k+1} = e^{-L^+} C_{2k+1}, & (k\geqslant 0), \end{cases}$$

satisfy the recursion relations

$$(6.6)\qquad \Gamma^-_0 = [e^{-sx}\,e^{l^+}]^-,$$

(thus, $\Gamma^-_0=1$ if $x=0$), and

$$(6.7)\qquad \begin{cases} \Gamma^+_{2k+1} = T_+\,\Gamma^-_{2k}, & (k\geqslant 0), \\ \Gamma^-_{2k+2} = T_-\,\Gamma^+_{2k+1}, & (k\geqslant 0). \end{cases}$$

In proving this, observe first that $l^-\in B^-$, $l^+\in B^+$ and

$$(6.8)\qquad (1-t\varphi)^{-1} = \exp \sum_{n=1}^{\infty} (t^n/n)\,\varphi^n = e^{l^- + l^+}.$$

Similarly, $L^-\in B^-$, $L^+\in B^+$ and

$$(6.9)\qquad (1-\varrho t\Phi)^{-1} = e^{L^- + L^+}.$$

Multiplying (4.4) by $e^{L^-} = \sum_{n=0}^{\infty} (L^-)^n/n! \in B^-$, one obtains

$$e^{-L^+} C_{2k+1} - \varrho t\varphi\, e^{L^-} C_{2k} \in B^-.$$

Here, from (4.3) and $e^{-L^+}-1 \in B^+$, the first term denotes a function in B^+, therefore,

$$e^{-L^+} C_{2k+1} = \varrho t\, [\varphi\, e^{L^-} C_{2k}]^+.$$

Using the definitions (6.4) and (6.5) this implies the first relation (6.7). Similarly, (4.5) implies the second relation (6.7), while (4.6) implies (6.6).

The usefulness of the scheme (6.5)–(6.7) depends to a large extent on the possibility of obtaining useful explicit formulae for the operator T_+ restricted to B^- or for the operator T_- restricted to B^+. As will be shown in the sections 8 and 9, such explicit formulae can be found when the cumulative d.f. $F(y)$ corresponding to $\varphi(s)$ has a certain simple behavior for $y>0$. However, before restricting ourselves to this special case, let us first make a few remarks on the number N_∞ of changes of sign in the entire sequence $\{z_n\}$.

7. *Total number of changes of sign.* From the definition of N_n, we have $N_{n+1} \geqslant N_n$, hence, the limit

$$N_\infty = \lim_{n\to\infty} N_n$$

always exists, $N_\infty \leqslant \infty$. Further,

$$\{N_{n+1} \geqslant k\} \geqslant \{N_n \geqslant k\},$$

thus, as $n \to \infty$ we have

$$\Pr(N_n \geqslant k) \to \Pr(N_\infty \geqslant k) \text{ and } \Pr(N_n = k) \to \Pr(N_\infty = k).$$

It follows from (3.2), (3.4) that

$$\Pr(N_\infty = k) = \lim_{t\uparrow 1}\, (1-t)\, [C_k(0)]_{\varrho=1}, \tag{7.1}$$

$k=0, 1, 2, \ldots$; remember that $z_0 = -x \leqslant 0$.

For the moment, let us restrict ourselves to the special case that

$$\varphi(s) \equiv \Phi(s) \not\equiv 1. \tag{7.2}$$

Then the $X_n = z_n - z_{n-1}$ are independent random variables with

$$E(e^{sX_n}) = \varphi(s),$$

thus $\Pr(X_n \neq 0) > 0$. Let further $S_n = X_1 + \ldots + X_n$, hence, $E(e^{sS_n}) = \varphi(s)^n$ and

$$[\varphi(s)^n]^- = E(\{S_n \leqslant 0\}\, e^{sS_n}), \quad [\varphi(s)^n]^+ = E(\{S_n > 0\}\, e^{sS_n}).$$

From (6.1) and Abel's theorem,

$$\begin{cases} \lim\limits_{t\uparrow 1} l^-(0) = \sum\limits_{n=1}^{\infty} n^{-1} \Pr(S_n \leqslant 0) = A, & \text{(say)}, \\ \lim\limits_{t\uparrow 1} l^+(0) = \sum\limits_{n=1}^{\infty} n^{-1} \Pr(S_n > 0) = B, & \text{(say)}. \end{cases} \tag{7.3}$$

Clearly, $A + B = \sum n^{-1} = \infty$.

