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The vulnerability of the diameter of folded n-cubes

E. Simó*, J.L.A. Yebra

Department of Applied Mathematics and Telematics, Polytechnical University of Catalonia, Apdo. 30002, 08080-Barcelona, Spain

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Abstract

Fault tolerance concern in the design of interconnection networks has arisen interest in the study of graphs such that the subgraphs obtained by deleting some vertices or edges have a moderate increment of the diameter. Besides the general problem, several particular families of graphs are worthy of consideration. Both the odd graphs and the n-cubes have been studied in this context. In this paper we deal with folded n-cubes, a much interesting family because: (i) like the n-cubes, their order is a power of 2, (ii) their diameter is half the diameter of the n-cube of the same order, while their degree only increases by one, and (iii) as we show, in a folded n-cube of degree Δ, the deletion of less than $\lfloor \frac{1}{2}\Delta \rfloor - 1$ vertices or edges does not increase the diameter of the graph, and the deletion of up to $\Delta - 1$ vertices or edges increases it by at most one. This last property means that interconnection networks modelled by folded n-cubes are extremely robust.

1. Introduction

The designer of interconnection networks must allow for the fact that machines and/or communication links may malfunction or cease to function. In this event, it is important that communication can still be achieved with reasonable efficiency. It may be required, for instance, that between any two nodes of the remaining network there still exists a path of length not exceeding some fixed value, see [7].

The modellisation in terms of graphs leads to the search for large graphs such that the deletion of some vertices or edges moderately increases the diameter of the graph. In the literature this is called the (Δ, D, D', s)-problem. It consists of finding large graphs with maximum degree Δ and diameter D, such that the subgraphs obtained by deleting any set of up to s vertices or edges have diameter $\leqslant D'$, see, for instance, [1] or [3]. The main interest is in families of graphs such that $D' - D$ is small. Families

* Corresponding author. E-mail: ester@mat.upc.es.

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of graphs that have been studied in this context are the odd graphs, see [5] or [8], and the n-cubes, see [6].

In this paper we concentrate on folded n-cubes. This is a much interesting family because: (i) like the n-cubes, their order is a power of 2, (ii) their diameter is half the diameter of the n-cube of the same order, while their degree only increases by one, and (iii) as we show, in a folded n-cube of degree Δ, the deletion of less than $\lfloor \frac{1}{2}\Delta \rfloor - 1$ vertices or edges does not increase the diameter of the graph, and the deletion of up to $\Delta - 1$ vertices or edges increases it by at most one. This last property means that interconnection networks modelled by folded n-cubes are extremely robust. We begin by defining the folded n-cubes in the next section. Afterwards, in Section 3 we prove the main results of the paper.

2. The folded n-cubes

We recall that the n-cube, Q_n, also called n-dimensional hypercube, is the graph of order 2^n whose vertices can be labelled as the n-length sequences of 0's and 1's, two vertices being adjacent whenever their labels differ in just one digit. By defining the Hamming distance between two sequences as the number of digits in which they differ, we can alternatively say that two vertices of the n-cube are adjacent when the Hamming distance between the corresponding sequences is one. It follows that the Hamming distance measures the distance between vertices in the n-cube.

The *folded* $(n+1)$*-cube*, denoted by $\square_{n+1}$, is the graph obtained from the hypercube in either one of the two following equivalent ways (where as usual we write $\bar{0} = 1$ and $\bar{1} = 0$):

1. consider the $(n+1)$-cube and identify opposite vertices

$$A = (a_0 a_1 \cdots a_n) \equiv (\bar{a}_0 \bar{a}_1 \cdots \bar{a}_n) = \overline{A};$$

2. consider the n-cube and add an edge between any two opposite vertices

$$A = (a_1 a_2 \cdots a_n) \sim (\bar{a}_1 \bar{a}_2 \cdots \bar{a}_n) = \overline{A}.$$

Fig. 1 illustrates how the folded 4-cube is obtained in these two ways. The resulting graph has 2^n vertices, is regular of degree $n+1$ and has diameter $\lfloor \frac{1}{2}(n+1) \rfloor$, since for any vertex A of the $(n+1)$-cube, all other vertices are at distance $\leqslant \lfloor \frac{1}{2}(n+1) \rfloor$ of either A or $\overline{A}$. It is well known that the folded n-cube is distance transitive, see, for instance, [2, p. 178] or [4, Ch. 4.1.F]. We shall only use the fact that it is vertex transitive.