Theorem 7.1. *Suppose that* (7.2) *holds true. Then*

(i) *If* $A=B=\infty$ *then* $\Pr(N_\infty=\infty)=1$.

(ii) *If* $A<\infty$ *then* $\Pr(N_\infty=\infty)=0$ *and*

$$(7.4)\qquad \begin{cases} \Pr(N_\infty=2k+1) = e^{-A}\,[\Gamma^+_{2k+1}(0)]_{\varrho=1,\,t=1}, \\ \Pr(N_\infty=2k) = 0, \end{cases}$$

$(k=0,1,2,\ldots)$.

(iii) *If* $B<\infty$ *then* $\Pr(N_\infty=\infty)=0$ *and*

$$(7.5)\qquad \begin{cases} \Pr(N_\infty=2k) = e^{-B}\,[\Gamma^-_{2k}(0)]_{\varrho=1,\,t=1}, \\ \Pr(N_\infty=2k+1) = 0, \end{cases}$$

$(k=0,1,2,\ldots)$.

(iv) *If* $E(|X|)<\infty$ *these three cases correspond to* $E(X)=0$, $E(X)>0$ *and* $E(X)<0$, *respectively.*

Proof. Let us take $\varrho=1$. From (7.2), we have $L^+=l^+$. From (6.5), (6.8) and $\varphi(0)=1$,

$$(1-t)\,C_{2k+1}(0) = e^{-l^-(0)}\,\Gamma^-_{2k+1}(0),$$
$$(1-t)\,C_{2k}(0) = e^{-l^+(0)}\,\Gamma_{2k}(0).$$

Hence, (7.1) and (7.3) imply the first parts of (7.4) and (7.5).

The remaining assertions are an immediate consequence of known results, cf. [5] p. 331 or [4]. In particular, if $A=B=\infty$ then $\sup z_n=+\infty$ and $\inf z_n=-\infty$ with probability 1. The same is true if $E(|X|)<\infty$ and $E(X)=0$.

Further, $A<\infty$ implies that $\lim z_n=+\infty$ with probability 1, similarly, $B<\infty$ implies that $\lim z_n=-\infty$ with probability 1. By the strong law of large numbers, the same is true if $E(|X|)<\infty$ and $E(X)>0$ or $E(X)<0$, respectively.

8. *A special case.* The remaining part of this paper is concerned with the application of (6.7) to a certain special case, where all the Γ^+_{2k+1} are situated in a finite-dimensional subspace of B^+. Letting

$$(8.1)\qquad \varphi(s)=\int e^{sy}\,dF(y), \qquad \Phi(s)=\int e^{sy}\,dG(y),$$

$(F(-\infty)=G(-\infty)=0)$, this will turn out to be the case when, for $y>0$, $F(y)$ can be expressed as an exponential polynomial and also when both F and G correspond to an integer valued random variable, such that, for j as a sufficiently large integer, the jump of $F(y)$ at j can be expressed as an exponential polynomial in j.

In order to treat these two situations simultaneously, we consider a fixed closed subgroup R_0 of the reals, having at least two distinct elements. Hence, either R_0 coincides with the group R of all real numbers or with a discrete group of the form

$$I_d = \{jd;\ j=0,\pm 1,\pm 2,\ldots\}.$$

Let B_0 consist of all those elements $\hat{\mu} \in B$ for which the corresponding measure $\mu \in \mathfrak{M}$ is carried by R_0. Let further

$$B_0^- = B_0 \cap B^-, \qquad B_0^+ = B_0 \cap B^+.$$

Then each of B_0, B_0^- and B_0^+ is a closed linear manifold of B which is also closed under multiplication. Clearly, if $\chi \in B_0$ then $\chi^- \in B_0^-$ and $\chi^+ \in B_0^+$.

Let us assume that

$$x \in R_0, \quad \varphi \in B_0, \quad \Phi \in B_0. \tag{8.2}$$

From (6.1) and (6.2), l^+ and L^+ are in B_0^+, l^- and L^- are in B_0^-. Hence from (6.3) and (6.4), $T_- B_0 \subset B_0^-$ and $T_+ B_0 \subset B_0^+$, thus, from (6.6) and (6.7), $\Gamma_{2k}^- \in B_0^-$ and $\Gamma_{2k+1}^+ \in B_0^+$.

We now introduce the crucial assumption that the space $T_+ B_0^-$ be finite-dimensional. In other words, there exist finitely many elements $\lambda_\nu^+ (\nu = 1, \ldots, r)$ in B_0^+ such that

$$T_+ \chi^- = \sum_{\nu=1}^{r} a_\nu \{\chi^-\} \lambda_\nu^+ \quad \text{if} \quad \chi^- \in B_0^-, \tag{8.3}$$

where $a_\nu\{\chi^-\}$ denotes a complex number independent of s; (the quantities $a_\nu\{\chi^-\}$ and λ_ν^+ will turn out to be rather simple, see section 9). Introducing

$$c_\nu^{(k)} = a_\nu \{\Gamma_{2k}^-\},$$

and using (6.6), (6.7), one easily obtains the following simple scheme for computing the Γ_{2k}^- and Γ_{2k+1}^+.

One has

$$\Gamma_0^- = [e^{-sx} e^{l^+}]^-, \tag{8.4}$$

($\Gamma_0^- = 1$ if $x = 0$). Let

$$c_\nu^{(0)} = a_\nu \{\Gamma_0^-\}, \qquad (\nu = 1, \ldots, r), \tag{8.5}$$

and

$$c_\mu^{(k+1)} = \sum_{\nu=1}^{r} a_{\mu\nu} c_\nu^{(k)}, \qquad (\mu = 1, \ldots, r;\ k \geqslant 0), \tag{8.6}$$

where

$$a_{\mu\nu} = a_\mu \{T_- \lambda_\nu^+\}. \tag{8.7}$$

Then

$$\Gamma_{2k+1}^+ = \sum_{\nu=1}^{r} c_\nu^{(k)} \lambda_\nu^+, \qquad (k \geqslant 0), \tag{8.8}$$

and

$$\Gamma_{2k+2}^- = \sum_{\nu=1}^{r} c_\nu^{(k)} T_- \lambda_\nu^+, \qquad (k \geqslant 0). \tag{8.9}$$

If desired, the generating functions

$$\gamma_\nu = \sum_{k=0}^{\infty} c_\nu^{(k)} u^k$$

may be computed first from the system of linear equations

$$\gamma_\mu - u \sum_{\nu=1}^{r} a_{\mu\nu} \gamma_\nu = c_\mu^{(0)}, \qquad (\mu = 1, \ldots, r).$$

In particular, this would be a good method for studying the moments of N_n through the derivatives of the γ_ν at $u=1$, cf. (6.5), (3.4) and (3.2).

9. *More details.* Let us now determine under what circumstances (8.3) is satisfied. Observing that $\chi \in B_0^-$ if and only if $e^{l^- + L^-}\chi \in B_0^-$, we have from the definition (6.4) of T_+ that (8.3) is equivalent to

$$[\varphi\chi^-]^+ = \sum_{\nu=1}^{r} b_\nu\{\chi^-\}\lambda_\nu^+ \qquad \text{if } \chi^- \in B_0^-, \tag{9.1}$$

provided that

$$a_\nu\{\chi^-\} = \varrho t\, b_\nu\{e^{l^- + L^-}\chi^-\}. \tag{9.2}$$

By the way, (6.8) implies

$$e^{-l^+} + t\varphi\, e^{l^-} = e^{l^-} \in B^-,$$

hence,

$$e^{-l^+} - 1 = [e^{-l^+}]^+ = [-t\varphi e^{l^-}]^+,$$

thus, from (9.1),

$$e^{-l^+} = 1 - t\sum_{\nu=1}^{r} b_\nu\{e^{l^-}\}\lambda_\nu^+. \tag{9.3}$$

We next observe that (9.1) is in turn equivalent to the special case where $\chi^-(s) = e^{sy}$ for some fixed $y \in R_0$, $y \leqslant 0$. More precisely, letting $b_\nu\{e^{sy}\} = \beta_\nu(y)$, $(y \leqslant 0)$, (9.1) implies