In what follows we use the second presentation, that is

$$V(\square_{n+1}) = \{(a_1 a_2 \cdots a_n),\ a_i \in \{0, 1\}\},$$

and each vertex $A = (a_1 a_2 \cdots a_n)$ is adjacent to the $n+1$ vertices

1. $(a_1 a_2 \cdots a_{i-1} \bar{a}_i a_{i+1} \cdots a_n) = B_i,\ 1 \leqslant i \leqslant n$;
2. $(\bar{a}_1 \bar{a}_2 \cdots \bar{a}_n) = \overline{A}$.

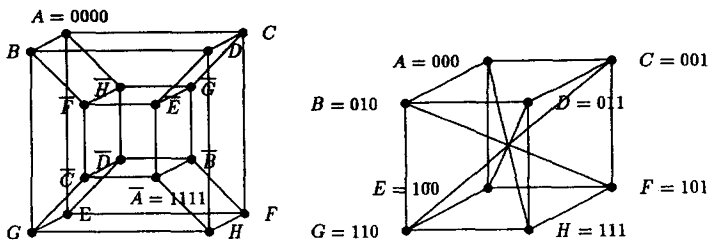

Fig. 1.

The adjacency rule readily leads to a proper edge-colouring of the graph by colouring 0 the edges $A \sim \overline{A}$ and colouring i the edges $A \sim B_i$. This in turn leads to a colouring of the paths in $\square_{n+1}$.

3. Deletion of vertices or edges

We study in this section the vulnerability of the diameter of the folded $(n+1)$-cube $\square_{n+1}$ under deletion of s, $1 \leqslant s \leqslant n$ vertices or edges. Our first result is

Proposition 1. *For $n \geqslant 2$ there exist $n+1$ independent paths of length $\leqslant \lfloor \frac{1}{2}(n+1) \rfloor + 1$ between any pair of vertices of $\square_{n+1}$.*

Proof. Let A and B be two vertices of $\square_{n+1}$ at Hamming distance $d_H(A,B) = k$. Without loss of generality we may set

$$A \equiv (a_1 a_2 \cdots a_k a_{k+1} \cdots a_n),$$

$$B \equiv (\overline{a}_1 \overline{a}_2 \cdots \overline{a}_k a_{k+1} \cdots a_n),$$

and distinguish three types of neighbours of A:

$$\begin{aligned} V_i &\equiv (a_1 a_2 \cdots \overline{a}_i \cdots a_k a_{k+1} \cdots a_n), \quad 1 \leqslant i \leqslant k, \\ W_i &\equiv (a_1 a_2 \cdots a_k a_{k+1} \cdots \overline{a}_i \cdots a_n), \quad k+1 \leqslant i \leqslant n, \\ \overline{A} &\equiv (\overline{a}_1 \overline{a}_2 \cdots \overline{a}_n). \end{aligned}$$

The next step is to construct shortest paths between each of these vertices and vertex B, thus assuring that the length of the paths between A and B is $\leqslant 1 + D = 1 + \lfloor \frac{1}{2}(n+1) \rfloor$. Each path will be identified by its colouring. We will consider separately the three cases:

$$k \leqslant \frac{n-1}{2}, \qquad \frac{n-1}{2} < k < \frac{n+3}{2} \quad \text{and} \quad k \geqslant \frac{n+3}{2}.$$