$$\int_{0+}^{\infty} e^{sz}\, dF(z-y) = \sum_{\nu=1}^{r} \beta_\nu(y)\,\lambda_\nu^+(s), \qquad (y \in R_0,\ y \leqslant 0), \tag{9.4}$$

where F denotes the d.f. defined by (8.1). Moreover, if (9.4) holds (with $\beta_\nu(y)$ as a bounded measurable function, $y \leqslant 0$), and $\chi^- \in B_0^-$,

$$\chi^- = \int_{-\infty}^{0+} e^{sy}\, dH(y),$$

(say), then (9.1) holds with

$$b_\nu\{\chi^-\} = \int_{-\infty}^{0+} \beta_\nu(y)\, dH(y). \tag{9.5}$$

One may assume that the cumulative d.f. $F(y)$ is continuous to the right. Let $\lambda_\nu^+ = \hat{\sigma}_\nu$ (say); denoting the σ_ν-measure of the interval $\{u : u > z\}$ by $\gamma_\nu(z)$, we have that (9.4) is equivalent to the functional equation

$$\bar{F}(y+z) = \sum_{\nu=1}^{r} \beta_\nu(-y)\,\gamma_\nu(z), \qquad (y \geqslant 0,\ z \geqslant 0,\ y \in R_0), \tag{9.6}$$

where $\bar{F}(y) = 1 - F(y)$. Clearly, $\gamma_\nu(z)$ is a function of bounded variation which is continuous to the right and tends to 0 when $z \to \infty$. Moreover, if $R_0 = I_d$ then, (from $\lambda_\nu^+ \in B_0^+$), $\gamma_\nu(z)$ is a step function with d, $2d$, $3d$, ... as the only possible discontinuities.

We shall now distinguish between the two cases $R_0 = R$ and $R_0 = I_d$.

Suppose first that $R_0 = R$. Then [1]), as may be seen from a slight modification of a proof of FENYÖ [3], (9.6) implies that, for $y > 0$, $\bar{F}(y) = 1 - F(y)$ is an exponential polynomial. More precisely, for $y > 0$, the derivative $F'(y)$ of $F(y)$ exists and admits the expression

$$F'(y) = \sum_{h=1}^{p} \sum_{k=1}^{k_h} d_{hk}\, y^{k-1}\, e^{-\alpha_h y} \qquad \text{if } y > 0. \tag{9.7}$$

Here, the k_h denote positive integers while the d_{hk} and α_h denote complex numbers, such that (9.7) defines a real valued and non-negative function with $\int_0^\infty F'(y)\, dy \leqslant 1$. Thus, we may assume that $\mathrm{Re}(\alpha_h) > 0$ and further that the α_h are distinct and $d_{h,k_h} \neq 0$, $(h = 1, \ldots, p)$.

Let the

$$r = \sum_{h=1}^{p} k_h$$

pairs (h, i), $(h = 1, \ldots, p;\ i = 1, \ldots, k_h)$, be enumerated as $\nu = 1, \ldots, r$. Then, (9.7) in turn implies (9.6) with

$$\beta_\nu(-y) = \sum_{k=i}^{k_h} d_{hk}\, y^{k-i}\, e^{-\alpha_h y}\, (k-1)!/(k-i)!\,,$$

$$\gamma_\nu'(z) = -\, z^{i-1}\, e^{-\alpha_h z}/(i-1)!\,.$$

Consequently, using (9.5), (9.7) implies that (9.1) holds with

$$\lambda_\nu^+(s) = \int_0^\infty e^{sy}\, z^{i-1}\, e^{-\alpha_h z}\, dz/(i-1)! = (\alpha_h - s)^{-i} \tag{9.8}$$

and

$$b_\nu\{\chi\} = \sum_{j=0}^{k_h - i} d_{h,\, i+j} (-1)^j\, \chi^{(j)}(\alpha_h)\, (i+j-1)!/j!\,, \tag{9.9}$$

when $\chi \in B^-$. Here, $\chi^{(j)}$ denotes the j–th derivative of the function $\chi(s)$. Note that for each $\chi \in B^-$,

$$\chi(s) = \int_{-\infty}^{0+} e^{sy}\, dH(y),$$

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[1]) A more elementary proof would be as follows. Using the result which we shall prove concerning (9.14), one obtains that for each $\varepsilon \geqslant 0$ and each integer $q \geqslant 1$ there exists an exponential polynomial $f_{\varepsilon, q}$ of the type (9.7), (with $\sum k_h \leqslant r$), such that $\bar{F}(\varepsilon + i/q!) = f_{\varepsilon, q}(\varepsilon + i/q!)$ when $i = r+1, r+2, \ldots$. Varying q, it follows that $f_{\varepsilon, q}$ does not depend on q. This yields an exponential polynomial f_ε such that $\bar{F}(\varepsilon + y) = f_\varepsilon(\varepsilon + y)$ for each positive *rational* number y, hence, for all $y > 0$, $\bar{F}(y)$ being monotone; (it can be shown that even measurability of $\bar{F}$ would suffice).

(say), there is a natural extension of $\chi(s)$ with $\mathrm{Re}(s)=0$ to a continuous function in $\mathrm{Re}(s)\geqslant 0$ which is analytic in $\mathrm{Re}(s)>0$; (clearly, such an extension is unique). In a similar way, each function $\chi(s)\in B^+$ with $\mathrm{Re}(s)=0$ can be extended to a continuous function in $\mathrm{Re}(s)\leqslant 0$ which is analytic in $\mathrm{Re}(s)<0$ and tends to 0 as $\mathrm{Re}(s)\to -\infty$.

These remarks apply in particular to the functions $l^-(s)\in B^-$ and $l^+(s)\in B^+$. Consequently, (assuming that (9.7) holds), (9.3) and (9.8) imply that $e^{-l^+(s)}$ is of the form

$$e^{-l^+} = \prod_{i=1}^{r} (\xi_i - s) \prod_{h=1}^{p} (\alpha_h - s)^{-k_h}, \tag{9.10}$$

$(\mathrm{Re}(s)\leqslant 0)$, where $r=\sum k_h$ and the ξ_i denote yet unknown complex numbers having a *positive* real part.

Further, from (9.7),

$$1 - t\,\varphi(s) = 1 - t\int_{-\infty}^{0+} e^{sy}\,dF(y) - t\sum_{h=1}^{p}\sum_{k=1}^{k_h}(k-1)!\,d_{hk}(\alpha_h - s)^{-k} \tag{9.11}$$

yielding an extension of the function $1-t\varphi(s)$, $(\mathrm{Re}(s)=0)$, to a continuous function in $\mathrm{Re}(s)\geqslant 0$ which is analytic in $\mathrm{Re}(s)>0$, except for the pole α_h $(h=1, \ldots, p)$ which is of order k_h, from $d_{h,k_h}\neq 0$ and $t\neq 0$. Finally, from (6.8),

$$e^{-l^+(s)} = e^{l^-(s)}\,(1-t\,\varphi(s))$$

when $\mathrm{Re}(s)=0$, hence, whenever $\mathrm{Re}(s)\geqslant 0$ provided that $e^{-l^+(s)}$ is there defined by (9.10). It follows that the ξ_i are distinct from the α_h and in fact that $\{\xi_1, \ldots, \xi_r\}$ is precisely the *full set of zeros* of the function $1-t\varphi(s)$ in $\mathrm{Re}(s)>0$, (each zero being counted as often as its multiplicity).