1. $k \leqslant \frac{1}{2}(n-1)$. The $n+1$ paths between A and B are k paths of length k through vertices V_i

$$\begin{array}{lcccccc}
\text{path } 1 & 1 & 2 & 3 & \cdots & k-1 & k \\
\text{path } 2 & 2 & 3 & 4 & \cdots & k & 1 \\
\vdots & & \vdots & & & \vdots & \\
\text{path } k & k & 1 & 2 & \cdots & k-2 & k-1
\end{array}$$

$n-k$ paths of length $k+2$ through vertices W_i

$$\begin{array}{lccccccc}
\text{path } k+1 & k+1 & 1 & 2 & \cdots & k-1 & k & k+1 \\
\text{path } k+2 & k+2 & 1 & 2 & \cdots & k-1 & k & k+2 \\
\vdots & \vdots & & & & & \vdots & k+2 \\
\text{path } n & n & 1 & 2 & \cdots & k-1 & k & n
\end{array}$$

and 1 path of length $k+2$ through vertex $\overline{A}$

$$\begin{array}{lccccccc}
\text{path } n+1 & 0 & 1 & 2 & \cdots & k-1 & k & 0
\end{array}$$

Note that $k+2 \leqslant \frac{1}{2}(n-1)+2 = \frac{1}{2}(n+1)+1$, so that $k+2 \leqslant D+1$.

2. $\frac{1}{2}(n-1) < k < \frac{1}{2}(n+3)$. The $n+1$ paths between A and B are k paths of length k through vertices V_i as in the preceding case, $n-k$ paths through vertices W_i of length $n-k+1$

$$\begin{array}{lccccccc}
\text{path } k+1 & k+1 & k+2 & k+3 & \cdots & n-1 & n & 0 \\
\text{path } k+2 & k+2 & k+3 & k+4 & \cdots & n & 0 & k+1 \\
\vdots & & \vdots & & & & \vdots & \\
\text{path } n & n & 0 & k+1 & \cdots & n-3 & n-2 & n-1
\end{array}$$

and 1 path of length $n-k+1$ through vertex $\overline{A}$

$$\begin{array}{lcccccccc}
\text{path } n+1 & 0 & k+1 & k+2 & \cdots & n-2 & n-1 & n
\end{array}$$

Again we have $k \leqslant D+1$ and $n-k+1 \leqslant D+1$. This case corresponds to $k = \frac{1}{2}n$ or $k = \frac{1}{2}n+1$ when n is even and to $k = \frac{1}{2}(n+1)$ when k is odd. When $k = \frac{1}{2}n$ the shortest paths are those through vertices V_i, when $k = \frac{1}{2}n+1$ the shortest paths are those through vertices W_i and A, and when $k = \frac{1}{2}(n+1)$ all paths are shortest paths between A and B.

3. $k \geqslant \frac{1}{2}(n+3)$. The $n+1$ paths between A and B are k paths of length $n-k+3$ through vertices V_i

$$\text{path } i \quad i \quad k+1 \quad k+2 \quad \cdots \quad n \quad 0 \quad i, \quad 1 \leqslant i \leqslant k$$

and the $n-k+1$ paths of length $n+k-1$ through vertices W_i and $\overline{A}$ of the preceding case.

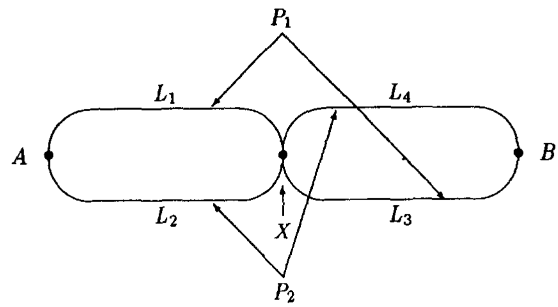

Fig. 2.