Also the function

$$\Gamma_0^-(s) = [e^{-sx}\,e^{l^+}]^-$$

can now easily be obtained by decomposing e^{l^+} as given by (9.10) into partial fractions. For instance, in case all the zeros $\xi_1, \ldots, \xi_r$ are distinct, letting

$$e^{l^+(s)} = \prod_{h=1}^{p}(\alpha_h - s)^{k_h}\prod_{i=1}^{r}(\xi_i - s)^{-1} = 1 + \sum_{i=1}^{r} D_i(\xi_i - s)^{-1},$$

we have

$$\left\{\begin{aligned} \Gamma_0^-(s) &= e^{-sx} + \sum_{i=1}^{r} D_i(e^{-sx} - e^{-\xi_i x})\,(\xi_i - s)^{-1} \\ &= e^{-sx}\,e^{l^+(s)} - \sum_{i=1}^{r} D_i\,e^{-\xi_i x}\,(\xi_i - s)^{-1}. \end{aligned}\right. \tag{9.12}$$

Next, let us consider the case $R_0 = I_d$; one might as well assume that $d=1$, thus, R_0 is the additive group of integers. Further, the standing condition (8.2) now means that x is a non-negative integer, while

$$\varphi(s) = \sum p_j\,e^{js}, \qquad \Phi(s) = \sum q_j\,e^{js}, \tag{9.13}$$

($p_j \geqslant 0$, $q_j \geqslant 0$, $\sum p_j = \sum q_j = 1$). In particular, $F(y)$ has a jump p_j at the integer j, hence, (9.6) implies in this case

$$p_{i+j} = \sum_{\nu=1}^{r} \beta_\nu(-i)\, \gamma_{\nu j}, \qquad (i=0, 1, \ldots; j=1, 2, \ldots), \tag{9.14}$$

(where $\gamma_{\nu j} = \gamma_\nu(j-0) - \gamma_\nu(j+0)$, thus, $\sum_j |\gamma_{\nu j}| < \infty$). Let $x_0, \ldots, x_r$ denote numbers, *not all zero*, such that

$$\sum_{i=0}^{r} \beta_\nu(-i)\, x_i = 0, \qquad (\nu=1, \ldots, r).$$

Let m and M denote the smallest and largest index i, respectively, for which $x_i \neq 0$; $0 \leqslant m \leqslant M \leqslant r$. Multiplying (9.14) by x_i and summing, we obtain

$$p_j = \sum_{\nu=1}^{M-m} a_\nu\, p_{j-\nu} \quad \text{if} \quad j \geqslant M+1; \tag{9.15}$$

here, $a_\nu = -x_{M-\nu}/x_M$, $a_{M-m} \neq 0$. If two solutions of (9.15) coincide for $j=m+1, \ldots, M$ they coincide for all $j>m$. Let the distinct roots of

$$\sum_{\nu=1}^{M-m} a_\nu\, \Theta^{-\nu} = 1$$

be denoted as $\Theta_1, \ldots, \Theta_p$ and let $k_1, \ldots, k_p$ denote the corresponding multiplicities. Then (9.15) admits the linearly independent solutions $j^{k-1}\Theta_h{}^j$, ($h=1, \ldots, p$; $k=1, \ldots, k_h$). It follows that there exist complex constants c_{hk} such that

$$p_j = \sum_{h=1}^{p} \sum_{k=1}^{k_h} c_{hk}\, j^{k-1}\, \Theta_h^j \quad \text{if} \quad j>m.$$

Here, $m + \sum k_h = M \leqslant r$. From $\sum p_j = 1$, we have $c_{hk}=0$ whenever $|\Theta_h| \geqslant 1$.

Conversely, let

$$p_j' = \sum_{h=1}^{p} \sum_{k=1}^{k_h} d_{hk}\, j^{k-1}\, e^{-\alpha_h j}, \tag{9.16}$$

with k_h as positive integers, d_{hk} and α_h as complex constants, $\mathrm{Re}(\alpha_h) > 0$, and suppose that

$$\begin{cases} p_j = p_j' & \text{if} \quad j>m, \\ \quad = p_j' + p_j'' & \text{if} \quad 1 \leqslant j \leqslant m, \end{cases} \tag{9.17}$$

where $m \geqslant 0$ is a fixed integer. Then, for each

$$\chi^-(s) = \sum_{j=-\infty}^{0} \chi_j\, e^{js}$$

in B_0^-, we have

$$[\varphi\chi^-]^+ = \sum_{j=-\infty}^{0} \chi_j \sum_{i=1}^{\infty} p_{i-j}\, e^{is} = \sum_{g=1}^{m} e^{sg} \sum_{j=0}^{m-g} \chi_{-j}\, p_{j+g}'' $$
$$+ \sum_{h=1}^{p} \sum_{k=1}^{k_h} d_{hk} [(-\partial/\partial\alpha)^{k-1}\, \chi^-(\alpha)\, (e^{\alpha-s}-1)^{-1}]_{\alpha=\alpha_h}.$$

Consequently, (9.17) implies (9.1) with $r = m + \sum k_h$,

$$(9.18)\qquad \begin{cases} \lambda_\nu^+(s) = \sum_{j=1}^{\infty} j^{i-1}\, e^{(s-\alpha_h)j}/(i-1)! \\ \qquad\;\; = (\partial/\partial s)^{i-1]}(e^{\alpha_h - s} - 1)^{-1}/(i-1)! \end{cases}$$

for $\nu = 1, \ldots, \sum k_h$; here, ν stands for one of the $\sum k_h$ pairs (h, i) with $1 \leqslant h \leqslant p$, $1 \leqslant i \leqslant k_h$. Further, for these indices ν, $b_\nu\{\chi\}$ is again given by formula (9.9). Finally,

$$(9.19)\qquad \lambda_\nu^+(s) = e^{sg}, \qquad b_\nu\{\chi\} = \sum_{j=0}^{m-g} \chi_{-j}\, p''_{j+g}$$

if $\nu = \sum k_h + g$ and $1 \leqslant g \leqslant m$.

We are now also in a position to determine an explicit formula for $e^{-l^+(s)}$. One may assume that in (9.16) the α_h are distinct and $d_{h,k_h} \neq 0$, $(h = 1, \ldots, p)$ and further that in (9.17) the number m is minimal in the sense that $p''_m \neq 0$ if $m > 0$.

Letting $w = e^s$ and using (9.3), (9.18), (9.19) and (9.17), it follows from a reasoning completely analogous to the proof of (9.10) that

$$(9.20)\qquad e^{-l^+(s)} = \prod_{i=1}^{r} (1 - w/\eta_i) \prod_{h=1}^{p} (1 - we^{-\alpha_h})^{-k_h}.$$

Here, $r = m + \sum k_h$. Further, $\{\eta_1, \ldots, \eta_r\}$ is precisely the full set of zeros of the function $1 - t\psi(w)$ in the region $|w| > 1$, (each zero counted as often as its multiplicity). Here,

$$\psi(w) = \sum_j p_j w^j, \qquad (|w| = 1),$$

which can be extended to a function which is continuous for $|w| \geqslant 1$, analytic for $|w| > 1$, except for the pole e^{α_h} which is of order k_h, $(h = 1, \ldots, p)$.

Decomposing $e^{l^+(s)}$ into partial fractions, one obtains an explicit formula for $\Gamma_0^- = [e^{-sx} e^{l^+}]^-$, $(x \geqslant 0$ a fixed integer). For instance, if all the $\eta_1, \ldots, \eta_r$ are distinct, letting

$$e^{l^+(s)} = \prod_{h=1}^{p} (1 - w\, e^{-\alpha_h})^{k_h} \prod_{i=1}^{r} (1 - w/\eta_i)^{-1} = 1 + \sum_{i=1}^{r} D_i w(\eta_i - w)^{-1},$$

one has

$$(9.21)\qquad \begin{cases} \Gamma_0^-(s) = w^{-x} + \sum_{i=1}^{r} D_i (w^{-x} - \eta_i^{-x})\, w(\eta_i - w)^{-1} \\ \qquad\quad = w^{-x}\, e^{l^+(s)} - \sum_{i=1}^{r} D_i\, \eta_i^{-x}\, w(\eta_i - w)^{-1}, \end{cases}$$

where $w = e^s$.

(To be continued)