Again $n-k+3 \leqslant n-\frac{1}{2}(n+3)+3=\frac{1}{2}(n+1)+1$, so that $n-k+3 \leqslant D+1$.

It remains to show that in each case the $n+1$ paths are independent. Suppose on the contrary that two paths P_1 and P_2 between A and B have a common vertex X as shown in Fig. 2.

Let L_1 and L_3 be the sections $A-X$ and $X-B$ of P_1 and L_2 and L_4 the sections $A-X$ and $X-B$ of P_2. Let $\bigcup L$ denote the union of the colours of path L and let $l=|L|$ denote its length. If the colour 0 does not appear in both paths P_1 and P_2 we must have $\bigcup L_1=\bigcup L_2$ and/or $\bigcup L_3=\bigcup L_4$ which is imposible by construction. The same conclusion is reached if the colour 0 appears in the same section of both P_1 and P_2. Finally, if the colour 0 appears in different sections of both P_1 and P_2, it must be

$$l_1+l_2 \geqslant n+1 \quad \text{and} \quad l_3+l_4 \geqslant n+1$$

since $n+1$ is the shortest length of any cycle which contains just one 0-coloured edge. Therefore

$$2n+2 \leqslant l_1+l_2+l_3+l_4=|P_1|+|P_2| \leqslant 2\left\lfloor\frac{n+1}{2}\right\rfloor+2,$$

a contradiction. □

A first consequence of this result is that the deletion of up to n vertices of $\square_{n+1}$ cannot increase the diameter of the graph by more than one. More precisely we have:

Theorem 1. *Let D'_s be the maximum value of the diameter of the subgraphs obtained from $\square_{n+1}$ when s vertices are deleted. Then*

$$D'_s=\begin{cases} D & \text{if } 0 \leqslant s<\left\lfloor\frac{1}{2}(n-1)\right\rfloor, \\ D+1 & \text{if } \left\lfloor\frac{1}{2}(n-1)\right\rfloor \leqslant s \leqslant n. \end{cases}$$

Proof. From Proposition 1 we know that $D'_s \leqslant D+1$, and from its proof it follows that the deletion of $s<\lfloor\frac{1}{2}(n-1)\rfloor$ vertices does not increase the diameter of the graph. On the other hand, for $s=\lfloor\frac{1}{2}(n-1)\rfloor$, that is for $s=\frac{1}{2}n-1$ when n is even and for

$s = \frac{1}{2}(n-1)$ when n is odd, the diameter could increase by one. We next show that this indeed happens.

When n is even, consider vertices A and B such that $d_H(A,B) = k = \frac{1}{2}n - 1$ which corresponds to case 1 in Proposition 1. If all vertices V_i are deleted from $\square_{n+1}$, the remaining shortest paths through vertices W_i or $\overline{A}$ have length $k+2 = \frac{1}{2}n + 1 = \lfloor \frac{1}{2}(n+1) \rfloor + 1 = D + 1$.

Analogously, when n is odd, consider vertices A and B such that $d_H(A,B) = k = \frac{1}{2}(n+3)$ which corresponds to case 3 in Proposition 1. If all vertices W_i and $\overline{A}$ are deleted from $\square_{n+1}$, the remaining shortest paths through the vertices V_i have length $n - k + 3 = \frac{1}{2}(n+1) + 1 = D + 1$. □

Clearly, the same result can be obtained when instead of vertices we delete edges, since the upper bounds are attained when the edges joining A to its selected neighbours are deleted from $\square_{n+1}$. Therefore, we now have

Theorem 2. *Let D''_s be the maximum value of the diameter of the subgraphs obtained from $\square_{n+1}$ when s edges are deleted. Then*

$$D''_s = \begin{cases} D & \text{if } 0 \leqslant s < \lfloor \frac{1}{2}(n-1) \rfloor, \\ D+1 & \text{if } \lfloor \frac{1}{2}(n-1) \rfloor \leqslant s \leqslant n. \end{cases}$$

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